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A Reduced Stability Criterion for Schur Stable Polynomials

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Abstract

In a recent paper, we advanced a systematic procedure for generating the coefficients of the continued fraction expansion of the test function associated with a Schur stable polynomial. The procedure we used involves long and elaborate calculations and the number of required iterations equal to the degree of the polynomial plus one. In the current work, we propose new continued fraction expansions which proceed in terms of some bilinear functions. the procedure to generate the coefficients of the new expansion is extremely simple. In addition, the number of iterations required to generate the coefficients of the new proposed expansion almost equal half the degree of the polynomial, which is a significant simplification in testing the stability of such polynomials. Illustrative examples are given to highlight the advantages of the new procedure compared to the old one.

Mathematics Subject Classification: Primary 37C75, 93C15. Secondary 93D, 30E

Keywords: Schur–Cohn stability; Schur polynomials; continued fractions; stability of discrete-time systems

1. Introduction

Polynomials having their zeros in the interior of the unit circle are said to be stable in the Schur-Cohn sense. We shall call them Schur stable polynomials. They are essential in a variety of applications, such as stability of discrete-time systems, digital signal processing, control theory, spectral analysis, numerical computations, and many others.

Recently, there has been renewed interest in the subject of continued fractions and its applications in stability theory [1, 2, 3, 6, 10]. In particular, we mention our relatively recent work [7] where we established interesting properties of the poles of positive functions which are Routh-Hurwitz related concepts. Symmetric properties were also established of the poles of complex discrete reactance functions which are the counterparts of positive functions in the Schur-Cohn stability type. The efforts initiated in [7] were thoroughly pursued in very recent works [8, and 9] where connections between continued fractions and Schur polynomials have been established.

In [9], we introduced a procedure to generate the coefficients of the continued fraction expansion of the test function associated with a Schur polynomial. Examples were given to illustrate the feasibility of the process but at the expense of very elaborate computations. This laborious work is best illustrated in the proofs of Theorems 2 and 3 in [9] as well as in the example given in Section 4 therein. The formulas given in the statements of these two theorems and their application in the given example show that the number of required iterations to generate the coefficients of the continued fraction expansion is equal to n+1, where n is the degree of the polynomial.

In the current work, we introduce new continued fraction expansions in terms of some bilinear functions in which the number of iterations is reduced from n+1 to n/2 when n is even and to (n+1)/2 when n is odd. More importantly, the burden of computations involved to generate the coefficients of the new expansions is highly reduced compared with the drudgery involved in the old expansion [9]. Illustrative examples are given to highlight the advantages of the new technique.

This paper is structured as follows. In Section 2, we lay out some definitions and notations. In Section 3, we introduce the new continued fraction expansions associated with a Schur stable polynomial of degree n and we prove that when n is even, we require n/2 iterations to generate the coefficients of the continued fraction. When n is odd, (n + 1)/2 iterations are required for the same purpose. Here, we stress the overwhelming simplicity to generate the coefficients of the new expansions compared with the laborious work of [9]. In section 4, we illustrate those advantages and the extreme simplicity in the computations by concrete examples.

2. Definitions and Notations

Throughout the paper, we shall use the same notations as in [9]. Below is a reminder of some of the definitions, notations, and results needed for the current work.

Definition 1. *A linear discrete-time system of difference equations is stable if and only if all its eigenvalues lie inside the unit disc. If*

$$g(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n \qquad (1)$$

is the characteristic polynomial of the system, then the system is stable if all zeros of g (z) lie inside the unit disc. Such polynomials are said to be Schur stable.

