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On Stević-Sharma Operators from the Analytic Besov Space into Bloch-Type Spaces

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Abstract

The boundedness, essential norm and compactness of Stević-Sharma operators from the the analytic Besov space B_1 into Bloch-type spaces are investigated in this paper.

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Keywords: Stević-Sharma operator; Bloch-type space; boundedness; compactness; essential norm

1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$ and $S(\mathbb{D})$ the family of all analytic self-maps of $\mathbb{D}$. Denote by $\mathbb{N}$ the set of positive integers.

The Bloch-type space, which is denoted by $\mathcal{B}_\mu$, consists of all $f \in H(\mathbb{D})$ such that $\|f\| = \sup_{z\in\mathbb{D}} \mu(z)|f'(z)| < \infty$, where μ is a weight, namely a strictly positive continuous function on $\mathbb{D}$. We also assume that μ is radial: $\mu(z) = \mu(|z|)$ for each $z \in \mathbb{D}$. Under the norm $\|f\|_{\mathcal{B}_\mu} = |f(0)| + \|f\|$, $\mathcal{B}_\mu$ becomes a Banach space.

The analytic Besov space B_1 consists of all $f \in H(\mathbb{D})$ which can be written as $f(z) = \sum_{n=1}^{\infty} a_n \sigma_{\lambda_n}(z)$ for some sequences $\{a_n\}_{n\in\mathbb{N}} \subset l^1$ and $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{D}$, where $\sigma_w(z) = \frac{w-z}{1-\overline{w}z}$ for $z, w \in \mathbb{D}$.

For $f \in B_1$, the norm is defined by

$$\|f\|_{\mathcal{B}_1} = \inf \left\{ \sum_{n=1}^{\infty} |a_n| : f(z) = \sum_{n=1}^{\infty} a_n \sigma_{\lambda_n}(z) \right\}.$$

The space B_1 was extensively studied in [1], where it was shown that if one defines appropriately the notion of a "Möbius invariant space", then B_1 is the smallest one. Therefore, B_1 is also called the minimal Möbius invariant space.

In [4, 5], Stević et al. introduced the following Stević-Sharma operator:

$$T_{\psi_1,\psi_2,\varphi} f(z) = \psi_1(z) f(\varphi(z)) + \psi_2(z) f'(\varphi(z)), \quad f \in H(\mathbb{D}),$$

where $\psi_1, \psi_2 \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Recently, the research of Stević-Sharma operator between analytic function spaces has aroused the interest of experts (see, for instance, [2, 3, 8] and also related references therein).

In this paper, we investigate the boundedness, compactness and essential norm of Stević-Sharma operator from B_1 space into Bloch-type spaces. Recall that the essential norm of a bounded linear operator $T : X \to Y$ is the distance from T to the compact operators $K : X \to Y$, namely

$$\|T\|_{e,X\to Y} = \inf \big\{ \|T - K\|_{X\to Y} : K \text{ is compact} \big\}.$$

Here X and Y are Banach spaces. Notice that $\|T\|_{e,X\to Y} = 0$ if and only if $T : X \to Y$ is compact.

Throughout this paper, for nonnegative quantities X and Y, we use the abbreviation $X \lesssim Y$ or $Y \gtrsim X$ if there exists a positive constant C independent of X and Y such that $X \le CY$. Moreover, we write $X \approx Y$ if $X \lesssim Y \lesssim X$.

2 Main Results

To prove the main results, we state several lemmas firstly.

Lemma 2.1. [6] *Let $k \in \mathbb{N}$, then for any $f \in B_1$, we have*

$$\|f\|_\infty \lesssim \|f\|_{B_1} \quad \textit{and} \quad (1 - |z|^2)^k |f^{(k)}(z)| \lesssim \|f\|_{B_1}.$$

For any $w \in \mathbb{D}$ and $j \in \mathbb{N}$, set

$$f_{j,w}(z) = \frac{(1 - |w|^2)^j}{(1 - \overline{w}z)^j}, \quad z \in \mathbb{D}. \tag{1}$$

It is known that $f_{j,w} \in B_1$, and for each $j \in \mathbb{N}$, $\|f_{j,w}\|_{B_1} \lesssim 1$. Moreover, $f_{j,w}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|w| \to 1$.

Similar to the proof of [2, Lemma 2], we have the following lemma.

