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On the Local Everywhere Hölder Continuity for Weak Solutions of a Class of Not Convex Vectorial Problems of the Calculus of Variations

Tiziano Granucci [1]

MIUR, Istituto Salvemini Duca D'Aosta
via Giusti, 50100, Firenze, Italy

Dedicated to my family, Elisa Cirri, Caterina Granucci and Delia Granucci

Abstract

In this paper we study the regularity of the local minima of the following integral functional

$$J(u,\Omega)=\int_{\Omega}\sum_{\alpha=1}^{n}|\nabla u^{\alpha}(x)|^{p}+G(x,u(x),\nabla u(x))\,dx \tag{0.1}$$

where Ω is a open subset of $\mathbb{R}^n$ and $u\in W^{1,p}(\Omega,\mathbb{R}^m)$ with $n\geq 2$, $m\geq 1$ and $1<p<n$. In particular, not convexity (quasi-convexity, policonvexity or rank one convexity) hypothesis will be made on the density G, neither structure hypothesis nor radial nor diagonal.

Mathematics Subject Classifications: 49N60, 35J50

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1. Introduction

In this paper we study the regularity of the local minima of the following integral functional

$$J(u,\Omega)=\int_{\Omega}\sum_{\alpha=1}^{n}|\nabla u^{\alpha}(x)|^{p}+G(x,u(x),\nabla u(x))\,dx \tag{1.1}$$

where Ω is a open subset of $\mathbb{R}^n$ and $u\in W^{1,p}(\Omega,\mathbb{R}^m)$ with $n\geq 2$, $m\geq 1$ and $1<p<n$.

Moreover the following hypotheses hold

H.1.1: $G:\Omega\times\mathbb{R}^m\times\mathbb{R}^{n\times m}\to\mathbb{R}$ is a Caratheodory function such that

$$G(x,0,0)\in L^1_{loc}(\Omega)$$

and

$$|G(x,s_1,\xi_1)-G(x,s_2,\xi_2)|\leq a(x)(|\xi_1|+|\xi_2|+1)^{q_1}+b(x)(|s_1|+|s_2|+1)^{q_2}$$

for $\mathcal{L}^n$ almost every $x\in\Omega$ and for every $s_1,s_2\in\mathbb{R}^m$ and $\xi_1,\xi_2\in\mathbb{R}^{n\times m}$, where $0<q_1<p$, $0<q_2<p^*$, $a\in L^{\sigma_1}_{loc}(\Omega)$ is a not negative function, $b\in L^{\sigma_2}_{loc}(\Omega)$ is a not negative function, $\sigma_1>\frac{p}{p-q_1}$, $\sigma_2>\frac{p^*}{p^*-q_2}$,

$$\frac{q_1}{p}+\frac{1}{\sigma_1}<\frac{p}{n}$$

and

$$\frac{q_2}{p^*}+\frac{1}{\sigma_2}<\frac{p}{n}$$

The main result of this article is the following regularity theorem:

Theorem 1. *If $u\in W^{1,p}(\Omega,\mathbb{R}^m)$ is a minimizer of (1.1) and H.1.1 holds then $u\in C^{0,\delta}_{loc}(\Omega,\mathbb{R}^m)$.*

Theorem1 generalizes the author's results presented in [28, 32 and 33],these results arise from previous articles by Cupini, Focardi, Leonetti and Mascolo [8] and by the author and M. Randolfi [25]. Theorem 1 has no hypothesis either of structure or form, or of regularity or convexity on the density G. Finally, the proof of Theorem 1 is particularly simple, in fact the previous Theorem 1 derives from the following Cacciopoli inequalities using the techniques introduced by E. De Giorgi in [13].

