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Blow-up Phenomenon to a Generalized Camassa-Holm Equation

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Abstract

Considered in this paper is a generalized Camassa-Holm equation proposed by Novikov. Firstly, a blow-up criterion is established. Then, a new blow-up phenomenon is derived for the equation.

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Keywords: Blow-up phenomenon; Blow-up criterion; A generalized Camassa-Holm equation

1 Introduction

One of the most celebrated models of shallow water wave is the Camassa-Holm (CH)equation

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{1}$$

which was derived first by Fokas and Fuchssteiner [1] as a bi-Hamiltonian generalization of the KdV equation. It describes a certain non-Newtonian fluids and models finite length, small amplitude radial deformation waves in cylindrical hyperelastic rods [2]. The physical derivation and the discovery of soliton for the equation were done by Camassa and Holm [3]. It is shown in [3] that equation (1) possesses a Lax pair and infinitely many conserved integrals. The remarkable features of the CH equation can be found in [4, 5] and the references therein.

In this paper, we consider the Cauchy problem of integrable dispersive wave equation

$$\begin{cases} u_t - u_{txx} - 4uu_x + 6u_x u_{xx} + 2uu_{xxx} - 2u_x^2 - 2uu_{xx} = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{2}$$

which is presented in Novikov [6]. It is shown in [6] that Eq.(2) admits a hierarchy of local higher symmetries. Eq.(2) is regarded as a generalized Camassa-Holm equation (or a genenralized Degasperis-Procesi equation [7]) because it has similar structure with them. In [7], Li and Yin established the local existence and uniqueness of strong solutions for the problem (2) in nonhomogeneous Besov spaces by using the Littlewood-Paley theory. The well-posedness of (2) was studied in [8] for the periodic and the nonperiodic cases in the sense of Hadamard. In addition, nonuniform dependence was proved by using the method of approximate solutions and well-posedness estimates. However, to our best knowledge, the blow-up mechanisms and travelling waves have not been investigated yet.

Inspired by the works [7,8], our aim in this paper is to investigate whether or not equations (2) with nonlocal nonlinearities has similar remarkable properties as Eq. (1). More precisely, we firstly establish a blow-up criterion, then a new blow-up phenomenon for the problem (2) is derived. One of difficult issues in our blow-up phenomenon analysis is that there is not the estimate of the norm $\| u \|_{H^1}$ for the problem (2). To overcome the difficult, we subtly select to track the blow-up quantities $P(t) = \big(\sqrt{2}u + (\frac{\sqrt{2}}{2} + \frac{\sqrt{10}}{2})u_x\big)(t, q(t, x_1))$ and $Q(t) = \big(\sqrt{2}u + (\frac{\sqrt{2}}{2} - \frac{\sqrt{10}}{2})u_x\big)(t, q(t, x_1))$ along the characteristics. In fact, in the blow-up analysis, the interaction between u and u_x plays a key role, which motivates us to carry out a refined analysis of the characteristic dynamics of P and Q. For the problem (2), the estimates of P and Q can be closed in the form of

$$P'(t) \leq \alpha PQ, \quad Q'(t) \geq -\beta PQ, \tag{3}$$

where $\alpha, \beta \geq 0$ can be constants. From (3) the monotonicity of P and Q can be established, and hence the finite-time blow-up follows.

2 Preliminary

We write the equivalent form of the problem (2) as follows

$$\begin{cases} u_t - 2uu_x = \partial_x(1-\partial_x^2)^{-1}(u^2 + (u^2)_x), \\ u(0,x) = u_0(x) \end{cases} \tag{4}$$

The characteristics $q(t, x)$ relating to (4) is governed by

$$\begin{cases} q_t(t,x) = -2u(t, q(t,x)), \quad t \in [0, T), \\ q(0,x) = x, \quad x \in \mathbb{R}. \end{cases}$$

Applying the classical results in the theory of ordinary differential equations, one can obtain that the characteristics $q(t,x) \in C^1([0,T)\times\mathbb{R})$ with $q_x(t,x) > 0$ for all $(t,x) \in [0,T) \times \mathbb{R}$. Furthermore, it is shown from [7] that the potential $m = u - u_{xx}$ satisfies

$$m(t,q(t,x))q_x^2(t,x) \geq m_0(x)e^{-\int_0^t (2u_x-2u)(\tau,q(\tau,x))d\tau}. \tag{5}$$

2.1 Notation

We firstly give some notations.