Lemma 2.2. *For any $0 \neq w \in \mathbb{D}$ and $i, k \in \{0,1,2\}$, there exist constants $c_{i,j}, j \in \{1,2,3\}$ such that the function*

$$g_{i,w}(z) := \sum_{j=1}^{3} c_{i,j} f_{j,w}(z) \in B_1 \quad \text{and} \quad g_{i,w}^{(k)}(w) = \frac{\overline{w}^k \delta_{ik}}{(1-|w|^2)^k},$$

where δ_{ik} is Kronecker delta.

Now we characterize the boundedness of Stević-Sharma operator $T_{\psi_1,\psi_2,\varphi} : B_1 \to \mathcal{B}_\mu$. To simplify notation of this paper, we set

$$A_0(z) = \psi_1'(z), \quad A_1(z) = \psi_1(z)\varphi'(z) + \psi_2(z), \quad A_2(z) = \psi_2(z)\varphi'(z).$$

Theorem 2.3. *Let $\psi_1, \psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$ and μ be a radial weight. Then the following statements are equivalent.*

(i) *The operator $T_{\psi_1,\psi_2,\varphi} : B_1 \to \mathcal{B}_\mu$ is bounded.*

(ii) *For each $i \in \{0,1,2\}$,*

$$\sup_{w\in\mathbb{D}} \|T_{\psi_1,\psi_2,\varphi} f_{i+1,w}\|_{\mathcal{B}_\mu} < \infty \quad \text{and} \quad \sup_{z\in\mathbb{D}} \mu(z)|A_i(z)| < \infty,$$

where $f_{i+1,w}$ is defined in (1).

(iii) *For each $i \in \{0,1,2\}$,*

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^i} < \infty. \tag{2}$$

Proof. (i)⇒(ii). Assume that $T_{\psi_1,\psi_2,\varphi} : B_1 \to \mathcal{B}_\mu$ is bounded. For each $w \in \mathbb{D}$ and $i \in \{0,1,2\}$, we have $\sup_{w\in\mathbb{D}} \|f_{i+1,w}\|_{B_1} \lesssim 1$. Hence,

$$\sup_{w\in\mathbb{D}} \|T_{\psi_1,\psi_2,\varphi} f_{i+1,w}\|_{\mathcal{B}_\mu} \leq \|T_{\psi_1,\psi_2,\varphi}\|_{B_1\to\mathcal{B}_\mu} \sup_{w\in\mathbb{D}} \|f_{i+1,w}\|_{B_1} < \infty.$$

Taking $f_0(z) = 1 \in B_1$, by the boundedness of $T_{\psi_1,\psi_2,\varphi} : B_1 \to \mathcal{B}_\mu$ we get

$$\sup_{z\in\mathbb{D}} \mu(z)|A_0(z)| = \sup_{z\in\mathbb{D}} \mu(z)|(T_{\psi_1,\psi_2,\varphi} f_0)'(z)| \leq \|T_{\psi_1,\psi_2,\varphi} f_0\|_{\mathcal{B}_\mu} < \infty. \tag{3}$$

Applying the operator $T_{\psi_1,\psi_2,\varphi}$ to $f_1(z) = z \in B_1$, we obtain

$$\sup_{z\in\mathbb{D}} \mu(z)|A_0(z)\varphi(z) + A_1(z)| \leq \sup_{z\in\mathbb{D}} \mu(z)|(T_{\psi_1,\psi_2,\varphi} f_1)'(z)| \leq \|T_{\psi_1,\psi_2,\varphi} f_1\|_{\mathcal{B}_\mu} < \infty,$$

which along with (3), the fact that $|\varphi(z)| < 1$ and the triangle inequality yields

$$\sup_{z\in\mathbb{D}} \mu(z)|A_1(z)| < \infty.$$

By using the function $f_2(z) = z^2 \in B_1$, in the same manner we obtain

$$\sup_{z\in\mathbb{D}} \mu(z)|A_2(z)| < \infty.$$

(ii)⇒(iii). Note that we only need to show that for $i \in \{1,2\}$, (2) holds. By Lemma 2.2, for each $i \in \{1,2\}$ and $\varphi(w) \neq 0$, there exist constants $c_{i,1}, c_{i,2}, c_{i,3}$ such that

$$g_{i,\varphi(w)}(z) = \sum_{j=1}^{3} c_{i,j} f_{j,\varphi(w)}(z) \in B_1 \quad \text{and} \quad g^{(k)}_{i,\varphi(w)}(z) = \frac{\overline{\varphi(w)}^k \delta_{ik}}{(1-|\varphi(w)|^2)^k},$$

where $k \in \{0,1,2\}$. Then we have

$$\begin{aligned} \sum_{j=1}^{3} |c_{i,j}| \sup_{w\in\mathbb{D}} \|T_{\psi_1,\psi_2,\varphi} f_{j,\varphi(w)}\|_{\mathcal{B}_\mu} &\geq \sup_{w\in\mathbb{D}} \|T_{\psi_1,\psi_2,\varphi} g_{i,\varphi(w)}\|_{\mathcal{B}_\mu} \\ &\geq \frac{\mu(w)|A_i(w)||\varphi(w)|^i}{(1-|\varphi(w)|^2)^i}. \end{aligned} \tag{4}$$