Definition 2. *The reciprocal of g is defined by* $g^{\tau}(z) = z^n \overline{g(1/\bar{z})}$. *Then* g^{τ} *can be written as* $g^{\tau}(z) = \bar{a}_n + \bar{a}_{n-1} z + \bar{a}_{n-2} z^2 + \cdots + \bar{a}_0 z^n$ *where* $\bar{a}_k$ *denotes the complex conjugate of* a_k *for* $k = 0, 1, \ldots, n.$

Definition 3. *The test function of the given discrete − time system is defined by*

$$\Psi(z) = \frac{g(z) + g^{\tau}(z)}{g(z) - g^{\tau}(z)} \qquad (2)$$

We note that in [8], we flipped the numerator and denominator of $\Psi(z)$, but that will not make any difference, since the expression of $\Psi(z)$ and $1/\Psi(z)$ as continued fractions are equivalent.

Theorem 1. [8, Theorem 4]) *The linear discrete-time system of difference equations characterized by (1) is stable if and only if the test function* $\Psi(z)$ *defined by* (2) *can be written in the continued fraction expansion*

$$\Psi(z) = h_0 \frac{z-1}{z+1} + k_0 + \cfrac{1}{h_1 \frac{z-1}{z+1} + k_1 + \cfrac{\vdots}{+ \cfrac{1}{h_n \frac{z-1}{z+1} + k_n}}} \qquad (3)$$

where $h_0 \geq 0, h_1 > 0, \ldots, h_n > 0$ *and* k_j are *imaginary or zero for* $0 \leq j \leq n.$

It is clear from Theorem 1 that the number of iterations required to generate the coefficients of (3) is equal to n + 1, one for each pair h_i, k_i for i = 0, 1, …, n.

3. The New Procedure

We now introduce new continued fraction expansions in which the number of iterations is reduced from n+1 to n/2 when n is even and to (n+1)/2 when n is odd which is a tremendous simplification in stability testing.

Theorem 2. *If n is even, the linear discrete-time system of difference equations characterized by* (1) *is stable if and only if the test function*

$$\Psi(z) = \frac{g(z) + g^{\tau}(z)}{g(z) - g^{\tau}(z)}$$

can be written in the form

$$\Psi(z) = h_1 \frac{z-1}{z+1} + k_1 \frac{z+1}{z-1} + \Psi_1(z) \qquad (4)$$

where h_1 *and* k_1 *are positive and* $\Psi_1(z)$ *has degree* $n-2$ *and has* 1 *and* -1 *as zeros.*

Proof. By [5, Prop. 2 P.101], $\Psi(z)$ can always be written as

$$\Psi(z) = k_1 \frac{z+1}{z-1} + \Psi_0(z)$$

where k_1 is positive and $1/\Psi_0(z)$ has a pole at z = 1 .

Now, since n is even z = –1 and z = 1 are both poles of $\Psi(z)$, and by [4, Condition 2a, P. 87] they are both simple, and that leads to the following expansion of $\Psi(z)$

$$\Psi(z) = h_1 \frac{z-1}{z+1} + k_1 \frac{z+1}{z-1} + \Psi_1(z)$$

where h_1, k_1 and $\Psi_1(z)$ satisfy the requirements of the theorem, and $\frac{1}{\Psi_1(z)}$ has two poles $z = \pm 1$.

Corollary 1. *If n is even, the linear discrete-time system of difference equations characterized by* (1) *is stable if and only if the test function*

$$\Psi(z) = \frac{g(z) + g^{\tau}(z)}{g(z) - g^{\tau}(z)}$$

can be written in the continued fraction expansion form

$$\Psi(z) = h_1 \frac{z-1}{z+1} + k_1 \frac{z+1}{z-1} + \cfrac{1}{h_2 \frac{z-1}{z+1} + k_2 \frac{z+1}{z-1} + \begin{array}{l} \vdots \\ + \cfrac{1}{h_{n/2} \frac{z-1}{z+1} + k_{n/2} \frac{z+1}{z-1}} \end{array}} \qquad (5)$$

where h_i *and* k_i *are positive for* $1 \le i \le n/2$.

Proof. By Theorem 2, $\Psi(z)$ can be written in the form

$$\Psi(z) = h_1 \frac{z-1}{z+1} + k_1 \frac{z+1}{z-1} + \Psi_1(z)$$

where h_1 and k_1 are positive and $\Psi_1(z)$ has degree $n - 2$ and $\frac{1}{\Psi_1(z)}$ has two poles $z = \pm 1$.