Let $\mathbb{R}$ denote real number set. The space of all infinitely differentiable functions $\phi(t,x)$ with compact support in $[0,+\infty) \times \mathbb{R}$ is denoted by C_0^∞. Let $L^p = L^p(\mathbb{R})(1 \leq p < +\infty)$ be the space of all measurable functions h such that $\| h \|_{L^P}^P = \int_\mathbb{R} |h(t,x)|^p dx < \infty$. We define $L^\infty = L^\infty(\mathbb{R})$ with the standard norm $\| h \|_{L^\infty} = \inf_{m(e)=0} \sup_{x\in R\backslash e} |h(t,x)|$. For any real number s, $H^s = H^s(\mathbb{R})$ denotes the Sobolev space with the norm defined by

$$\| h \|_{H^s} = \left(\int_\mathbb{R} (1+|\xi|^2)^s |\hat{h}(t,\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty,$$

where $\hat{h}(t,\xi) = \int_\mathbb{R} e^{-ix\xi} h(t,x)dx$.

We denote by $*$ the convolution, and the convolution product on $\mathbb{R}$ is defined by

$$(f * g)(x) = \int_\mathbb{R} f(y)g(x-y)dy. \tag{6}$$

Using the Green function $g(x) = \frac{1}{2}e^{-|x|}$, we have $(1-\partial_x^2)^{-1}f = g(x) * f$ for all $f \in L^2$, and $g * (u - u_{xx}) = u$. For $T > 0$ and nonnegative number s, $C([0,T);H^s(\mathbb{R}))$ denotes the Frechet space of all continuous H^s-valued functions on $[0,T)$. For simplicity, throughout this article, we let C denote any positive constant

2.2 Several Lemmas

In this section, we firstly give some Lemmas.

Lemma 2.1.(Kato and Pronce [9]) *Let $r > 0$.If $u \in H^r \cap W^{1,\infty}$ and $v \in H^{r-1} \cap L^\infty$, then*

$$\| [\Lambda^r, u]v \|_{L^2} \leq c(\| \partial_x u \|_{L^\infty} \| \Lambda^{r-1} v \|_{L^2} + \| \Lambda^r u \|_{L^2} \| v \|_{L^\infty}), \tag{7}$$

where $[\Lambda^r, u]v = \Lambda^r(uv) - u\Lambda^r v$.

Lemma 2.2. (Kato and Pronce [9])*If $r > 0$, then $H^r \cap L^\infty$ is an algebra. Moreover,*

$$\| uv \|_{H^r} \leq c(\| u \|_{L^\infty} \| v \|_{H^r} + \| u \|_{H^r} \| v \|_{L^\infty}), \tag{8}$$

where c is a constant depending only on r.

Lemma 2.3. *Let $u_0(x) \in H^1(\mathbb{R})$. Then the following inequality holds*

$$\| u(t,x) \|_{H^1(\mathbb{R})} \leq \| u_0(x) \|_{H^1(\mathbb{R})} e^{2\int_0^t \|u_x\|_{L^\infty} d\tau}. \quad (9)$$

Proof.

Multiplying both sides of (2) by u and integrating with respect to x on $\mathbb{R}$, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(u^2+u_x^2)dx &= \int_{\mathbb{R}} u_x^3 dx - \int_{\mathbb{R}} 2uu_x^2 dx \\ &\leq 2 \| u_x \|_{L^\infty} \int_{\mathbb{R}}(u^2+u_x^2)dx. \end{aligned} \quad (10)$$

Using Gronwall's inequality, we obtain (9).