Theorem 2. *If $u\in W^{1,p}(\Omega,\mathbb{R}^m)$ is a minimizer of (1.1) and H.1.1 holds then, for every $\Sigma\subset\Omega$ compact, two positive constants $C_{Cac,1}$, $C_{Cac,2}$ (depending only on Σ, p and n) and a radius $R_0>0$ exist such that for every $0<\varrho<R<R_0$ for every $x_0\in\Sigma$ and for every $k\in\mathbb{R}$ it follows*

$$\int_{A^{\alpha}_{k,\varrho}}|\nabla u^{\alpha}|^{p}\,dx\leq\frac{C_{Cac,1}}{(R-\varrho)^{p}}\int_{A^{\alpha}_{k,R}}(u^{\alpha}-k)^{p}\,dx+C_{Cac,2}\left|A^{\alpha}_{k,R}\right|^{1-\frac{p}{N}+\epsilon}$$

and

$$\int_{B_{k,\varrho}^{\alpha}} |\nabla u^{\alpha}|^{p}\ dx \leq \frac{C_{Cac,1}}{(R-\varrho)^{p}} \int_{B_{k,R}^{\alpha}} (k-u^{\alpha})^{p}\ dx + C_{Cac,2} \left|B_{k,R}^{\alpha}\right|^{1-\frac{p}{N}+\epsilon}$$

where $A_{k,s}^{\alpha} = \{u^{\alpha} > k\} \cap B_{s}(x_{0})$ *and* $B_{k,s}^{\alpha} = \{u^{\alpha} < k\} \cap B_{s}(x_{0})$ *with* $\alpha = 1, ..., m$.

Theorem 1 is interesting for a few reasons. We know that in the vector case there are many counter examples, refer to [14, 19, 21, 23], and in general the minima are not everywhere regular, refeer to [16, 37]. Moreover, starting from the end of the 1970s, using suitable hypotheses of convexity and regularity on the density Φ for the minima of integral functionals depending only on the modulus of the gradient, some regularity theorems have been proved, refer to [1, 2, 4-7, 15, 17, 18, 20, 21, 34, 42-44]. Our results can therefore be framed within a vast area of research called everywhere regulairy that was born with the fundamental works of Uhlenbeck [44], Tolksdorf [42, 43] and Acerbi - Fusco [1].

2. Preliminary results

Before giving the proofs of Theorem 1 and Theorem 2, for completeness we introduce a list of results that we will use during the proof.

2.1. Lemmata.

Lemma 1 (Young Inequality). *Let* $\varepsilon > 0$, $a, b > 0$ *and* $1 < p, q < +\infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *then it follows*

$$ab \leq \varepsilon \frac{a^{p}}{p} + \frac{b^{q}}{\varepsilon^{\frac{q}{p}} q} \tag{2.1}$$

Lemma 2 (Hölder Inequality). *Assume* $1 \leq p, q \leq +\infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *then if* $u \in L^{p}(\Omega)$ *and* $v \in L^{p}(\Omega)$ *it follows*

$$\int_{\Omega} |uv|\ dx \leq \left(\int_{\Omega} |u|^{p}\ dx\right)^{\frac{1}{p}} \left(\int_{\Omega} |v|^{q}\ dx\right)^{\frac{1}{q}} \tag{2.2}$$

Lemma 3. *Let* $Z(t)$ *be a nonnegative and bounded function on the set* $[\varrho, R]$*; if for every* $\varrho \leq t < s \leq R$ *we get*

$$Z(t) \leq \theta Z(s) + \frac{A}{(s-t)^{\lambda}} + \frac{B}{(s-t)^{\mu}} + C \tag{2.3}$$

where $A, B, C \geq 0$, $\lambda > \mu > 0$ *and* $0 \leq \theta < 1$ *then it follows*

$$Z(\varrho) \leq C(\theta, \lambda) \left(\frac{A}{(R-\varrho)^{\lambda}} + \frac{B}{(R-\varrho)^{\mu}} + C\right) \tag{2.4}$$

where $C(\theta,\lambda) > 0$ is a real constant depending only on θ and λ.

Refer to [12, 24].

2.2. **Polyconvex, Quasi-Convex and Rank-one Convex functions.**

Definition 1. *A function $f : \mathbb{R}^{nm} \to \mathbb{R} \cup \{+\infty\}$ is said to be rank one convex if*

$$f(\lambda A + (1-\lambda) B) \leq \lambda f(A) + (1-\lambda) f(B)$$

for every $\lambda \in [0,1]$, $A,B \in \mathbb{R}^{nm}$ with rank $\{A-B\} \leq 1$.