Rewrite $\Psi(z)$ as

$$\Psi(z) = h_1 \frac{z-1}{z+1} + k_1 \frac{z+1}{z-1} + \frac{1}{\left(\frac{1}{\Psi_1(z)}\right)}.$$

We now operate on $\frac{1}{\Psi_1(z)}$ in the same way as we operated before on $\Psi(z)$ to get

$$\Psi(z) = h_1 \frac{z-1}{z+1} + k_1 \frac{z+1}{z-1} + \frac{1}{h_2 \frac{z-1}{z+1} + k_2 \frac{z+1}{z-1} + \Psi_2(z)}$$

where h_2 and k_2 are positive and $\Psi_2(z)$ has degree $n - 4$ and $\frac{1}{\Psi_2(z)}$ has two poles $z = \pm 1$.

We now operate on $\frac{1}{\Psi_2(z)}$ in the same way as we did with $\frac{1}{\Psi_1(z)}$.

The process continues in the same way till the end.

As mentioned above, Theorem 2 applies only when the degree of $g(z)$ in (1) is even. According to expansion (5), only n/2 iterations are needed, one for each pair h_i, k_i for i = 1, …, n/2.

Now, when n is odd, we refer again to Proposition 2 of [5], which states the following.

Theorem 3. [5, Proposition 2] *The linear discrete-time system of difference equations characterized by* (1) *is stable if and only if the test function*

$$\Psi(z) = \frac{g(z) + g^{\tau}(z)}{g(z) - g^{\tau}(z)}$$

can be written in the form

$$\Psi(z) = s_1 \frac{z+1}{z-1} + \Psi_1(z) \qquad (6)$$

where

1. s_1 *is real and positive,*
2. *the function* $1/\Psi_1(z)$ *has a pole at* $z = 1$.

By [5, P. 101] s_1 can be calculated as

$$s_1 = \lim_{z \to 1} \left(\frac{z-1}{z+1} \cdot \Psi(z) \right)$$

The last theorem applies independently of the parity of the degree of $g(z)$.

Corollary 2. *If n is odd, the linear discrete-time system of difference equations characterized by* (1) *is stable if and only if the test function*

$$\Psi(z) = \frac{g(z) + g^{\tau}(z)}{g(z) - g^{\tau}(z)}$$

can be written in the continued fraction expansion form

$$\Psi(z) = s_1 \frac{z+1}{z-1} + \cfrac{1}{r_2 \frac{z-1}{z+1} + s_2 \frac{z+1}{z-1} + \atop \vdots \qquad + \cfrac{1}{r_{(n+1)/2} \frac{z-1}{z+1} + s_{(n+1)/2} \frac{z+1}{z-1}}} \qquad (7)$$

where s_i *are positive for* $1 \le i \le (n+1)/2$, *and* r_i *are positive for* $2 \le i \le (n+1)/2$.

Proof. By Theorem 3, $\Psi(z)$ can be expressed as in (6)

$$\Psi(z) = s_1 \frac{z+1}{z-1} + \Psi_1(z)$$

After isolating the term $s_1 \frac{z+1}{z-1}$ of $\Psi(z)$, the characteristic polynomial corresponding to test function $\Psi_1(z)$ is now of even degree, and Corollary (1) can now be applied to generate the remaining terms of the expansion (7).

As a result of the above argument, s_2 up to $s_{(n+1)/2}$ can be calculated in the same way as s_1

namely,

$$s_2 = \lim_{z \to 1} \left(\frac{z-1}{z+1} \cdot \frac{1}{\Psi_1(z)} \right), \text{where}$$

$$\Psi_1(z) = \Psi(z) - s_1 \frac{z+1}{z-1}$$

and so on for s_3 up to $s_{(n+1)/2.}$

Similar considerations apply on r_2 and we have

$$r_2 = \lim_{z \to -1} \left(\frac{z+1}{z-1} \cdot \frac{1}{\Psi_1(z)} \right), \text{where also}$$

$$\Psi_1(z) = \Psi(z) - s_1 \frac{z+1}{z-1}$$

and so on for r_3 up to $r_{(n+1)/2}$.

