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On the Algebra of Diagonal Operators

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Abstract

Let $\mathcal{A}$ be the algebra of all bounded diagonal operators on an infinite dimensional separable complex Hilbert space $\mathcal{H}$. In this paper, we characterize the algebra $N(\mathcal{A})$ of unbounded operators affiliated with $\mathcal{A}$ and the unbounded Borel functions of these operators.

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1 The bounded case

Let $B(\mathcal{H})$ be the algebra of all bounded operators on an infinite dimensional separable complex Hilbert space $\mathcal{H}$ and $\{e_n : n \in \mathbb{N}\}$ an orthonormal basis for $\mathcal{H}$. Given a sequence $\{a_n\}$ in $\mathbb{C}$ and $x \in \mathcal{H}$, the operator $A \in B(\mathcal{H})$ defined by

$$Ax = \sum_{n=1}^{\infty} a_n(x, e_n)e_n, \quad \text{or equivalently,} \quad Ae_n = a_n e_n \ \ (n \in \mathbb{N})$$

is called a *diagonal operator* on $\mathcal{H}$.

Note that for each $j = 1, 2, ...$, we have

$$Ae_j = \sum_{i=1}^{\infty}(Ae_j, e_i)e_i = \sum_{i=1}^{\infty} a_j(e_j, e_i)e_i = \sum_{i=1}^{\infty} a_j\delta_{ij}e_i,$$

where δ_{ij} is the Kronecher delta: $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq i$. In this way, we associate to each diagonal operator an infinite diagonal matrix having

a_n as diagonal elements.

Let $l^\infty = l^\infty(\mathbb{N}, \mathbb{C})$ be the C^*-algebra of all bounded complex sequences $\{a_n\}$ with norm

$$||\{a_n\}|| = \sup_n |a_n|$$

We denote by

$$\mathcal{A} = \{A \in B(\mathcal{H}) : Ae_n = a_n e_n,\ where\ \{a_n\} \in l^\infty\} \tag{1}$$

the sub-algebra of $B(\mathcal{H})$ of all bounded diagonal operators on $\mathcal{H}$.

Proposition 1.1. *The mapping* $\Psi : \mathcal{A} \to l^\infty$ *defined by* $\Psi(A) = \{a_n\}$, *is an isometric* $*$*-isomorphism from* $\mathcal{A}$ *onto* l^∞.

Proof. For each $A \in \mathcal{A}$ and $n \in \mathbb{N}$, we have

$$|a_n| = ||a_n e_n|| = ||Ae_n|| \leq ||A||||e_n|| = ||A||.$$

Hence, $\sup_n |a_n| \leq ||A||$.

On the other hand, for every $x \in H$, we have

$$||Ax||^2 = ||\sum_{n=1}^{\infty}(x, e_n)Ae_n||^2 = ||\sum_{n=1}^{\infty}(x, e_n)a_n e_n||^2 = \sum_{n=1}^{\infty}|(x, e_n)|^2|a_n|^2$$

$$\leq \sup_n |a_n|^2 \sum_{n=1}^{\infty}|(x, e_n)|^2 = \sup_n |a_n|^2||x||^2,$$

(the last equality by Parseval's identity). Therefore, $||A|| \leq \sup_n |a_n|$.

Thus,

$$||A|| = \sup_n |a_n| = ||\{a_n\}||,$$

and Ψ is isometric. Finally, it is easy to see that Ψ is a $*$-isomorphism and makes $\mathcal{A}$ an abelian C^*-subalgebra of $B(\mathcal{H})$ with $A^* e_n = \overline{a_n} e_n$. □

Let X_{l^∞} be the Gelfand space of l^∞ (the space of all non-zero multipicative linear functional on l^∞). Now, if ρ is a non-zero multiplicative linear functional on l^∞, the composite mapping $\rho \circ \Psi$ is a non-zero multiplicative linear functional on $\mathcal{A}$. That is, $\rho \in \mathrm{X}_{\mathcal{A}}$ (the Gelfand space of $\mathcal{A}$). Accordingly, we can define a mapping $\Psi^\natural : X_{l^\infty} \to X_{\mathcal{A}}$ by

$$\Psi^\natural(\rho) = \rho \circ \Psi \qquad (\rho \in X_{l^\infty}).$$

Then the mapping $\Psi^\natural$ is a homeomorhism (both spaces X_{l^∞} and $X_{\mathcal{A}}$ equipped with the weak*-topology). Thus, we may identify $X_{\mathcal{A}}$ with X_{l^∞}.

