Blow-up Phenomenon to a Generalized Camassa-Holm Equation

Ying Wang

Department of Mathematics
Zunyi Normal University, 563006, Zunyi, China

Abstract
Considered in this paper is a generalized Camassa-Holm equation proposed by Novikov. Firstly, a blow-up criterion is established. Then, a new blow-up phenomenon is derived for the equation.

Mathematics Subject Classification: 35D05, 35G25, 35L05, 35Q35

Keywords: Blow-up phenomenon; Blow-up criterion; A generalized Camassa-Holm equation

1 Introduction

One of the most celebrated models of shallow water wave is the Camassa-Holm (CH) equation

\[ u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \quad (1) \]

which was derived first by Fokas and Fuchssteiner [1] as a bi-Hamiltonian generalization of the KdV equation. It describes a certain non-Newtonian fluids and models finite length, small amplitude radial deformation waves in cylindrical hyperelastic rods [2]. The physical derivation and the discovery of soliton for the equation were done by Camassa and Holm [3]. It is shown in [3] that equation (1) possesses a Lax pair and infinitely many conserved integrals. The remarkable features of the CH equation can be found in [4, 5] and the references therein.
In this paper, we consider the Cauchy problem of integrable dispersive wave equation

\[
\begin{aligned}
&\begin{cases}
  u_t - u_{txx} - 4uu_x + 6u_xu_{xx} + 2uu_{xxx} - 2u_x^2 - 2uu_{xx} = 0, \\
  u(0, x) = u_0(x),
\end{cases}
\end{aligned}
\]

which is presented in Novikov [6]. It is shown in [6] that Eq. (2) admits a hierarchy of local higher symmetries. Eq. (2) is regarded as a generalized Camassa-Holm equation (or a generalized Degasperis-Procesi equation [7]) because it has similar structure with them. In [7], Li and Yin established the local existence and uniqueness of strong solutions for the problem (2) in nonhomogeneous Besov spaces by using the Littlewood-Paley theory. The well-posedness of (2) was studied in [8] for the periodic and the nonperiodic cases in the sense of Hadamard. In addition, nonuniform dependence was proved by using the method of approximate solutions and well-posedness estimates. However, to our best knowledge, the blow-up mechanisms and travelling waves have not been investigated yet.

Inspired by the works [7,8], our aim in this paper is to investigate whether or not equations (2) with nonlocal nonlinearities has similar remarkable properties as Eq. (1). More precisely, we firstly establish a blow-up criterion, then a new blow-up phenomenon for the problem (2) is derived. One of difficult issues in our blow-up phenomenon analysis is that there is not the estimate of the norm \( \| u \|_{H^1} \) for the problem (2). To overcome the difficult, we subtly select to track the blow-up quantities

\[
P(t) = \left( \sqrt{2}u + \left( \frac{\sqrt{2}}{2} + \frac{\sqrt{10}}{2} \right)u_x \right)(t, q(t, x_1))
\]

and

\[
Q(t) = \left( \sqrt{2}u + \left( \frac{\sqrt{2}}{2} - \frac{\sqrt{10}}{2} \right)u_x \right)(t, q(t, x_1))
\]

along the characteristics. In fact, in the blow-up analysis, the interaction between \( u \) and \( u_x \) plays a key role, which motivates us to carry out a refined analysis of the characteristic dynamics of \( P \) and \( Q \). For the problem (2), the estimates of \( P \) and \( Q \) can be closed in the form of

\[
P'(t) \leq \alpha PQ, \quad Q'(t) \geq -\beta PQ,
\]

where \( \alpha, \beta \geq 0 \) can be constants. From (3) the monotonicity of \( P \) and \( Q \) can be established, and hence the finite-time blow-up follows.