From (4) and (ii), for each $i \in \{1,2\}$, we have

$$\begin{aligned} \sup_{z\in\mathbb{D}} \frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^i} &\leq \sup_{|\varphi(w)|>\frac{1}{2}} \frac{\mu(w)|A_i(w)|}{(1-|\varphi(w)|^2)^i} + \sup_{|\varphi(w)|\leq\frac{1}{2}} \frac{\mu(w)|A_i(w)|}{(1-|\varphi(w)|^2)^i} \\ &\lesssim \sum_{j=1}^{3} \sup_{w\in\mathbb{D}} \|T_{\psi_1,\psi_2,\varphi} f_{j,\varphi(w)}\|_{\mathcal{B}_\mu} + \sup_{w\in\mathbb{D}} \mu(w)|A_i(w)| < \infty. \end{aligned}$$

(iii)⇒(i). Suppose that (iii) holds. For any $f \in B_1$, by Lemma 2.1 we have

$$\begin{aligned} \mu(z)|(T_{\psi_1,\psi_2,\varphi} f)'(z)| &\leq \sum_{i=0}^{2} \mu(z)|A_i(z)||f^{(i)}(\varphi(z))| \\ &\lesssim \|f\|_{B_1} \sum_{i=0}^{2} \frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^i}. \end{aligned} \tag{5}$$

Moreover,

$$\begin{aligned} |(T_{\psi_1,\psi_2,\varphi} f)(0)| &\leq |\psi_1(0)||f(\varphi(0))| + |\psi_2(0)|f'(\varphi(0))| \\ &\lesssim \left(|\psi_1(0)| + \frac{|\psi_2(0)|}{1-|\varphi(0)|^2}\right)\|f\|_{\mathcal{B}_1}. \end{aligned}$$

Thus $T_{\psi_1,\psi_2,\varphi} : B_1 \to \mathcal{B}_\mu$ is bounded. The proof is completed. □

Next, we give some estimations of the essential norm of Stević-Sharma operator acting from B_1 space to Bloch-type spaces.

Theorem 2.4. *Let $\psi_1, \psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$ and μ be a radial weight such that $T_{\psi_1,\psi_2,\varphi} : B_1 \to \mathcal{B}_\mu$ is bounded. Then*

$$\|T_{\psi_1,\psi_2,\varphi}\|_{e,B_1\to\mathcal{B}_\mu} \approx \max\{\rho_i\}_{i=0}^2 \approx \max\{\tau_l\}_{l=1}^2,$$

where

$$\rho_i = \limsup_{|w|\to 1} \|T_{\psi_1,\psi_2,\varphi} f_{i+1,w}\|_{\mathcal{B}_\mu}, \quad \tau_l = \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|A_l(z)|}{(1-|\varphi(z)|^2)^l}.$$

Proof. It is evident that for each $i \in \{0,1,2\}$, $\sup_{w\in\mathbb{D}} \|f_{i+1,w}\|_{B_1} \lesssim 1$ and $f_{i+1,w}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|w| \to 1$. For any compact operator K from B_1 into $\mathcal{B}_\mu$, by using [7, Lemma 2] we have $\lim_{|w|\to 1} \|K f_{i+1,w}\|_{\mathcal{B}_\mu} = 0$. Therefore, for each $i \in \{0,1,2\}$,

$$\|T_{\psi_1,\psi_2,\varphi} - K\|_{B_1\to\mathcal{B}_\mu} \gtrsim \limsup_{|w|\to 1} \|T_{\psi_1,\psi_2,\varphi} f_{i+1,w}\|_{\mathcal{B}_\mu} - \limsup_{|w|\to 1} \|K f_{i+1,w}\|_{\mathcal{B}_\mu} = \rho_i.$$

Hence,

$$\|T_{\psi_1,\psi_2,\varphi}\|_{e,B_1\to\mathcal{B}_\mu} = \inf_K \|T_{\psi_1,\psi_2,\varphi} - K\|_{B_1\to\mathcal{B}_\mu} \gtrsim \max\{\rho_i\}_{i=0}^2. \tag{6}$$

