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The Quasi-Boundary Regularization Method for Identifying the Source Term of Time-Fractional Diffusion Wave Equation

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Abstract

In this paper, we consider the inverse problem for identifying the source term of time-fractional diffusion wave equation. We prove that the inverse problem is ill posed and use the quasi boundary regularization method to solve the ill posed nature of the equation solution. Based on prior bounds, we provide corresponding error estimates to verify the effectiveness of the method.

Keywords: Time-fractional diffusion wave equation; Inverse problem; Source term; Regularization method

1 Introduction

In this paper, the inverse problem of inversion of source term from time-fractional diffusion wave equation is considered. The time-fractional telegraph equation can be expressed as follows: [1]

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = k\frac{\partial^2 u(x,t)}{\partial x^2} + f(x), & 0 \leqslant x \leqslant L, t > 0, 1 < \alpha < 2, \\ u(0,t) = u(L,t) = 0, & t > 0, \\ u(x,0) = 0, & 0 \leqslant x \leqslant L, \\ u_t(x,0) = 0, & 0 \leqslant x \leqslant L, \\ u(x,T) = g(x), & 0 \leqslant x \leqslant L, \end{cases} \tag{1.1}$$

where k is a constant, $u(x,0)$ is the initial data, $D_t^\alpha u(x,t)$ is Caputo fractional derivative. The Caputo fractional derivative of order α is defined as [2]:

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^m(\tau)}{(t-\tau)^{1+\alpha-m}} d\tau, & m-1<\alpha<m \\ \frac{d^m}{dt^m} f(t), & \alpha = m, \end{cases}$$

where $\Gamma(\cdot)$ is a gamma function.

Problem (1.1) is a forward problem when the function $f(x)$ is given appropriately. The inverse problem here is to determine the source term $f(x)$ based on Problem (1.1) and an additional condition

$$u(x,T) = g(x), \quad 0<x<L, \tag{1.2}$$

In practical applications, the input data $g(x)$ is given by measurement, measurement data are often inconsistent with accurate data, so we actually have the measured data $g(x)$ which satisfies

$$\|g(\cdot) - g^\delta(\cdot)\| \leqslant \delta, \tag{1.3}$$

where $\|\cdot\|$ is the $L^2(\Omega)$ norm and $\delta > 0$ is the measurement error.
The traditional diffusion equation, often described by Fick's second diffusion law, is widely used to understand diffusion phenomena [3–7]. However, for processes with non-local, nonlinear, and non-Markov properties, such as fractional diffusion, the fractional diffusion wave equation is more appropriate. This equation finds applications in various fields like physics, biology, and finance [9–11].

This paper focuses on employing the quasi-boundary regularization method to solve the inverse problem of fractional diffusion wave equations. Chapter 2 presents the lemma and problem solution, while Chapter 3 introduces the method, providing the regularization solution and error estimation proof for the source term. Finally, we validate the rationality of the regularized solution through prior error estimation.

2 The solution of the problem(1.1) and the result of conditional stability

In this section, we mainly give the proof of ill posedness of problems(1.1) , the solution of the problem (1.1) and the result of conditional stability. Let $\lambda_n = k(\frac{n\pi}{L})^2$ and $X_n(x) = sin\frac{n\pi}{L}x$ be the Dirichlet eigenvalues and eigenfunctions of $-\frac{\partial^2 u}{\partial x^2}$ on the domain $[0,L]$, satisfies

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} \chi_n(x) = -\frac{\lambda_n}{k} X_n(x), & 0 \le x \le L, \\ X_n(0) = X_n(L) = 0, \end{cases} \tag{2.1}$$

where $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$, $\lim_{n\to\infty} \lambda_n = +\infty$ and $\chi_n(x) \in H^2[0, L] \cap H_0^1[0, L]$, then $\{X_n\}_{n=1}^{\infty}$ can be normalized as the orthonormal basis in space $L^2[0, L]$.