\section{Preliminary}

We write the equivalent form of the problem (2) as follows

\[
\begin{aligned}
&\begin{cases}
  u_t - 2uu_x = \partial_x(1 - \partial_x^2)^{-1}(u^2 + (u^2)_x), \\
  u(0, x) = u_0(x)
\end{cases}
\end{aligned}
\]

The characteristics \( q(t, x) \) relating to (4) is governed by

\[
\begin{aligned}
&\begin{cases}
  q_t(t, x) = -2u(t, q(t, x)), \quad t \in [0, T), \\
  q(0, x) = x, \quad x \in \mathbb{R}.
\end{cases}
\end{aligned}
\]
Applying the classical results in the theory of ordinary differential equations, one can obtain that the characteristics \( q(t, x) \in C^1([0, T) \times \mathbb{R}) \) with \( q_u(t, x) > 0 \) for all \((t, x) \in [0, T) \times \mathbb{R}\). Furthermore, it is shown from [7] that the potential \( m = u - u_{xx} \) satisfies
\[
m(t, q(t, x))q_x^2(t, x) \geq m_0(x)e^{-\int_0^t (2u_x - 2u)(\tau, q(\tau, x))d\tau}.
\]

### 2.1 Notation

We firstly give some notations.

Let \( \mathbb{R} \) denote real number set. The space of all infinitely differentiable functions \( \phi(t, x) \) with compact support in \([0, +\infty) \times \mathbb{R}\) is denoted by \( C_0^\infty \). Let \( L^p = L^p(\mathbb{R})(1 \leq p < +\infty) \) be the space of all measurable functions \( h \) such that
\[
\| h \|_{L^p} = \int_\mathbb{R} |h(t, x)|^pdx < \infty.
\]
We define \( L^\infty = L^\infty(\mathbb{R}) \) with the standard norm \( \| h \|_{L^\infty} = \sup_{x \in \mathbb{R}}|h(t, x)| \). For any real number \( s \), \( H^s = H^s(\mathbb{R}) \) denotes the Sobolev space with the norm defined by
\[
\| h \|_{H^s} = \left( \int_\mathbb{R} (1 + |\xi|^2)^s |\hat{h}(t, \xi)|^2d\xi \right)^{\frac{1}{2}} < \infty,
\]
where \( \hat{h}(t, \xi) = \int_\mathbb{R} e^{i\xi \tau} h(t, \tau)d\tau \).

We denote by \( * \) the convolution, and the convolution product on \( \mathbb{R} \) is defined by
\[
(f * g)(x) = \int_\mathbb{R} f(y)g(x - y)dy.
\]

Using the Green function \( g(x) = \frac{1}{2}e^{-|x|} \), we have \((1 - \partial_x^2)^{-1} f = g(x) * f \) for all \( f \in L^2 \), and \( g(u - u_{xx}) = u \). For \( T > 0 \) and nonnegative number \( s \), \( C([0, T); H^s(\mathbb{R})] \) denotes the Frechet space of all continuous \( H^s \)-valued functions on \([0, T)\). For simplicity, throughout this article, we let \( C \) denote any positive constant.

### 2.2 Several Lemmas

In this section, we firstly give some Lemmas.

**Lemma 2.1.** (Kato and Pronce [9]) Let \( r > 0 \). If \( u \in H^r \cap W^{1,\infty} \) and \( v \in H^{r-1} \cap L^\infty \), then
\[
\| [A^r, u]v \|_{L^2} \leq c(\| \partial_x u \|_{L^\infty} \| A^{r-1} v \|_{L^2} + \| A^r u \|_{L^2} \| v \|_{L^\infty}),
\]
where \([A^r, u]v = A^r(uv) - uA^rv\).

**Lemma 2.2.** (Kato and Pronce [9]) If \( r > 0 \), then \( H^r \cap L^\infty \) is an algebra. Moreover,
\[
\| uv \|_{H^r} \leq c(\| u \|_{L^\infty} \| v \|_{H^r} + \| u \|_{H^r} \| v \|_{L^\infty}),
\]
where $c$ is a constant depending only on $r$.

**Lemma 2.3.** Let $u_0(x) \in H^1(\mathbb{R})$. Then the following inequality holds

$$\| u(t, x) \|_{H^1(\mathbb{R})} \leq \| u_0(x) \|_{H^1(\mathbb{R})} e^{2 \int_0^t \| u_x \|_{L^\infty} dt}.$$  \hspace{1cm} (9)

**Proof.**

Multiplying both sides of (2) by $u$ and integrating with respect to $x$ on $\mathbb{R}$, we get

$$\frac{1}{2} \frac{d}{dt} \int_\mathbb{R} (u^2 + u_x^2) dx = \int_\mathbb{R} u_x^2 dx - \int_\mathbb{R} 2uu_x^2 dx \leq 2 \| u_x \|_{L^\infty} \int_\mathbb{R} (u^2 + u_x^2) dx.$$ \hspace{1cm} (10)

Using Gronwall’s inequality, we obtain (9).