Let $\{z_j\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. Since $T_{\psi_1,\psi_2,\varphi} : B_1 \to \mathcal{B}_\mu$ is bounded, using (4) for any compact operator $K : B_1 \to \mathcal{B}_\mu$ and $l \in \{1,2\}$, we obtain

$$\|T_{\psi_1,\psi_2,\varphi} - K\|_{B_1\to\mathcal{B}_\mu} \gtrsim \limsup_{j\to\infty} \frac{\mu(z_j)|A_l(z_j)||\varphi(z_j)|^l}{(1-|\varphi(z_j)|^2)^l},$$

Thus we have

$$\|T_{\psi_1,\psi_2,\varphi}\|_{e,B_1\to\mathcal{B}_\mu} \gtrsim \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|A_l(z)|}{(1-|\varphi(z)|^2)^l} = \tau_l,$$

and consequently

$$\|T_{\psi_1,\psi_2,\varphi}\|_{e,B_1\to\mathcal{B}_\mu} \gtrsim \max\{\tau_l\}_{l=1}^2. \tag{7}$$

Define $K_r f(z) = f_r(z) = f(rz)$, $0 \le r < 1$. Then $K_r : B_1 \to \mathcal{B}_1$ is a compact operator with $\|K_r\| \le 1$. Moreover, it is easily seen that $f_r \to f$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1$. Let $\{r_j\} \subset (0,1)$ be a sequence such that $r_j \to 1$ as $j \to \infty$. Hence, for any $j \in \mathbb{N}$, $T_{\psi_1,\psi_2,\varphi} K_{r_j} : B_1 \to \mathcal{B}_\mu$ is compact, and so

$$\|T_{\psi_1,\psi_2,\varphi}\|_{e,B_1\to\mathcal{B}_\mu} \le \limsup_{j\to\infty} \|T_{\psi_1,\psi_2,\varphi} - T_{\psi_1,\psi_2,\varphi} K_{r_j}\|_{B_1\to\mathcal{B}_\mu}.$$

Therefore, we only need to show that

$$\limsup_{j\to\infty}\|T_{\psi_1,\psi_2,\varphi}-T_{\psi_1,\psi_2,\varphi}K_{r_j}\|_{B_1\to\mathcal{B}_\mu}\lesssim\min\big\{\max\{\rho_i\}_{i=0}^2,\max\{\tau_l\}_{l=1}^2\big\}. \tag{8}$$

For any $f\in B_1$ such that $\|f\|_{B_1}\le 1$, we have

$$\begin{aligned}
&\|(T_{\psi_1,\psi_2,\varphi}-T_{\psi_1,\psi_2,\varphi}K_{r_j})f\|_{\mathcal{B}_\mu}\\
=&|(T_{\psi_1,\psi_2,\varphi}f-T_{\psi_1,\psi_2,\varphi}f_{r_j})(0)|+\sup_{z\in\mathbb{D}}\mu(z)|(T_{\psi_1,\psi_2,\varphi}f-T_{\psi_1,\psi_2,\varphi}f_{r_j})'(z)|\\
\lesssim&\underbrace{|(f-f_{r_j})(\varphi(0))|+|(f-f_{r_j})'(\varphi(0))|}_{E_0}+\underbrace{\sup_{z\in\mathbb{D}}\mu(z)|(f-f_{r_j})(\varphi(z))A_0(z)|}_{E_1}\\
&+\underbrace{\sup_{|\varphi(z)|\le r_N}\mu(z)\sum_{l=1}^{2}|(f-f_{r_j})^{(l)}(\varphi(z))A_l(z)|}_{E_2}\\
&+\underbrace{\sup_{|\varphi(z)|>r_N}\mu(z)\sum_{l=1}^{2}|(f-f_{r_j})^{(l)}(\varphi(z))A_l(z)|}_{E_3},
\end{aligned}\tag{9}$$

where $N\in\mathbb{N}$ such that $r_j\ge\frac{1}{2}$ for all $j\ge N$. Moreover, for any nonnegative integer s, $(f-f_{r_j})^{(s)}\to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j\to\infty$. Theorem 2.3 now implies

$$\limsup_{j\to\infty}E_0=\limsup_{j\to\infty}E_2=0. \tag{10}$$

From [7, Lemma 3] it follows that

$$\lim_{j\to\infty}E_1\lesssim\lim_{j\to\infty}\sup_{z\in\mathbb{D}}|(f-f_{r_j})(z)|=0, \tag{11}$$

where we used the condition (2). Finally, we estimate E_3.