Before we illustrate the above results through concrete examples, we mention that the coefficients in the expansion of $\Psi(z)$ corresponding to n even given by (5), namely

$$\Psi(z) = h_1 \frac{z-1}{z+1} + k_1 \frac{z+1}{z-1} + \cfrac{1}{h_2 \frac{z-1}{z+1} + k_2 \frac{z+1}{z-1} + \atop \vdots \atop + \cfrac{1}{h_{n/2} \frac{z-1}{z+1} + k_{n/2} \frac{z+1}{z-1}}}$$

can also be obtained in the same way. We have

$$h_1 = \lim_{z \to -1} \left(\frac{z+1}{z-1} \cdot \Psi(z) \right), \text{and}$$

$$h_2 = \lim_{z \to -1} \left(\frac{z+1}{z-1} \cdot \frac{1}{\Psi_1(z)} \right) \text{ where}$$

$$\Psi_1(z) = \Psi(z) - h_1 \frac{z-1}{z+1} - k_1 \frac{z+1}{z-1}$$

and so on for h_3 up to $h_{n/2}$.

Similarly,

$$k_1 = \lim_{z \to 1} \left(\frac{z-1}{z+1} \cdot \Psi(z) \right), \text{and}$$

$$k_2 = \lim_{z \to 1} \left(\frac{z-1}{z+1} \cdot \frac{1}{\Psi_1(z)} \right) \text{ where}$$

$$\Psi_1(z) = \Psi(z) - h_1 \frac{z-1}{z+1} - k_1 \frac{z+1}{z-1}$$

and so on for k_3 up to $k_{n/2}$.

4. Feasibility of the New Procedure

We will consider both cases when n is even or odd. For each case, we shall compare the old procedure [9] with the new one.

<u>Case 1</u>: n even

Consider the polynomial with degree n = 4,

$$g(z) = 8z^4 - 8z^3 + 2z^2 + 2z - 1$$

(a) The old procedure [9]:

The reciprocal of g is

$$g^{\tau}(z) = z^n\overline{g(1/\bar{z})} = -z^4 + 2z^3 + 2z^2 - 8z + 8$$

Therefore, the test function can be written as

$$\Psi(z) = \frac{g(z) + g^{\tau}(z)}{g(z) - g^{\tau}(z)} = \frac{7z^4 - 6z^3 + 4z^2 - 6z + 7}{9z^4 - 10z^3 + 10z - 9}$$

Now we proceed as in [9, Section 4, P. 6] by applying the transformation

$$T(s) = \Psi(2s-1).$$

Then,

$$T(s) = \Psi(2s-1) = \frac{7(2s-1)^4 - 6(2s-1)^3 + 4(2s-1)^2 - 6(2s-1) + 7}{9(2s-1)^4 - 10(2s-1)^3 + 10(2s-1) - 9}.$$

$T(s)$ can now be written as

$$T(s) = \frac{56s^4 - 136s^3 + 128s^2 - 60s + 15}{8(9s^4 - 23s^3 + 21s^2 - 7s)}.$$

We would like to expand $T(s)$ in the form

$$\Psi(z) = h_0\left(\frac{s-1}{s}\right) + k_0 + \cfrac{1}{h_1\left(\frac{s-1}{s}\right) + k_1 + \cfrac{1}{h_2\left(\frac{s-1}{s}\right) + k_2 + \cfrac{1}{h_3\left(\frac{s-1}{s}\right) + k_3}}} \qquad (8)$$

We seek the values of $h_0, k_0, h_1, k_1, h_2, k_2$ *and* h_3, k_3 by the above procedure used in [1].