**Definition 2.1.** Given initial data $u_0 \in H^s$, $s > \frac{3}{2}$, the function $u$ is said to be a weak solution to the initial-value problem (21) if it satisfies the following identity

$$\int_0^T \int_\mathbb{R} u \varphi_t - u^2 \varphi_x - p^* (u^2 + 2uu_x) \varphi_x dx dt + \int_\mathbb{R} u_0(x) \varphi(0, x) dx = 0$$ \hspace{1cm} (11)

for any smooth test function $\varphi(t, x) \in C^\infty_c([0, T] \times \mathbb{R})$. If $u$ is a weak solution on $[0, T)$ for every $T > 0$, then it is called a global weak solution.

## 3 Blow-up

### 3.1 Blow-up criterion

The blow-up criterion was listed as follows

**Theorem 3.1.** Let $u_0 \in H^r(\mathbb{R})$ with $r > \frac{3}{2}$. Then the corresponding solution $u$ to problem (2) blows up in finite time if and only if

$$\lim_{{t \to T^-}} \inf_{x \in \mathbb{R}} \{|u_x|\} = +\infty.$$ \hspace{1cm} (12)

**Proof.** Applying $\Lambda^r$ to two sides of Eq. (21) and multiplying by $\Lambda^r u$ and integrating on $\mathbb{R}$

$$\frac{1}{2} \frac{d}{dt} \int_\mathbb{R} (\Lambda^r u)^2 = 2 \int_\mathbb{R} \Lambda^r (uu_x) \Lambda^r u dx + \int_\mathbb{R} \Lambda^r f(u) \Lambda^r u dx,$$ \hspace{1cm} (13)

where $f(u) = \partial_x (1 - \partial_x^2)^{-1} \left[ u^2 + (u^2)_x \right]$. 

Thanks to Lemma 2.1 and 2.2, we get

\[
\int_{\mathbb{R}} \Lambda^r(uu_x)\Lambda^r udx = \int_{\mathbb{R}} [\Lambda^r, u]u_x \Lambda^r udx + \int_{\mathbb{R}} u\Lambda^r u_x\Lambda^r u
\]

\[
\leq \| [\Lambda^r, u]u_x \|_{L^2} \| \Lambda^r u \|_{L^2} + c \| u_x \|_{L^\infty} \| u \|_{H^r}^2
\]

\[
\leq c \| u \|_{H^r} (\| u_x \|_{L^\infty} \| u_x \|_{H^{r-1}} + \| u \|_{H^r} \| u_x \|_{L^\infty})
\]

\[
+ c \| u_x \|_{L^\infty} \| u \|_{H^r}^2
\]

\[
\leq c \| u_x \|_{L^\infty} \| u \|_{H^r}^2. \tag{14}
\]

Similarly, we have

\[
| \int_{\mathbb{R}} \Lambda^r f(u)\Lambda^r udx | \leq \| u \|_{H^r} \| \Lambda^r f(u) \|_{L^2} \tag{15}
\]

and

\[
\| \Lambda^r f(u) \|_{L^2} \leq \| \partial_x(1 - \partial^2_x)^{-1} \left[ u^2 + 2uu_x \right] \|_{H^r}
\]

\[
\leq \| u^2 \|_{H^{r-1}} + \| u^2 \|_{H^r}
\]

\[
\leq c \| u^2 \|_{H^r} \leq c \| u \|_{L^\infty} \| u \|_{H^r}, \tag{16}
\]

where we have used Lemma 2.1.

It follows from (13), (14), (16) and Lemma 2.3 that

\[
\frac{d}{dt} \| u \|_{H^r}^2 \leq c \| u \|_{H^r}^2 (1 + \| u_x \|_{L^\infty} + \| u_0(x) \|_{H^1(\mathbb{R})} e^{2\int_{0}^{t}\|u_x\|_{L^\infty} ds}). \tag{17}
\]

Therefore, if there exists a positive number \( M \) such that \( \| u_x \|_{L^\infty} \leq M \), then Gronwall’s inequality gives rise to

\[
\| u \|_{H^r}^2 \leq c \| u_0 \|_{H^r}^2 e^{\int_{0}^{t}(1+M+\|u_0(x)\|_{H^1(\mathbb{R})})e^{2Ms}ds}, \tag{18}
\]

which implies that \( u \) does not blow up. This completes the proof of Theorem 3.1.