For any $p > 0$, we define the space

$$H^p[0, L] = \Big\{\phi \in L^2[0, L]\Big| \sum_{n=1}^{\infty} \lambda_n^p |(\phi, X_n)|^2 < \infty\Big\}, \tag{2.2}$$

where $(\cdot, \cdot)$ is the inner product in $L^2[0, L]$, then $H^p[0, L]$ is a Hilbert space with the norm

$$\|\phi\|_{H^p(\Omega)} := \Big(\sum_{n=1}^{\infty} \lambda_n^p |(\phi, X_n)|^2\Big)^{\frac{1}{2}}. \tag{2.3}$$

Using characteristic function, variable separation method and Laplace transform, the solution of the problem (1.1) is obtained

$$u(x, t) = \sum_{n=1}^{\infty} t^{\alpha} E_{\alpha,\alpha+1}(-\lambda_n t^{\alpha}) f_n X_n(x), \tag{2.4}$$

where $f_n = (f(x), \chi_n(x))$ is the Fourier coefficient. Using $u(x, T) = g(x)$, and according to (1.1) and (1.2), we have:

$$u(x, T) = \sum_{n=1}^{\infty} T^{\alpha} E_{\alpha,\alpha+1}(-\lambda_n T^{\alpha}) f_n X_n(x), \quad g_n = T^{\alpha} E_{\alpha,\alpha+1}(-\lambda_n T^{\alpha}) f_n, \tag{2.5}$$

where $g_n = (g(x), x_n(x))$ is the Fourier coefficient. So we get the exact solution of the source term from (2.5)

$$f(x) = \sum_{n=1}^{\infty} \frac{g_n}{T^{\alpha} E_{\alpha,\alpha+1}(-\lambda_n T^{\alpha})}, \tag{2.6}$$

To address the problem more effectively, we introduce the following definitions and lemmas.

Definition 2.1. *[11] Mittag-leffler The function is defined as follows:*

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, z \in C \tag{2.7}$$

where $\alpha > 0$ and $\beta \in R$ are arbitrary constants.

Lemma 2.1. *[12]If $\lambda > 0$, then the following equation holds:*

$$\int_0^\infty e^{-pt}t^{\alpha m+\beta-1}E_{\alpha,\beta}^{(m)}\left(\pm at^\alpha\right)dt = \frac{m!p^{\alpha-\beta}}{\left(p^\alpha \mp a\right)^{m+1}}, \mathrm{Re}(p) > |a|^{\frac{1}{\alpha}}, \tag{2.8}$$

*where $E_{\alpha,\beta}^{(m)}(y) := \frac{d^m}{dy^m}E_{\alpha,\beta}(y)$. **Lemma** 2.1 means that the Laplace transformation of*
$t^{\alpha m+\beta-1}E_{\alpha,\beta}^{(m)}\left(\pm at^\alpha\right)$ is $\frac{m!p^{\alpha-\beta}}{(p^\alpha \mp a)^{m+1}}$.

Lemma 2.2. *[12]For any $\alpha > 0, \gamma \in R$, the following conclusions hold:*

$$E_{\alpha,\gamma}(z) = zE_{\alpha,\alpha+\gamma}(z) + \frac{1}{\Gamma(\gamma)}, \quad z \in \mathbb{C}, \tag{2.9}$$

Lemma 2.3. *[13]For $2 > \alpha > 0$, $t > 0$, we have $0 < E_{\alpha,1} < 1$. More over, $E_{\alpha,1}(-t)$ is completely monotonic, that is:*

$$(-1)^n \frac{d^n}{dt^n}E_{\alpha,1}(-t) \geq 0 \tag{2.10}$$

Lemma 2.4. *[13]For $1 < \alpha < 2$ and any λ_n satisfying $\lambda_n > \lambda_1 > 0$, there exists a positive constant C_1 such that :*

$$\frac{1}{\lambda_n T^\alpha} \leq \left|E_{\alpha,\alpha+1}\left(-\lambda_n T^\alpha\right)\right| \leq \frac{C_1}{\lambda_n T^\alpha}. \tag{2.11}$$

Lemma 2.5. *For any constants $p > 0$, $\mu > 0$, $T > 0$, C_1 and $0 < \lambda_1 \leqslant s$, the following inequality holds:*

$$A(s) = \frac{\mu s^{1-\frac{p}{2}}}{\mu s+1} \leqslant \begin{cases} C_2\mu^{\frac{p}{2}}, & 0 < p < 2, \\ C_3\mu, & p \geqslant 2, \end{cases} \tag{2.12}$$

where $C_2 := \frac{\left(\frac{(2-p)}{P}\right)^{1-\frac{p}{2}}}{\frac{2}{p}}$, $C_3 := \lambda_1^{1-\frac{p}{2}}, s = \lambda_n$.