By [9, Formula (5)], we have $T_j(s) = \frac{\sum_{l=0}^{n-j} a_{j,j+l}s^l}{\sum_{l=1}^{n-j} b_{j,j+l}s^l}$,

and By [9, Formula (4)], $T(s) = T_0(s)$, so

$$T(s) = T_0(s) = \frac{\sum_{l=0}^{4} a_{0,l}s^l}{\sum_{l=1}^{4} b_{0,l}s^l} = \frac{a_{0,0} + a_{0,1}s + a_{0,2}s^2 + a_{0,3}s^3 + a_{0,4}s^4}{b_{0,1}s + b_{0,2}s^2 + b_{0,3}s^3 + b_{0,4}s^4},$$

which compared with $T(s) = \dfrac{56s^4 - 136s^3 + 128s^2 - 60s + 15}{72s^4 - 184s^3 + 168s^2 - 56s}$ leads to

$a_{0,0} = 15,\ \ a_{0,1} = -60,\ \ a_{0,2} = 128,\ \ a_{0,3} = -136,\ \ a_{0,4} = 56$
$b_{0,1} = -56,\ \ b_{0,2} = 168,\ \ b_{0,3} = -184,\ \ b_{0,4} = 72.$

Calculation of h_0 and k_0:

By [9, Theorem 1], $h_0 = -\dfrac{a_{0,0}}{b_{0,1}} = -\dfrac{15}{-56} = \dfrac{15}{56}$

By [9, Theorem 2], $k_0 = \dfrac{a_{0,1} - h_0(b_{0,1} - b_{0,2})}{b_{0,1}} = \dfrac{-60 - \frac{15}{56}(-56 - 168)}{-56} = 0$

Calculation of h_1 and k_1:

Again by [9, Theorem 1], $h_1 = -\dfrac{a_{1,1}}{b_{1,2}}$.

By the formulas of [1, Theorem 3], namely

$$a_{j+1,j+l} = b_{j,j+l},$$

and

$$b_{j+1,j+l} = a_{j,j+l} - h_j\left(b_{j,j+l} - b_{j,j+l+1}\right) - k_j b_{j,j+l},$$

We get, $a_{1,1} = b_{0,1} = -56$.

$b_{1,2} = a_{0,2} - h_0\left(b_{0,2} - b_{0,3}\right) - k_0 b_{0,2} = 128 - \dfrac{15}{56}(168 + 184) - 0 = \dfrac{236}{7}$.

Then,

$$h_1 = -\frac{a_{1,1}}{b_{1,2}} = \frac{56}{\frac{236}{7}} = \frac{98}{59}.$$

Also by [9, Theorem 2], $k_1 = \dfrac{a_{1,2} - h_1(b_{1,2} - b_{1,3})}{b_{1,2}}$.

$b_{1,2} = \dfrac{236}{7}$, already calculated.

$a_{j+1,j+l} = b_{j,j+l}$ leads to $a_{1,2} = b_{0,2} = 168$.

$b_{j+1,j+l} = a_{j,j+l} - h_j\left(b_{j,j+l} - b_{j,j+l+1}\right) - k_j b_{j,j+l}$ leads to $b_{1,3}$

$= a_{0,3} - h_0\left(b_{0,3} - b_{0,4}\right) - k_0 b_{0,3}$

$b_{1,3} = -136 - \dfrac{15}{56}(-184 - 72) - 0 \cdot b_{0,3} = -\dfrac{472}{7}$.

Therefore, $k_1 = \dfrac{a_{1,2} - h_1(b_{1,2} - b_{1,3})}{b_{1,2}} = \dfrac{168 - \frac{98}{59}\left(\frac{236}{7} + \frac{472}{7}\right)}{\frac{236}{7}} = 0$.

Calculation of h_2 and k_2:

Now, $h_2 = -\dfrac{a_{2,2}}{b_{2,3}}$, where

$a_{2,2} = b_{1,2} = \dfrac{236}{7}$, already calculated.