Since $X_{l^\infty} = \beta(\mathbb{N})$ (*the β-compactification of* $\mathbb{N}$, see e.g. [1, Exercise 3.5.5]), we obtain

$$X_{\mathcal{A}} \approx \beta(\mathbb{N}).$$

Let $l^2 = l^2(\mathbb{N}, \mathbb{C})$ be the Hilbert space of square summable complex sequences $\{a_n\}$ with norm

$$||\{a_n\}|| = \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{\frac{1}{2}}$$

and $U : \mathcal{H} \mapsto l^2$ be the unitary isomorphism

$$Ux = \{(x, e_n)\}_{n=1}^{\infty}.$$

Then, as is easily seen,

$$U\mathcal{A}U^{-1} = \mathcal{M},$$

where

$$\mathcal{M} = \{M_{a_n} \in B(l^2) : \{a_n\} \in l^\infty\}$$

is the multiplication algebra acting on l^2. Moreover, $\mathcal{M}$ is maximal abelian i.e. $\mathcal{M}' = \mathcal{M}$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$. Therefore $\mathcal{A}$ is also maximal abelian ($\mathcal{A}' = \mathcal{A}$). Hence, $\mathcal{A}'' = \mathcal{A}$, and the double commutant theorem tells us that $\mathcal{A}$ is an abelian von Neumann algebra. Thus, in view of the Gelfand-Naimark theorem ([1, Theorem 4.4.3]), $\mathcal{A} \cong C(X)$, where $X = X_{\mathcal{A}}$. Moreover, by ([1, Theorem 5.2.1]), $\mathrm{X}_{\mathcal{A}} \approx \beta(\mathbb{N})$ is extremely disconnected compact Hausdorff space.

2 The unbounded case

A closed linear operator A defined on a dense linear subspace $\mathcal{D}(A)$ of $\mathcal{H}$ is said to *commute* with the bounded operator $T \in B(\mathcal{H})$, if $TA \subseteq AT$. This means that for each $x \in \mathcal{D}(A)$, we have $Tx \in \mathcal{D}(A)$ and $TAx = ATx$. A projection E on $\mathcal{H}$ such that $EA \subseteq AE$ and $AE \in B(\mathcal{H})$ is called a *bounding projection* for A. A *bounding sequence* for A is a non-decreasing sequence $\{E_n\}_{n\in\mathbb{N}}$ of projections on $\mathcal{H}$ such that $\bigvee_{n=1}^{\infty} E_n = I$, $E_nA \subseteq AE_n$ and $AE_n \in B(\mathcal{H})$ for all $n \in \mathbb{N}$.

Let $\{A\}' = \{T \in B(\mathcal{H}) : TA \subseteq AT\}$. It is easy to see that $\{A\}'$ is a strongly closed sub-algebra of $B(\mathcal{H})$, and $T \in \{A\}'$ if and only if $T^* \in \{A^*\}'$. Hence, $\{A\}' \cap \{A^*\}'$ is a von Neumann algebra. A closed densely defined operator A is *affiliated* with a von Neumann algebra $\mathcal{U}$, denoted by $A\eta\mathcal{U}$ if $\mathcal{U}' \subseteq \{A\}'$. The algebra $W^*(A) = \left\{\{A\}' \cap \{A^*\}'\right\}'$ is the smallest von Neumann algebra with which A is affiliated, and is referred to it as the *von Neumann algebra generated* by A. In fact, an operator A is normal ($A^*A = AA^*$) if and only if it is affiliated with an abelian von Neumann algebra ([1, Theorem 5.6.18]).