Definition 2. *A Borel measurable function and locally integrable function $f : \mathbb{R}^{nm} \to \mathbb{R}$ is said to be quasiconvex if*

$$f(A) \leq \frac{1}{|D|} \int_D f(A + \nabla\varphi)\, dx$$

for every bounded domain $D \subset \mathbb{R}^n$, for every $A \in \mathbb{R}^{nm}$ and for every $\varphi \in W_0^{1,\infty}(D;\mathbb{R}^{nm})$.

Definition 3. *A function $f : \mathbb{R}^{nm} \to \mathbb{R} \cup \{+\infty\}$ is said to be polyconvex if there exists a function $g : \mathbb{R}^{\tau(n,m)} \to \mathbb{R} \cup \{+\infty\}$ convex such that*

$$f(A) = g(T(A))$$

where $T : \mathbb{R}^{nm} \to \mathbb{R}^{\tau(n,m)}$ is such that

$$T(A) = (A, adj_2(A), ..., adj_{n\wedge m}(A))$$

where $adj_s(A)$ stands for the matrix of all $s \times s$ minors of the matrix $A \in \mathbb{R}^{nm}$, $2 \leq s \leq n \wedge m = \min\{n,m\}$ and

$$\tau(n,m) = \sum_{s=1}^{n\wedge m} \sigma(s)$$

where $\sigma(s) = \frac{n!m!}{(s!)^2 (m-s)!(n-s)!}$.

In particular we recall the following theorem.

Theorem 3. (1) *Let $f : \mathbb{R}^{nm} \to \mathbb{R}$ then*

f convex $\Longrightarrow$ f polyconvex $\Longrightarrow$ f quasiconvex $\Longrightarrow$ f rank one convex.

(2) *If $m = 1$ or $n = 1$ then all thess notions are equivalent.*

(3) *If $f \in C^2(\mathbb{R}^{nm})$ then rank one convexity is equivalent to Legendre-Hadamard condition*

$$\sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 f}{\partial A_\alpha^i \partial A_\beta^j}(A)\, \lambda^i \lambda^j \mu_\alpha \mu_\beta \geq 0$$

for every $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}^n$, $A = (A_\alpha^i)_{1\leq i\leq m, 1\leq\alpha\leq n} \in \mathbb{R}^{nm}$.

(4) *If $f : \mathbb{R}^{nm} \to \mathbb{R}$ is convex, polyconvex, quasiconvex or rank one convex then f is locally Lipschitz.*

Refer to [12, 24].

2.3. Sobolev Spaces.

Theorem 4 (Sobolev Inequality). *Let Ω be an open subset of $\mathbb{R}^N$ if $u \in W_0^{1,p}(\Omega)$ with $1 \le p < N$ there exists a real positive constant C_{SN}, depending only on p and N, such that*

$$\|u\|_{L^{p^*}(\Omega)} \le C_{SN} \|\nabla u\|_{L^p(\Omega)} \tag{2.5}$$

where $p^* = \frac{Np}{N-p}$.

Theorem 5. *(Rellich-Sobolev Immersion Theorem) Let Ω be an open bounded subset of $\mathbb{R}^N$ with lipschitz boundary then if $u \in W^{1,p}(\Omega)$ with $1 \le p < N$ there exists a real positive constant $C_{IS,\Omega}$, depending only on p, N and Ω, such that*

$$\|u\|_{L^{p^*}(\Omega)} \le C_{IS,\Omega} \|u\|_{W^{1,p}(\Omega)} \tag{2.6}$$

where $p^* = \frac{Np}{N-p}$.

Refer to [3, 12, 24, 40, 41].