$b_{j+1,j+l} = a_{j,j+l} - h_j\left(b_{j,j+l} - b_{j,j+l+1}\right) - k_j b_{j,j+l}$ leads to $b_{2,3}$

$= a_{1,3} - h_1\left(b_{1,3} - b_{1,4}\right) - k_1 b_{1,3}$

$a_{1,3} = b_{0,3} = -184$.

$b_{1,3} = -\dfrac{472}{7}$ already calculated.

$b_{j+1,j+l} = a_{j,j+l} - h_j\left(b_{j,j+l} - b_{j,j+l+1}\right) - k_j b_{j,j+l}$ implies $b_{1,4}$

$= a_{0,4} - h_0\left(b_{0,4} - b_{0,5}\right) - 0 \cdot b_{1,3}$

$b_{1,4} = 56 - \dfrac{15}{56}(72 - 0) = \dfrac{257}{7}$.

Then, $b_{2,3} = a_{1,3} - h_1\left(b_{1,3} - b_{1,4}\right) - k_1 b_{1,3} = -184 - \dfrac{98}{59}\left(-\dfrac{472}{7} - \dfrac{257}{7}\right) - 0 \cdot b_{1,3}$

$= -\dfrac{650}{59}$

Therefore, $h_2 = -\dfrac{a_{2,2}}{b_{2,3}} = -\dfrac{\frac{236}{7}}{-\frac{650}{59}} = \dfrac{6962}{2275}$.

$k_2 = \dfrac{a_{2,3} - h_2(b_{2,3} - b_{2,4})}{b_{2,3}}$.

$a_{j+1,j+l} = b_{j,j+l}$ leads to $a_{2,3} = b_{1,3} = -\dfrac{472}{7}$

We know that, $b_{2,3} = -\dfrac{650}{59}$.

$b_{j+1,j+l} = a_{j,j+l} - h_j\left(b_{j,j+l} - b_{j,j+l+1}\right) - k_j b_{j,j+l}$ implies $b_{2,4}$

$= a_{1,4} - h_1\left(b_{1,4} - b_{1,5}\right) - k_1 b_{1,4}$.

By the formula $a_{j+1,j+l} = b_{j,j+l}$, we get $a_{1,4} = b_{0,4} = 72$

$b_{1,4} = \dfrac{257}{7}$ already calculated.

$b_{j+1,j+l} = a_{j,j+l} - h_j\left(b_{j,j+l} - b_{j,j+l+1}\right) - k_j b_{j,j+l}$ leads to $b_{1,5}$

$= a_{0,5} - h_0\left(b_{0,5} - b_{0,6}\right) - k_0 b_{0,5}$

So, $b_{1,5} = 0 - \dfrac{15}{56}(0 - 0) - 0 \cdot b_{0,5} = 0$.

So, $b_{2,4} = a_{1,4} - h_1\left(b_{1,4} - b_{1,5}\right) - k_1 b_{1,4} = 72 - \dfrac{98}{59}\left(\dfrac{257}{7} - 0\right) - 0 \cdot b_{1,4} = \dfrac{4550}{413}$.

$$k_2 = \frac{a_{2,3} - h_2(b_{2,3} - b_{2,4})}{b_{2,3}} = \frac{-\frac{472}{7} - \frac{6962}{2275}\left(-\frac{650}{59} - \frac{4550}{413}\right)}{-\frac{650}{59}} = 0.$$

Calculation of h_3 and k_3:

By repeatedly applying the formulas in [9, theorems 3 and 4] as we did in the previous calculations above, we get:

$$h_3 = \frac{650}{177} \text{ and } k_3 = 0.$$

Substituting $h_1, k_{1,}, h_2, k_{2,},$ and h_3, k_3 in expansion (8), we get

$$\Psi(z) = \frac{15}{56}\left(\frac{s-1}{s}\right) + \cfrac{1}{\frac{98}{59}\left(\frac{s-1}{s}\right) + \cfrac{1}{\frac{6962}{2275}\left(\frac{s-1}{s}\right) + \cfrac{1}{\frac{650}{177}\left(\frac{s-1}{s}\right)}}}$$