Proof: When $0 < p < 2$, due to $\lim_{s\to 0} A(s) = 0$ and $\lim_{s\to\infty} A(s) = 0$, then we obtain

$$A(s) \leqslant \sup_{s\geqslant\lambda_1} A(s) \leqslant A(s^*).$$

Here s^* is the root of equation $A'(s) = 0$ and its value is $s^* = \frac{(2-p)}{p\mu}$.
so we have

$$A(s) \leqslant A\left(s^*\right) = \frac{\mu\left(\frac{(2-p)}{P\mu}\right)^{1-\frac{p}{2}}}{\left(\frac{2-p}{p}\right)+1} =: C_2(p)\mu^{\frac{p}{2}}.$$

When $p \geqslant 2$,

$$A(s) = \frac{\mu s^{1-\frac{p}{2}}}{\mu s + 1} = \frac{\mu}{(\mu s + 1) s^{\frac{p}{2}-1}} \leqslant \mu \lambda_1^{1-\frac{p}{2}} =: C_3\left(p, \lambda_1\right) \mu.$$

Lemma 2.6. *For any constants $p > 0$, $\mu > 0$, C_1 and $0 < \lambda_1 \leqslant s$, the following inequality holds:*

$$B(s) = \frac{\mu\left(C_1\right)^{\frac{1}{2}}}{\lambda_n \mu + 1} \lambda_n^{\frac{2-p}{4}} \leqslant \begin{cases} C_4 \mu^{\frac{p+2}{4}}, & 0 < p < 2, \\ C_5 \mu, & p \geqslant 2, \end{cases} \tag{2.13}$$

where $C_4 := \frac{(C_1)^{\frac{1}{2}}\left(\frac{(2-p)}{p+2}\right)^{\frac{2-p}{4}}}{\frac{(2-p)}{p+2}+1}$, $C_5 := \frac{(C_1)^{\frac{1}{2}}}{\lambda_1^{\frac{p-2}{4}}}$.

Proof: When $0 < p < 2$, due to $\lim_{s \to 0} B(s) = 0$ and $\lim_{s \to \infty} B(s) = 0$, then we obtain

$$B(s) \leqslant \sup_{s \geqslant \lambda_1} B(s) \leqslant B(s_0).$$

Here s_0 is the root of equation $B^{'}(s) = 0$ and its value is $s_0 = \left(\frac{(2-p)}{\mu(p+2)}\right)$. so we have

$$G(s) \leqslant G\left(s_0\right) = \frac{\mu (C_1)^{\frac{1}{2}} \left(\frac{(2-p)}{(p+2)\mu}\right)^{\frac{2-p}{4}}}{\frac{(2-p)}{p+2} + 1} =: C_4(p, C_1) \mu^{\frac{p+2}{4}}.$$

When $p \geqslant 2$,

$$G(s) = \frac{\mu (C_1)^{\frac{1}{2}} s^{\frac{2-p}{4}}}{s\mu + 1} \leqslant \frac{\mu (C_1)^{\frac{1}{2}}}{\lambda_1^{\frac{p-2}{4}}} =: C_5\left(p, C_1, \lambda_1\right) \mu.$$

Define operator $K: f(\cdot) \longrightarrow g(\cdot)$, then problem (1.1) can be transformed into the following operator equation: $Kf(x) = g(x), x \in [0, L]$, where K satisfies $Kf(x) = g(x) = \sum_{n=1}^{\infty} T^{\alpha} E_{\alpha,\alpha+1}(-\lambda_n T^{\alpha}) f_n X_n(x)$. Obviously, K is a linear self adjoint operator, and its eigenvalues and eigenvectors are respectively: $K_n = T^{\alpha} E_{\alpha,\alpha+1}(-\lambda_n T^{\alpha})$ and $X_n(x)$. Due to $g_n = f_n \cdot T^{\alpha} E_{\alpha,\alpha+1}(-\lambda_n T^{\alpha})$, thus $f_n = K_n^{-1} \cdot g_n$. Therefore, we have

$$f(x) = \sum_{n=1}^{\infty} \frac{g_n}{T^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda_n T^{\alpha}\right)} X_n(x) \tag{2.14}$$

According to (2.6), a small disturbance of $g(x)$ will cause significant changes in the source term $f(x)$ Therefore, this is an ill posed problem that cannot be solved using classical methods and needs to be solved through regularization methods. Firstly, we provide a prior bound for the exact solution $f(x)$

$$\|f(\cdot)\|_{H^p(\Omega)} = \left(\sum_{n=1}^{\infty} \lambda_n^p \left|(f, \chi_n)\right|^2\right)^{\frac{1}{2}} \leqslant E, \tag{2.15}$$

where E and p are normal numbers.