For completeness we remember that if Ω is an open subset of $\mathbb{R}^N$ and u is a Lebesgue measurable function then $L^p(\Omega)$ is the set of the class of the Lebesgue measurable function such that $\int_\Omega |u|^p \, dx < +\infty$ and $W^{1,p}(\Omega)$ is the set of the function $u \in L^p(\Omega)$ such that its weak derivate $\partial_i u \in L^p(\Omega)$. The spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$ are Banach spaces with the respective norms

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{p}} \tag{2.7}$$

and

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^{N} \|\partial_i u\|_{L^p(\Omega)} \tag{2.8}$$

We say that the function $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}^n$ belong to $W^{1,p}(\Omega, \mathbb{R}^n)$ if $u^\alpha \in W^{1,p}(\Omega)$ for every $\alpha = 1, ..., n$, where u^α is the α component of the vector-valued function u; we end by remembering that $W^{1,p}(\Omega, \mathbb{R}^n)$ is a Banach space with the norm

$$\|u\|_{W^{1,p}(\Omega,\mathbb{R}^n)} = \sum_{\alpha=1}^{n} \|u^\alpha\|_{W^{1,p}(\Omega)} \tag{2.9}$$

Definition 4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $v : \Omega \to \mathbb{R}$, we say that $v \in W_{loc}^{1,p}(\Omega)$ belongs to the De Giorgi class $DG^+(\Omega, p, \lambda, \lambda_*, \chi, \varepsilon, R_0, k_0)$ with $p > 1$, $\lambda > 0$, $\lambda_* > 0$, $\chi > 0$, $\varepsilon > 0$, $R_0 > 0$ and $k_0 \ge 0$ if*

$$\int_{A_{k,\varrho}} |\nabla v|^p \, dx \le \frac{\lambda}{(R-\varrho)^p} \int_{A_{k,R}} (v-k)^p \, dx + \lambda_* \left(\chi^p + k^p R^{-N\varepsilon} \right) |A_{k,R}|^{1-\frac{p}{N}+\varepsilon} \tag{2.10}$$

for all $k \geq k_0 \geq 0$ *and for all pair of balls* $B_\varrho(x_0) \subset B_R(x_0) \subset\subset \Omega$ *with* $0 < \varrho < R < R_0$ *and* $A_{k,s} = B_s(x_0) \cap \{v > k\}$ *with* $s > 0$.

Definition 5. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded open set and* $v : \Omega \to \mathbb{R}$*, we say that* $v \in W^{1,p}_{loc}(\Omega)$ *belongs to the De Giorgi class* $DG^-(\Omega, p, \lambda, \lambda_*, \chi, \varepsilon, R_0, k_0)$ *with* $p > 1$, $\lambda > 0$, $\lambda_* > 0$, $\chi > 0$ *and* $k_0 \geq 0$ *if*

$$\int_{B_{k,\varrho}} |\nabla v|^p \, dx \leq \frac{\lambda}{(R-\varrho)^p} \int_{B_{k,R}} (k-v)^p \, dx + \lambda_* \left(\chi^p + |k|^p R^{-N\varepsilon}\right) |B_{k,R}|^{1-\frac{p}{N}+\varepsilon} \tag{2.11}$$

for all $k \leq -k_0 \leq 0$ *and for all pair of balls* $B_\varrho(x_0) \subset B_R(x_0) \subset\subset \Omega$ *with* $0 < \varrho < R < R_0$ *and* $B_{k,s} = B_s(x_0) \cap \{v < k\}$ *with* $s > 0$.

Definition 6. *We set* $DG(\Omega, p, \lambda, \lambda_*, \chi, \varepsilon, R_0, k_0) = DG^+(\Omega, p, \lambda, \lambda_*, \chi, \varepsilon, R_0, k_0) \cap DG^-(\Omega, p, \lambda, \lambda_*, \chi, \varepsilon, R_0, k_0)$.

Theorem 6. *Let* $v \in DG(\Omega, p, \lambda, \lambda_*, \chi, \varepsilon, R_0, k_0)$ *and* $\tau \in (0,1)$*, then there exists a constant* $C > 1$ *depending only upon the data and not-dependent on* v *and* $x_0 \in \Omega$ *such that for every pair of balls* $B_{\tau\varrho}(x_0) \subset B_\varrho(x_0) \subset\subset \Omega$ *with* $0 < \varrho < R_0$

$$\|v\|_{L^\infty(B_{\tau\varrho}(x_0))} \leq \max\left\{\lambda_* \varrho^{\frac{N\varepsilon}{p}}; \frac{C}{(1-\tau)^{\frac{N}{p}}} \left[\frac{1}{|B_\varrho(x_0)|} \int_{B_\varrho(x_0)} |v|^p \, dx\right]^{\frac{1}{p}}\right\} \tag{2.12}$$