Replacing $\frac{s-1}{s}$ by $\frac{z-1}{z+1}$ we get:

$$\Psi(z) = \frac{15}{56}\left(\frac{z-1}{z+1}\right) + \cfrac{1}{\frac{98}{59}\left(\frac{z-1}{z+1}\right) + \cfrac{1}{\frac{6962}{2275}\left(\frac{z-1}{z+1}\right) + \cfrac{1}{\frac{650}{177}\left(\frac{z-1}{z+1}\right)}}}$$

Since $h_i > 0$ and $k_i = 0$ for $0 \leq i$
$\leq 3,$ the conditions of [8, Theorem 1] are satisfied,
and the polynomial $g(z) = 8z^4 - 8z^3 + 2z^2 + 2z - 1$ is Shur stable.
In fact the zeros of $g(z)$ are $\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} + \frac{1}{2}i, \frac{1}{2} - \frac{1}{2}i$ all lying inside theunit circle.
At the end of this example, two things should be noted.
1. In this expansion, we needed 4 iterations:
first for $h_1 = \frac{15}{56}$ and $k_1 = 0,$ second for $h_2 = \frac{98}{59}$ and $k_2 = 0,$
third for $h_3 = \frac{6962}{2275}$ and $k_3 = 0,$ and fourth for $h_4 = \frac{650}{177}$ and $k_4 = 0.$

2. More importantly, one should note the elaborate calculations to generates the coefficients
h_i and k_i for $0 \leq i \leq 3.$

(b) The new procedure:

Now, let us expand $\Psi(z)$ using the new procedure illustrated in expansion (5).
By Corollary 1, $\Psi(z)$ can be expanded as:

$$\Psi(z) = h_1\frac{z-1}{z+1} + k_1\frac{z+1}{z-1} + \cfrac{1}{h_2\frac{z-1}{z+1} + k_2\frac{z+1}{z-1}}$$

The coefficients $h_1,\ k_1,\ h_2$ and k_2 are now easily calculated using the new procedure.

$$h_1 = \lim_{z\to -1}\left(\frac{z+1}{z-1}\cdot \Psi(z)\right) = \frac{15}{56}$$
$$k_1 = \lim_{z\to 1}\left(\frac{z-1}{z+1}\cdot \Psi(z)\right) = \frac{3}{16}$$

Easy to verify that the function

$$\Psi_1(z) = \Psi(z) - \frac{15}{56}\left(\frac{z-1}{z+1}\right) - \frac{3}{16}\left(\frac{z+1}{z-1}\right)$$

can be written as:

$$\Psi_1(z) = \frac{325}{112}\left(\frac{z^2-1}{9z^2-10z+9}\right).$$

The coefficients h_2 and k_2 can now be calculated as

$$h_2 = \lim_{z\to -1}\left(\frac{z+1}{z-1}\cdot \frac{1}{\Psi_1(z)}\right) = \frac{784}{325}$$

$$k_2 = \lim_{z\to 1}\left(\frac{z-1}{z+1}\cdot \frac{1}{\Psi_1(z)}\right) = \frac{224}{325}$$

Finally,

$$\Psi(z) = \frac{15}{56}\left(\frac{z-1}{z+1}\right) + \frac{3}{16}\left(\frac{z+1}{z-1}\right) + \cfrac{1}{\frac{784}{325}\left(\frac{z-1}{z+1}\right) + \frac{224}{325}\left(\frac{z+1}{z-1}\right)}$$

Since $h_i > 0$ and $k_i > 0$ for $1 \le i \le 2$, the conditions of Corollary 1 are satisfied, and the polynomial $g(z) = 8z^4 - 8z^3 + 2z^2 + 2z - 1$ is Shur stable.