### 3.2 Blow-up phenomenon

In this section, we give a new blow-up phenomenon.

**Theorem 3.2.** Let \( u_0 \in H^s(\mathbb{R}), s > \frac{3}{2} \). There is a point \( x_1 \in \mathbb{R} \) such that

\[
|\sqrt{2}u_0(x_1) + \frac{\sqrt{2}}{2}u_{0,x}(x_1)| < \frac{\sqrt{10}}{2}u_{0,x}(x_1).
\]

Then the blow-up occurs in finite time \( T_0 \) with

\[
T_0 \leq \frac{\sqrt{5} + 1}{4\sqrt{-u_0^2(x_1) - u_0(x_1)u_{0,x}(x_1) + u_{0,x}^2(x_1)}} < \infty. \tag{19}
\]
Proof. Along with the trajectory of \( q(t, x) \) defined in (5), we have

\[
\frac{\partial u(t, q)}{\partial t} = \partial_x (1 - \partial_x^2)^{-1}(u^2 + (u_x)^2) \tag{20}
\]

and

\[
\frac{\partial u_x(t, q)}{\partial t} = 2u_x^2 - u^2 - 2uu_x + (1 - \partial_x^2)^{-1}(u^2 + (u_x)^2). \tag{21}
\]

At the point \((t, q(t, x_1))\), we select to track the dynamics of \( P(t) = (\sqrt{2}u + (\sqrt{2} + \sqrt{10})u_x)(t, q(t, x_1)) \) and \( Q(t) = (\sqrt{2}u + (\sqrt{2} - \sqrt{10})u_x)(t, q(t, x_1)) \) along the characteristics, we obtain

\[
P'(t) = \sqrt{2}\frac{\partial u(t, q(t, x_1))}{\partial t} + (\sqrt{2} + \sqrt{10})\frac{\partial u_x(t, q(t, x_1))}{\partial t}
\]

\[
= \sqrt{2}\partial_x (1 - \partial_x^2)^{-1}(u^2 + (u_x)^2) + (\sqrt{2} + \sqrt{10})(2u_x^2 - u^2 - 2uu_x + (1 - \partial_x^2)^{-1}(u^2 + (u_x)^2))
\]

\[
= -\left(\frac{\sqrt{10}}{2} + \frac{3\sqrt{2}}{2}\right)u^2 + \left(\frac{\sqrt{10}}{2} + \frac{\sqrt{2}}{2}\right)(-2uu_x + 2u_x^2) + \left(\frac{\sqrt{10}}{2} + \frac{3\sqrt{2}}{2}\right)\int_0^\infty e^{-|x-y|}u^2dy
\]

\[
\geq -\left(\frac{\sqrt{10}}{2} + \frac{\sqrt{2}}{2}\right)(-2uu_x + 2u_x^2)
\]

\[
\geq \left(\frac{\sqrt{10}}{2} + \frac{\sqrt{2}}{2}\right)(-2uu_x + 2u_x^2)
\]

\[
\geq -\left(\frac{\sqrt{10}}{2} + \frac{\sqrt{2}}{2}\right)PQ \tag{22}
\]

and

\[
Q'(t) = \sqrt{2}\frac{\partial u(t, q(t, x_1))}{\partial t} - (\sqrt{2} - \sqrt{10})\frac{\partial u_x(t, q(t, x_1))}{\partial t}
\]

\[
= \sqrt{2}\partial_x (1 - \partial_x^2)^{-1}(u^2 + (u_x)^2) - (\sqrt{2} - \sqrt{10})(2u_x^2 - u^2 - 2uu_x + (1 - \partial_x^2)^{-1}(u^2 + (u_x)^2))
\]

\[
= -\left(\frac{\sqrt{10}}{2} + \frac{3\sqrt{2}}{2}\right)u^2 - \left(\frac{\sqrt{10}}{2} - \frac{\sqrt{2}}{2}\right)(-2uu_x + 2u_x^2) + \left(\frac{\sqrt{10}}{2} + \frac{3\sqrt{2}}{2}\right)\int_0^\infty e^{-|x-y|}u^2dy
\]