Definition 2.1. *Given initial data $u_0 \in H^s$, $s > \frac{3}{2}$, the function u is said to be a weak solution to the initial-value problem (21) if it satisfies the following identity*

$$\int_0^T \int_{\mathbb{R}} u\varphi_t - u^2\varphi_x - p * (u^2 + 2uu_x)\varphi_x dxdt + \int_{\mathbb{R}} u_0(x)\varphi(0,x)dx = 0 \quad (11)$$

for any smooth test function $\varphi(t,x) \in C_c^\infty([0,T) \times \mathbb{R})$. If u is a weak solution on $[0,T)$ for every $T > 0$, then it is called a global weak solution.

3 Blow-up

3.1 Blow-up criterion

The blow-up criterion was listed as follows

Theorem 3.1.*Let $u_0 \in H^r(\mathbb{R})$ with $r > \frac{3}{2}$. Then the corresponding solution u to problem (2) blows up in finite time if and only if*

$$\lim_{t\to T^-} \inf_{x\in\mathbb{R}}\{|u_x|\} = +\infty. \quad (12)$$

Proof. Applying Λ^r to two sides of Eq. (21) and multiplying by $\Lambda^r u$ and integrating on $\mathbb{R}$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(\Lambda^r u)^2 = 2\int_{\mathbb{R}}\Lambda^r(uu_x)\Lambda^r u dx + \int_{\mathbb{R}}\Lambda^r f(u)\Lambda^r u dx, \quad (13)$$

where $f(u) = \partial_x(1-\partial_x^2)^{-1}\left[u^2 + (u^2)_x\right]$.

Thanks to Lemma 2.1 and 2.2, we get

$$\begin{aligned}\int_{\mathbb{R}} \Lambda^r(uu_x)\Lambda^r u dx &= \int_{\mathbb{R}} [\Lambda^r, u]u_x \Lambda^r u dx + \int_{\mathbb{R}} u\Lambda^r u_x \Lambda^r u \\ &\leq \| [\Lambda^r, u]u_x \|_{L^2} \| \Lambda^r u \|_{L^2} + c \| u_x \|_{L^\infty} \| u \|_{H^r}^2 \\ &\leq c \| u \|_{H^r} (\| u_x \|_{L^\infty} \| u_x \|_{H^{r-1}} + \| u \|_{H^r} \| u_x \|_{L^\infty}) \\ &+ c \| u_x \|_{L^\infty} \| u \|_{H^r}^2 \\ &\leq c \| u_x \|_{L^\infty} \| u \|_{H^r}^2 . \end{aligned} \quad (14)$$

Similarly, we have

$$| \int_R \Lambda^r f(u)\Lambda^r u dx | \leq \| u \|_{H^r} \| \Lambda^r f(u) \|_{L^2} \quad (15)$$

and

$$\begin{aligned}\| \Lambda^r f(u) \|_{L^2} &\leq \| \partial_x (1-\partial_x^2)^{-1} \left[u^2 + 2uu_x \right] \|_{H^r} \\ &\leq \| u^2 \|_{H^{r-1}} + \| u^2 \|_{H^r} \\ &\leq c \| u^2 \|_{H^r} \leq c \| u \|_{L^\infty} \| u \|_{H^r}, \end{aligned} \quad (16)$$

where we have used Lemma 2.1.

It follows from (13), (14), (16) and Lemma 2.3 that

$$\frac{d}{dt} \| u \|_{H^r}^2 \leq c \| u \|_{H^r}^2 (1 + \| u_x \|_{L^\infty} + \| u_0(x) \|_{H^1(\mathbb{R})} e^{2\int_0^t \|u_x\|_{L^\infty} d\tau}). \quad (17)$$

Therefore, if there exists a positive number M such that $\| u_x \|_{L^\infty} \leq M$, then Gronwall's inequality gives rise to

$$\| u \|_{H^r}^2 \leq c \| u_0 \|_{H^r}^2 e^{\int_0^t (1+M+\|u_0(x)\|_{H^1(\mathbb{R})} e^{2Ms}) ds}, \quad (18)$$

which implies that u does not blow up. This completes the proof of Theorem 3.1.

3.2 Blow-up phenomenon

In this section, we give a new blow-up phenomenon.