3 Quasi-Boundary Regularization Method and Its Convergence Estimation

In this section, we utilize the Quasi-boundary regularization method to solve problem (1.1) by introducing a penalty term to the final data. This approach transforms the original problem into finding the solution to a modified equation. We also provide H"older type error estimates between the exact solution and the regularization solution.

$$\begin{cases} \frac{\partial^\alpha u_\mu^\delta(x,t)}{\partial t^\alpha} = k\frac{\partial^2 u_\mu^\delta(x,t)}{\partial x^2} + f_\mu^\delta(x), & 0 \leqslant x \leqslant L, t > 0, 1 < \alpha < 2, \\ u_\mu^\delta(0,t) = u_\mu^\delta(L,t) = 0, & t > 0, \\ u_\mu^\delta(x,0) = 0, & 0 \leqslant x \leqslant L, \\ (u_\mu^\delta)_t(x,0) = 0, & 0 \leqslant x \leqslant L, \\ u_\mu^\delta(x,T) + \mu f_\mu^\delta(x) = g^\delta(x), & 0 \leqslant x \leqslant L, \end{cases} \tag{3.1}$$

where $\mu > 0$ is the regularization parameter. Similarly, the separation of variables method and the Laplace transform can be used to obtain solution u_μ^δ of formula (3.1)

$$u_\mu^\delta(x,t) = \sum_{n=1}^{\infty} t^\alpha E_{\alpha,\alpha+1}(-\lambda_n t^\alpha)(f_\mu^\delta)_n X_n(x), \tag{3.2}$$

Thus

$$(f_\mu^\delta)_n = \frac{g_n}{T^\alpha E_{\alpha,\alpha+1}\left(-\lambda_n T^\alpha\right) + \mu}, \tag{3.3}$$

where $\mu > 0$ is the regularization parameter.
Therefore, the regular solution expressions with and without errors can be written in the following form:

$$f_\mu^\delta(x) = \sum_{n=1}^{\infty} \frac{g_n^\delta}{T^\alpha E_{\alpha,\alpha+1}\left(-\lambda_n T^\alpha\right) + \mu} X_n(x) \qquad , f_\mu(x) = \sum_{n=1}^{\infty} \frac{g_n}{T^\alpha E_{\alpha,\alpha+1}\left(-\lambda_n T^\alpha\right) + \mu} X_n(x) \tag{3.4}$$

To invert the source term $f(x)$, the following integral equation needs to be solved:

$$(Kf)(x) := \int_{\Omega} k(x,\xi)f(\xi)d\xi = g(x), \qquad k(x,\xi) = \sum_{n=1}^{\infty} T^{\alpha}E_{\alpha,\alpha+1}\left(-\lambda_n T^{\alpha}\right)X_n(x)X_n(\xi) \tag{3.5}$$

3.1 The convergent error estimate with an a priori parameter choice rule

Theorem 3.1. *Assuming a priori bound* (2.15) *and a noise assumption* (1.3) *hold, then we have*
(1)If $0 < p < 2$ *and the regularization parameter* $\mu = (\frac{\delta}{E})^{\frac{2}{p+2}}$ *is selected, then there is*

$$\|f_{\mu}^{\delta}(\cdot) - f(\cdot)\| \leqslant (1 + C_2)E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}; \tag{3.6}$$

(2)If $p \geqslant 2$ *and the regularization parameter* $\mu = (\frac{\delta}{E})^{\frac{1}{2}}$ *is selected, then there is*

$$\|f_{\mu}^{\delta}(\cdot) - f(\cdot)\| \leqslant (1 + C_3)E^{\frac{1}{2}}\delta^{\frac{1}{2}}, \tag{3.7}$$

Proof: By means of a triangular inequality, we have

$$\|f_{\mu}^{\delta}(\cdot) - f(\cdot)\| \leqslant \|f_{\mu}^{\delta}(\cdot) - f_{\mu}(\cdot)\| + \|f_{\mu}(\cdot) - f(\cdot)\|. \tag{3.8}$$