\[
\leq -\left(\frac{\sqrt{10}}{2} + \frac{3\sqrt{2}}{2}\right)u^2 - \left(\frac{\sqrt{10}}{2} - \frac{\sqrt{2}}{2}\right)(-2uu_x + 2u_x^2) + (3\sqrt{2} - \sqrt{10})g * u^2
\]

\[
\leq 1/2(3\sqrt{2} - \sqrt{10})u_x^2 - \left(\frac{\sqrt{10}}{2} - \frac{\sqrt{2}}{2}\right)(-2uu_x + 2u_x^2)
\]

\[
\leq -\left(\frac{\sqrt{10}}{2} - \frac{\sqrt{2}}{2}\right)(-2uu_x + 2u_x^2)
\]

\[
\leq \left(\frac{\sqrt{10}}{2} - \frac{\sqrt{2}}{2}\right)PQ, \tag{23}
\]

where \(\|g\|_{L^1} = 1\) is applied.

Then we obtain

\[
P'(t) \geq -\left(\frac{\sqrt{10}}{2} + \frac{\sqrt{2}}{2}\right)PQ, \tag{24}
\]

\[
Q'(t) \leq \left(\frac{\sqrt{10}}{2} - \frac{\sqrt{2}}{2}\right)PQ. \tag{25}
\]
From assumptions of Theorem 3.2, the initial data satisfies
\begin{align*}
P(0) &= \sqrt{2}u_0(x_1) + \frac{\sqrt{2}}{2}u_{0,x}(x_1) + \frac{\sqrt{10}}{2}u_{0,x}(x_1) > 0, \\
Q(0) &= \sqrt{2}u_0(x_1) + \frac{\sqrt{2}}{2}u_{0,x}(x_1) - \frac{\sqrt{10}}{2}u_{0,x}(x_1) < 0, \\
P(0)Q(0) &= < 0. \quad (26)
\end{align*}
Therefore, using the continuity of $P(t)$ and $Q(t)$ along the characteristics emanating from $x_1$, the following inequalities
\begin{align*}
P(t) > P(0) > 0, \quad Q(t) < Q(0) < 0 \quad (27)
\end{align*}
and
\begin{align*}
P'(t) > 0, \quad Q'(t) < 0 \quad (28)
\end{align*}
hold.

Letting $h(t) = \sqrt{-PQ(t)}$ and using the estimate \( \frac{P-Q}{2} \geq h(t) \), we have
\begin{align*}
h'(t) &= -\frac{P'Q + PQ'}{2\sqrt{-PQ}} \\
&\geq \frac{(\sqrt{10} + \sqrt{2})PQ^2 - (\sqrt{10} - \sqrt{2})P^2Q}{2\sqrt{-PQ}} \\
&\geq -\frac{(\sqrt{10} - \sqrt{2})PQ(P - Q)}{2\sqrt{-PQ}} \\
&\geq (\frac{\sqrt{10} - \sqrt{2}}{2})h^2. \quad (29)
\end{align*}

Solving (29) gives rise to
\begin{align*}
\frac{1}{h} \leq \frac{1}{h(0)} - (\frac{\sqrt{10} - \sqrt{2}}{2})t, \quad (30)
\end{align*}
which implies that $h \rightarrow +\infty$ as $t \rightarrow T_0$ with $T_0$ given by
\begin{align*}
T_0 &\leq \frac{\sqrt{10} + \sqrt{2}}{4} \frac{1}{h(0)} = \frac{\sqrt{5} + 1}{4\sqrt{-u_0^2(x_1) - u_0(x_1)u_{0,x}(x_1) + u_{0,x}^2(x_1)}} < \infty. \quad (31)
\end{align*}
Observe that $h(t) = \sqrt{\frac{5}{2}u_x^2 - 2(u + \frac{1}{2}u_{x})} < |\frac{\sqrt{10}}{2} | u_x(t,q(t,x_1)) |$. Therefore, $h \rightarrow +\infty$ as $t \rightarrow T_0$ implies $| u_x(t,q(t,x_1)) | \rightarrow +\infty$ as $t \rightarrow T_0$.

The proof of Theorem 3.2 is completed.

**Acknowledgments.** This research was funded by the Guizhou Province Science and Technology Basic Project (Grant No. QianKeHe Basic [2020]1Y011).
References


Received: November 30, 2023; Published: December 19, 2023