Let $\mathcal{U}$ be an abelian von Neumann algebra. We denote by $\mathcal{N}(\mathcal{U})$ the abelian $*$-algebra (with unit I) of the closed densely defined operators affiliated with $\mathcal{U}$ ([1, Theorem 5.6.15]). The Gelfand space $X = X_{\mathcal{U}}$ is an extremely disconnected compact Hausdorff space. Let $\dot{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the one-point compactification of the complex plane $\mathbb{C}$. A function $f : X \to \dot{\mathbb{C}}$ is called *normal* if f is continuous and $\overline{U_f} = X$, where $U_f = \{x \in X : f(x) \neq \infty\}$. We denote by $N(X)$ the family of normal functions on X. If $f, g \in N(X)$, then the sum $f + g$ and product fg are both defined and continuous on $U_f \cap U_g$ and have unique continuous extensions on X denoted by $f \dot{+} g$ and $f \cdot g$ respectively ([2, Theorem 2.1]). Moreover, if $f \in N(X)$, then we define f^* to be the unique element of $N(X)$ that extends the function $\bar{f}$ defined on U_f. Now $(N(X), \dot{+}, \cdot)$ becomes a *-algebra containing $C(X)$ as subalgebra ([2, Proposition 2.2]), and the Gelfand *-isomorphism $\Gamma : \mathcal{U} \to C(X)$ extends to a $*$-isomorphism $\dot{\Gamma} : \mathcal{N}(\mathcal{U}) \to N(X)$ such that $\dot{\Gamma}(AE) = \dot{\Gamma}(A) \cdot \Gamma(E)$ for $A \in \mathcal{N}(\mathcal{U})$ and any bounding projection $E \in \mathcal{U}$ ([1, Theorem 5.6.19]).

The following theorem characterizes the unbounded operators affiliated with the algebra $\mathcal{A}$ of diagonal operators, viz, $N(\mathcal{A})$.

Theorem 2.1. *Let A be a closed densely defined operator on $\mathcal{H}$ and $\{e_n : n \in \mathbb{N}\}$ an orthonormal basis for $\mathcal{H}$. Then $A \in \mathcal{N}(\mathcal{A})$ if and only if there exists a sequence $\{a_n\}$ in $\mathbb{C}$ such that*

$$Ax = \sum_{n=1}^{\infty} a_n(x, e_n)e_n, \quad D(A) = \{x \in \mathcal{H} : \{a_n(x, e_n)\} \in l^2\} \tag{2}$$

Proof. Let A be the operator defined by (2). Clearly, if $\{a_n\} \in l^\infty$, then $A \in \mathcal{A}$. Note also that $e_m \in D(A)$ for each $m \in \mathbb{N}$ and A is a densely defined normal operator. Moreover, $D_0 = span\{e_n : n \in \mathbb{N}\}$ is a core for A, i.e., $A = \overline{A|_{D_0}}$, where the bar refers to the closure of the operator. To see this, let $x \in D(A)$ and take $x_n = \sum_{k=1}^{n}(x, e_k)e_k$. Then as $n \to \infty$, we have $x_n \to x$ and

$$Ax_n = \sum_{j=1}^{\infty} a_j(x_n, e_j)e_j = \sum_{j=1}^{\infty} a_j \left[(\sum_{k=1}^{n}(x, e_k)e_k, e_j)\right] e_j = \sum_{j=1}^{\infty} a_j \left[\sum_{k=1}^{n}(x, e_k)(e_k, e_j))\right] e_j$$

$$= \sum_{j=1}^{n} a_j(x, e_j)e_j \to \sum_{j=1}^{\infty} a_j(x, e_j)e_j = Ax.$$

We now show $\mathcal{A} \subseteq \{A\}'$. Let $T \in \mathcal{A}$. Since D_0 is a core for A, it is enough to show that $e_n \in D_0$ implies $Te_n \in D_0$ and $TAe_n = ATe_n$ for all $n \in \mathbb{N}$.

If $Te_n = t_n e_n$, then clearly $Te_n \in D_0$ and $TAe_n = ATe_n = a_n t_n e_n$. Thus, $A \in N(\mathcal{A})$.