Let us first give an estimate of the first term of (3.8). Through (3.4), (1.3) , we obtain

$$\begin{aligned} \left\|f_{\mu}^{\delta}(\cdot) - f_{\mu}(\cdot)\right\|^2 &= \left\|\sum_{n=1}^{\infty} \frac{g_n^{\delta} - g_n}{T^{\alpha}E_{\alpha,\alpha+1}\left(-\lambda_n T^{\alpha}\right) + \mu} X_n(x)\right\|^2 \\ &= \sum_{n=1}^{\infty} \left(\frac{g_n^{\delta} - g_n}{T^{\alpha}E_{\alpha,\alpha+1}\left(-\lambda_n T^{\alpha}\right) + \mu}\right)^2 \\ &\leqslant \left(\frac{\delta}{\mu}\right)^2. \end{aligned}$$

Then

$$\|f_{\mu}^{\delta}(\cdot) - f_{\mu}(\cdot)\| \leqslant \frac{\delta}{\mu}. \tag{3.9}$$

Now let us estimate the second term of equation (3.8). Using (2.13), and **Lemma** 2.5, we can deduce

$$\begin{aligned}\|f_\mu(\cdot)-f(\cdot)\|^2 &= \left\|\sum_{n=1}^{\infty}\frac{g_n}{T^\alpha E_{\alpha,\alpha+1}\left(-\lambda_n T^\alpha\right)+\mu}X_n(x)-\sum_{n=1}^{\infty}\frac{g_n}{T^\alpha E_{\alpha,\alpha+1}\left(-\lambda_n T^\alpha\right)}X_n(x)\right\|^2\\ &=\sum_{n=1}^{\infty}\left(\frac{g_n}{T^\alpha E_{\alpha,\alpha+1}\left(-\lambda_n T^\alpha\right)}\right)^2\lambda_n^p\left(\frac{\mu}{T^\alpha E_{\alpha,\alpha+1}\left(-\lambda_n T^\alpha\right)+\mu}\right)^2\lambda_n^{-p}\\ &\leqslant\sum_{n=1}^{\infty}\left(\frac{g_n}{T^\alpha E_{\alpha,\alpha+1}\left(-\lambda_n T^\alpha\right)}\right)^2\lambda_n^p\left(\frac{\mu\lambda_n^{1-\frac{p}{2}}}{1+\mu\lambda_n}\right)^2\\ &\leqslant E^2\sup_{n\geqslant 1}(A(n))^2,\end{aligned} \tag{3.10}$$

Applying **Lemma** 2.5, we obtain

$$A(n)=\frac{\mu\lambda_n^{1-\frac{p}{2}}}{\mu\lambda_n+1}=\frac{\mu s^{1-\frac{p}{2}}}{\mu s+1}\leqslant\begin{cases}C_2\mu^{\frac{p}{2}}, & 0<p<2,\\ C_3\mu, & p\geqslant 2.\end{cases} \tag{3.11}$$

Therefore, we have

$$\|f_\mu(\cdot)-f(\cdot)\|\leqslant\begin{cases}C_2E\mu^{\frac{p}{2}}, & 0<p<2,\\ C_3E\mu, & p\geqslant 2.\end{cases} \tag{3.12}$$

Combining (3.9) with (3.12), we obtain

$$\|f_\mu^\delta(\cdot)-f(\cdot)\|\leqslant\frac{\delta}{\mu}+\begin{cases}C_2E\mu^{\frac{p}{2}}, & 0<p<2,\\ C_3E\mu, & p\geqslant 2.\end{cases} \tag{3.13}$$

By choosing the regularization parameters $\mu=(\frac{\delta}{E})^{\frac{2}{p+2}}(0<p<2)$ and $\mu=(\frac{\delta}{E})^{\frac{1}{2}}(p\geqslant 2)$, we have the following results.

$$\|f_\mu^\delta(\cdot)-f(\cdot)\|\leqslant\begin{cases}(1+C_2)E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}, & 0<p<2,\\ (1+C_3)E^{\frac{1}{2}}\delta^{\frac{1}{2}}, & p\geqslant 2.\end{cases} \tag{3.14}$$

The proof of **Theorem** 3.1 is completed. □

3.2 Conclusion

The **Theorem** 3.1 indicates that the quasi boundary method successfully improves the stability of the solution, therefore the quasi boundary regularization method is very effective for the inverse source problem of fractional diffusion wave equations.

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