Theorem 3.2. *Let $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. There is a point $x_1 \in \mathbb{R}$ such that $|\sqrt{2}u_0(x_1) + \frac{\sqrt{2}}{2}u_{0,x}(x_1)| < \frac{\sqrt{10}}{2}u_{0,x}(x_1)$. Then the blow-up occurs in finite time T_0 with*

$$T_0 \leq \frac{\sqrt{5}+1}{4\sqrt{-u_0^2(x_1) - u_0(x_1)u_{0,x}(x_1) + u_{0,x}^2(x_1)}} < \infty. \quad (19)$$

Proof. Along with the trajectory of $q(t,x)$ defined in (5),we have

$$\frac{\partial u(t,q)}{\partial t} = \partial_x(1-\partial_x^2)^{-1}(u^2+(u^2)_x) \tag{20}$$

and

$$\frac{\partial u_x(t,q)}{\partial t} = 2u_x^2 - u^2 - 2uu_x + (1-\partial_x^2)^{-1}(u^2+(u^2)_x). \tag{21}$$

At the point $(t, q(t,x_1))$, we select to track the dynamics of $P(t) = \big(\sqrt{2}u + (\frac{\sqrt{2}}{2}+\frac{\sqrt{10}}{2})u_x\big)(t,q(t,x_1))$ and $Q(t) = \big(\sqrt{2}u + (\frac{\sqrt{2}}{2}-\frac{\sqrt{10}}{2})u_x\big)(t,q(t,x_1))$ along the characteristics, we obtain

$$\begin{aligned}
P'(t) &= \sqrt{2}\frac{\partial u(t,q(t,x_1))}{\partial t} + (\frac{\sqrt{2}}{2}+\frac{\sqrt{10}}{2})\frac{\partial u_x(t,q(t,x_1))}{\partial t}\\
&= \sqrt{2}\partial_x(1-\partial_x^2)^{-1}(u^2+(u^2)_x) + (\frac{\sqrt{2}}{2}+\frac{\sqrt{10}}{2})[2u_x^2-u^2-2uu_x+(1-\partial_x^2)^{-1}(u^2+(u^2)_x)]\\
&= -(\frac{\sqrt{10}}{2}+\frac{3\sqrt{2}}{2})u^2 + (\frac{\sqrt{10}}{2}+\frac{\sqrt{2}}{2})(-2uu_x+2u_x^2) + (\frac{\sqrt{10}}{2}+\frac{3\sqrt{2}}{2})\int_x^\infty e^{-|x-y|}u^2dy\\
&\geq -(\frac{\sqrt{10}}{2}+\frac{3\sqrt{2}}{2})u^2 + (\frac{\sqrt{10}}{2}+\frac{\sqrt{2}}{2})(-2uu_x+2u_x^2)\\
&\geq (\frac{\sqrt{10}}{2}+\frac{\sqrt{2}}{2})(-2u^2-2uu_x+2u_x^2)\\
&\geq -(\frac{\sqrt{10}}{2}+\frac{\sqrt{2}}{2})PQ
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
Q'(t) &= \sqrt{2}\frac{\partial u(t,q(t,x_1))}{\partial t} - (\frac{\sqrt{10}}{2}-\frac{\sqrt{2}}{2})\frac{\partial u_x(t,q(t,x_1))}{\partial t}\\
&= \sqrt{2}\partial_x(1-\partial_x^2)^{-1}(u^2+(u^2)_x) - (\frac{\sqrt{10}}{2}-\frac{\sqrt{2}}{2})[2u_x^2-u^2-2uu_x+(1-\partial_x^2)^{-1}(u^2+(u^2)_x)]\\
&= -(-\frac{\sqrt{10}}{2}+\frac{3\sqrt{2}}{2})u^2 - (\frac{\sqrt{10}}{2}-\frac{\sqrt{2}}{2})(-2uu_x+2u_x^2) + (\frac{-\sqrt{10}}{2}+\frac{3\sqrt{2}}{2})\int_x^\infty e^{-|x-y|}u^2dy\\
&\leq -(-\frac{\sqrt{10}}{2}+\frac{3\sqrt{2}}{2})u^2 - (\frac{\sqrt{10}}{2}-\frac{\sqrt{2}}{2})(-2uu_x+2u_x^2) + (3\sqrt{2}-\sqrt{10})g*u^2\\
&\leq \frac{1}{2}(3\sqrt{2}-\sqrt{10})u^2 - (\frac{\sqrt{10}}{2}-\frac{\sqrt{2}}{2})(-2uu_x+2u_x^2)\\
&\leq -(\frac{\sqrt{10}}{2}-\frac{\sqrt{2}}{2})(-2u^2-2uu_x+2u_x^2)\\
&\leq (\frac{\sqrt{10}}{2}-\frac{\sqrt{2}}{2})PQ,
\end{aligned} \tag{23}$$