$$E_3\le\sum_{l=1}^{2}\underbrace{\sup_{|\varphi(z)|>r_N}\mu(z)|f^{(l)}(\varphi(z))A_l(z)|}_{F_l}+\sum_{l=1}^{2}\underbrace{\sup_{|\varphi(z)|>r_N}\mu(z)|r_j^l f^{(l)}(r_j\varphi(z))A_l(z)|}_{G_l}. \tag{12}$$

For each $l\in\{1,2\}$, from Lemma 2.1, (4) and (5) it follows that

$$\begin{aligned}
F_l=&\sup_{|\varphi(z)|>r_N}\frac{(1-|\varphi(z)|^2)^l|f^{(l)}(\varphi(z))|}{|\varphi(z)|^l}\frac{\mu(z)|A_l(z)||\varphi(z)|^l}{(1-|\varphi(z)|^2)^l}\\
\lesssim&\|f\|_{B_1}\sup_{|\varphi(z)|>r_N}\|T_{\psi_1,\psi_2,\varphi}g_{l,\varphi(z)}\|_{\mathcal{B}_\mu}\\
\lesssim&\sum_{j=0}^{2}\sup_{|w|>r_N}\|T_{\psi_1,\psi_2,\varphi}f_{j+1,w}\|_{\mathcal{B}_\mu}.
\end{aligned}\tag{13}$$

On the other hand,

$$F_l = \sup_{|\varphi(z)|>r_N} (1-|\varphi(z)|^2)^l |f^{(l)}(\varphi(z))| \frac{\mu(z)|A_l(z)|}{(1-|\varphi(z)|^2)^l}$$
$$\lesssim \|f\|_{B_1} \sup_{|\varphi(z)|>r_N} \frac{\mu(z)|A_l(z)|}{(1-|\varphi(z)|^2)^l}. \tag{14}$$

Taking the limits as $N \to \infty$ in (13) and (14), we obtain

$$\limsup_{j\to\infty} F_l \lesssim \sum_{j=0}^{2} \limsup_{|w|\to 1} \|T_{\psi_1,\psi_2,\varphi} f_{j+1,w}\|_{\mathcal{B}_\mu} \lesssim \max\{\rho_i\}_{i=0}^2, \tag{15}$$

and

$$\limsup_{j\to\infty} F_l \lesssim \max\{\tau_l\}_{l=1}^2. \tag{16}$$

Similarly, we have

$$\limsup_{j\to\infty} G_l \lesssim \max\{\rho_i\}_{i=0}^2 \quad \text{and} \quad \limsup_{j\to\infty} G_l \lesssim \max\{\tau_l\}_{l=1}^2. \tag{17}$$

Therefore, by (9)–(12) and (15)–(17), we get

$$\limsup_{j\to\infty} \|T_{\psi_1,\psi_2,\varphi} - T_{\psi_1,\psi_2,\varphi} K_{r_j}\|_{B_1\to\mathcal{B}_\mu}$$
$$= \limsup_{j\to\infty} \sup_{\|f\|_{\mathcal{B}_1}\le 1} \|(T_{\psi_1,\psi_2,\varphi} - T_{\psi_1,\psi_2,\varphi} K_{r_j}) f\|_{\mathcal{B}_\mu}$$
$$\lesssim \min\big\{ \max\{\rho_i\}_{i=0}^3, \max\{\tau_l\}_{l=1}^3 \big\}.$$

That is, (8) holds. The proof is completed. □

From Theorem 2.4, we immediately obtain the following corollary, which characterizes the compactness of $T_{\psi_1,\psi_2,\varphi} : B_1 \to \mathcal{B}_\mu$.

Corollary 2.5. *Let $\psi_1, \psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$ and μ be a radial weight such that $T_{\psi_1,\psi_2,\varphi} : B_1 \to \mathcal{B}_\mu$ is bounded. Then the following statements are equivalent.*

(i) *The operator $T_{\psi_1,\psi_2,\varphi} : B_1 \to \mathcal{B}_\mu$ is compact.*

(ii) *For each $i \in \{0,1,2\}$,*

$$\limsup_{|w|\to 1} \|T_{\psi_1,\psi_2,\varphi} f_{i+1,w}\|_{\mathcal{B}_\mu} = 0.$$

(iii) *For each $l \in \{1,2\}$,*

$$\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|A_l(z)|}{(1-|\varphi(z)|^2)^l} = 0.$$

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