Next, we find the normal function $\varphi = \dot{\Gamma}(A)$. First note that, for each $n \in \mathbb{N}$, the functional $\rho_n : \mathcal{A} \mapsto \mathbb{C}$ defined by $\rho_n(T) = (Te_n, e_n)$ is a multiplicative linear functional on $\mathcal{A}$. Moreover, the set $\{\rho_n : n \in \mathbb{N}\}$ is a dense open subset of X. For this first note that if $(Te_n, e_n) = 0$ with $T \in \mathcal{A}$, then $T = 0$. Hence, if $f = \Gamma(T) \in C(X)$, then $f \equiv 0$ on X. Next suppose that $\overline{\{\rho_n : n \in \mathbb{N}\}} \neq X$ (the closure in X), and let $q \in X \setminus \overline{\{\rho_n : n \in \mathbb{N}\}}$. Since X is a compact Hausdorff space it is completely regular. Therefore, there exists $f \in C(X)$, $0 \leq f \leq 1$ such that $f(q) = 1$ and $f \equiv 0$ on $\overline{\{\rho_n : n \in \mathbb{N}\}}$. In particular, $f(\rho_n) = 0$, and so $f \equiv 0$ on X, a contradiction. Now, let $f \in C(X)$ be such that $f(\rho_m) = 1$ and $f(\rho_n) = 0$ for all $n \neq m$. Then $X = \{\rho_m\} \cup \overline{\{\rho_n : n \neq m\}}$. Since f is continuous, it follows that $f \equiv 0$ on $\overline{\{\rho_n : n \neq m\}}$. Hence, $f = \chi_{\{\rho_m\}}$ (the characteristic function of $\{\rho_m\}$), and so $\{\rho_m\}$ is an open (clopen) subset of X. Thus, $\{\rho_n : n \in \mathbb{N}\}$ is an open dense subset of X. If P_m is the projection onto *span*$\{e_m\}$, then $P_m \in \mathcal{A}$ and $\Gamma(P_m) = \chi_{\{\rho_m\}}$. Moreover, each P_m is a bounding projection for A and $\{E_n : n \in \mathbb{N}\}$, where $E_n = \sum_{m=1}^{n} P_m$, is a bounding sequence for A.

Finally,

$$\dot{\Gamma}(AP_m) = \Gamma(AP_m) = \dot{\Gamma}(A) \cdot \Gamma(P_m) = \varphi \cdot \chi_{\{\rho_m\}}.$$

At the same time, $\Gamma(AP_m) = a_m \Gamma(P_m) = a_m \chi_{\{\rho_m\}}$. Thus, $\varphi(\rho_n) = a_n$ for each $n \in \mathbb{N}$.

To prove the converse, suppose $A \in N(\mathcal{A})$ and let $\dot{\Gamma}(A) = \varphi$. Take $\{a_n\} = \{\varphi(\rho_n)\}$, and consider the operator $A_0 e_n = a_n e_n$ for all $n \in \mathbb{N}$. Then, arguing as above, we get $A_0 \in N(\mathcal{A})$ and $\dot{\Gamma}(A_0) = \varphi_0$ where $\varphi_0(\rho_n) = a_n$. Thus, $\varphi = \varphi_0$, and so $A = A_0$.

□

Next we characterize the Borel functional calculus for diagonal operators.

Theorem 2.2. *Let $A \in N(\mathcal{A})$ be $Ae_n = a_n e_n$ and $B_u(\sigma(A))$ the algebra of unbounded Borel functions on the spectrum $\sigma(A)$. If $f \in B_u(\sigma(A))$, then $f(A)e_n = f(a_n)e_n$ with $D(f(A)) = \{x \in \mathcal{H} : \{f(a_n)(x, e_n)\} \in l^2\}$ for all $n \in \mathbb{N}$.*

Proof. Let $\varphi = \dot{\Gamma}(A)$. Then $\sigma(A) = Range(\varphi) \cup \{\infty\} = \{a_n : n \in \mathbb{N}\} \cup \{\infty\} = \overline{\{a_n : n \in \mathbb{N}\}}$ (see [1, Proposition 5.6.20]). Moreover, the function $f \circ \varphi$ lies in $B_u(X)$ (the algebra of Borel functions on X). Since the complement of $\{\rho_n : n \in \mathbb{N}\}$ is a meager (nowhere dense) set in X, $f \circ \varphi$ agrees with a unique normal function g on $\{\rho_n : n \in \mathbb{N}\}$ ([1, Lemma 5.6.22]). Hence, $g(\rho_n) = f(\varphi(\rho_n)) = f(a_n)$.

By definition of the unbounded Borel functional calculus, $f(A) = \dot{\Gamma}^{-1}(g) \in N(\mathcal{A})$ ([1, Remark 5.6.25]), and so $\dot{\Gamma}(f(A)) = g$. Thus, in view of Theorem 2.1, we get $f(A)e_n = g(\rho_n)e_n = f(a_n)e_n$ and $D(f(A)) = \{x \in \mathcal{H} : \{f(a_n)(x, e_n)\} \in l^2\}$ for all $n \in \mathbb{N}$. $\square$

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