where $\| g \|_{L^1} = 1$ is applied.

Then we obtain

$$P'(t) \geq -(\frac{\sqrt{10}}{2}+\frac{\sqrt{2}}{2})PQ, \tag{24}$$

$$Q'(t) \leq (\frac{\sqrt{10}}{2}-\frac{\sqrt{2}}{2})PQ. \tag{25}$$

From assumptions of Theorem 3.2, the initial data satisfies

$$\begin{aligned}
&P(0) = \sqrt{2}u_0(x_1) + \frac{\sqrt{2}}{2}u_{0,x}(x_1) + \frac{\sqrt{10}}{2}u_{0,x}(x_1) > 0,\\
&Q(0) = \sqrt{2}u_0(x_1) + \frac{\sqrt{2}}{2}u_{0,x}(x_1) - \frac{\sqrt{10}}{2}u_{0,x}(x_1) < 0,\\
&P(0)Q(0) < 0. \qquad (26)
\end{aligned}$$

Therefore, using the continuity of $P(t)$ and $Q(t)$ along the characteristics emanating from x_1, the following inequalities

$$P(t) > P(0) > 0, \quad Q(t) < Q(0) < 0 \qquad (27)$$

and

$$P'(t) > 0, \quad Q'(t) < 0 \qquad (28)$$

hold.

Letting $h(t) = \sqrt{-PQ(t)}$ and using the estimate $\frac{P-Q}{2} \geq h(t)$, we have

$$\begin{aligned}
h'(t) &= -\frac{P'Q + PQ'}{2\sqrt{-PQ}}\\
&\geq \frac{(\frac{\sqrt{10}}{2} + \frac{\sqrt{2}}{2})PQ^2 - (\frac{\sqrt{10}}{2} - \frac{\sqrt{2}}{2})P^2Q}{2\sqrt{-PQ}}\\
&\geq \frac{-(\frac{\sqrt{10}}{2} - \frac{\sqrt{2}}{2})PQ(P-Q)}{2\sqrt{-PQ}}\\
&\geq (\frac{\sqrt{10}}{2} - \frac{\sqrt{2}}{2})h^2. \qquad (29)
\end{aligned}$$

Solving (29) gives rise to

$$\frac{1}{h} \leq \frac{1}{h(0)} - (\frac{\sqrt{10}}{2} - \frac{\sqrt{2}}{2})t, \qquad (30)$$

which implies that $h \to +\infty$ as $t \to T_0$ with T_0 given by

$$T_0 \leq \frac{\sqrt{10}+\sqrt{2}}{4}\frac{1}{h(0)} = \frac{\sqrt{5}+1}{4\sqrt{-u_0^2(x_1) - u_0(x_1)u_{0,x}(x_1) + u_{0,x}^2(x_1)}} < \infty. \qquad (31)$$

Observe that $h(t) = \sqrt{\frac{5}{2}u_x^2 - 2(u + \frac{1}{2}u_x)^2} <\mid \frac{\sqrt{10}}{2} \mid u_x(t, q(t, x_1)) \mid$. Therefore, $h \to +\infty$ as $t \to T_0$ implies $\mid u_x(t, q(t, x_1)) \mid \to +\infty$ as $t \to T_0$.

The proof of Theorem 3.2 is completed.

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