Also, we should note two things here:

1. In the last expansion, we needed only 2 iterations (half the degree of g): one for h_1 and k_1, another for h_2 and k_2.
2. More importantly, one should note the extremely simple calculations to generate the coefficients h_1, k_1 and h_2, k_2 compared to the highly elaborate computations of the old procedure.

<u>Case 2</u>: n odd.

We reconsider the same example we addressed in [9, P. 6],

Consider the polynomial

$$g(z) = 4z^3 - 6z^2 + 4z - 1$$

(a) The old procedure [9]:

The reciprocal of g is

$$g^\tau(z) = z^n\overline{g(1/\bar{z})} = -z^3 + 4z^2 - 6z + 4$$

Therefore, the test function can be written as

$$\Psi(z) = \frac{g(z) - g^{\tau}(z)}{g(z) + g^{\tau}(z)} = \frac{5z^3 - 10z^2 + 10z - 5}{3z^3 - 2z^2 - 2z + 3}$$

Using the old procedure of [9, P.6-8], $\Psi(z)$ given by (3) above can be expanded as follows:

$$\Psi(z) = \frac{15}{11}\left(\frac{z-1}{z+1}\right) + \cfrac{1}{\frac{121}{40}\left(\frac{z-1}{z+1}\right) + \cfrac{1}{\frac{40}{11}\left(\frac{z-1}{z+1}\right)}}$$

Since $h_i > 0$ and $k_i = 0$ for $0 \leq i \leq 2,$ the conditions of [8, Theorem 1] are satisfied, and the polynomial $g(z) = 4z^3 - 6z^2 + 4z - 1$is Shur stable.

In fact the zeros of $g(z)$ are $\frac{1}{2}, \frac{1}{2} + \frac{1}{2}i, \frac{1}{2} - \frac{1}{2}i$ all lying inside the unit circle.

In this expansion, we needed 3 (degree of g) iterations:

first for $h_1 = \frac{15}{11}, k_1 = 0,$ second for $h_2 = \frac{121}{40}, k_2 = 0,$ and third for $h_3 = \frac{40}{11}, k_3 = 0.$

(b) The new procedure:

Now, let us expand $\Psi(z)$ using the new procedure illustrated in expansion (7), namely

$$\Psi(z) = k_1\frac{z+1}{z-1} + \cfrac{1}{h_2\frac{z-1}{z+1} + k_2\frac{z+1}{z-1}}$$

The coefficients $k_1,\ h_2$ and k_2 are now easily calculated using the new procedure.

$$k_1 = \lim_{z\to 1}\left(\frac{z-1}{z+1}\cdot\Psi(z)\right) = \frac{1}{5}$$

Easy to verify that the function

$$\Psi_1(z) = \Psi(z) - \frac{1}{5}\left(\frac{z+1}{z-1}\right)$$

can be written as:

$$\Psi_1(z) = \frac{2}{5}\left(\frac{z^2-1}{z^2-z+1}\right)$$

and the coefficients h_2 and k_2 can now be calculated as

$$h_2 = \lim_{z \to -1}\left(\frac{z+1}{z-1} \cdot \frac{1}{\Psi_1(z)}\right) = \frac{15}{8}$$

$$k_2 = \lim_{z \to 1}\left(\frac{z-1}{z+1} \cdot \frac{1}{\Psi_1(z)}\right) = \frac{5}{8}$$

Finally,

$$\Psi(z) = \frac{1}{5}\left(\frac{z+1}{z-1}\right) + \frac{1}{\frac{15}{8}\left(\frac{z-1}{z+1}\right) + \frac{5}{8}\left(\frac{z+1}{z-1}\right)}$$

In this expansion, we needed only 2 iterations ((degree+1)/2: one for k_1, another for h_2 and k_2.

We also note the extremely simple calculations to generate the coefficients $k_1, h_2,$ and k_2 compared to the highly elaborate computations of the old procedure [9, P. 6 − 8].

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