moreover, there exists $\tilde{\alpha} \in (0,1)$ *depending only upon the data and not-dependent on* v *and* $x_0 \in \Omega$ *such that*

$$osc(v, B_\varrho(x_0)) \leq C \max\left\{\lambda_* \varrho^{\frac{N\varepsilon}{p}}; \left(\frac{\varrho}{R}\right)^{\tilde{\alpha}} osc(v, B_R(x_0))\right\} \tag{2.13}$$

where $osc(v, B_s(x_0)) = ess\ \sup_{B_s(x_0)}(v) - ess\ inf_{B_s(x_0)}(v)$*. Therefore* $v \in C^{0,\tilde{\alpha}_0}_{loc}(\Omega)$ *with* $\tilde{\alpha}_0 = \min\left\{\tilde{\alpha}; \frac{N\varepsilon}{p}\right\}$.

For more details on De Giorgi's classes and for the proof of the Theorem 8 refer to [22, 24] (see also [13, 38, 39] for the De Giorgi–Moser-Nash Theorem).

3. The proof of Theorem 2

Let us consider $y \in \Omega$ then we fix $R_0 = \frac{1}{4}\min\left\{\frac{1}{\sqrt[N]{\varpi_N}}, dist(\partial\Omega, y)\right\}$, where $\varpi_N = |B_1(0)|$, and we define $\Sigma = \{x \in \Omega : |x-y| \leq R_0\}$. We fix $x_0 \in \Sigma$, $R_1 = \frac{1}{4} dist(\partial\Sigma, x_0)$, $0 < \varrho \leq t < s \leq R < R_1$, $B_z(x_0) = \{x : |x - x_0| < z\}$ and we choose $\eta \in C^\infty_c(B_s(x_0))$ such that $\eta = 1$ on $B_t(x_0)$, $0 \leq \eta \leq 1$ on $B_s(x_0)$ and $|\nabla\eta| \leq \frac{2}{s-t}$ on $B_s(x_0)$. Let us define

$$\varphi = -\eta^p w$$

where $w \in W^{1,p}(\Sigma, \mathbb{R}^n)$ with

$$w^1 = \max\left(u^1 - k, 0\right), w^\alpha = 0, \alpha = 2, ..., n$$

Let us observe that $\varphi = 0$ $\mathcal{L}^N$-a.e. in $\Omega \backslash (\{\eta > 0\} \cap \{u^1 > k\})$ thus

$$\nabla u + \nabla \varphi = \nabla u \tag{3.1}$$

$\mathcal{L}^N$-a.e. in $\Omega \backslash (\{\eta > 0\} \cap \{u^1 > k\})$. Since u is a local minimizer of the functional (1.1) then we get

$$J(u, \Sigma) \leq J(u + \varphi, \Sigma) \tag{3.2}$$

it is

$$\begin{aligned} &\int_\Sigma \sum_{\alpha=1}^n |\nabla u^\alpha|^p + G(x, u, \nabla u)\ dx \\ &\leq \int_\Sigma \sum_{\alpha=1}^n |\nabla u^\alpha + \nabla \varphi^\alpha|^p + G(x, u + \varphi, \nabla u + \nabla \varphi)\ dx \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &\int_\Sigma \sum_{\alpha=2}^n |\nabla u^\alpha|^p\ dx + \int_\Sigma |\nabla u^1|^p + G(x, u, \nabla u)\ dx \\ &\leq \int_\Sigma \sum_{\alpha=2}^n |\nabla u^\alpha|^p\ dx + \int_\Sigma |\nabla u^1 + \nabla \varphi^1|^p + G(x, u + \varphi, \nabla u + \nabla \varphi)\ dx \end{aligned} \tag{3.4}$$

From (3.4) proceeding as in [28, 32 and 33], using H.1, we deduce

$$\begin{aligned} &\int_{E^1_{k,s}} |\nabla u^1|^p\ dx \\ &\leq 2^{p-1} \int_{E^1_{k,s}} (1 - \eta^p) |\nabla u^1|^p\ dx + +2^{2p-1} p^p \int_{E^1_{k,s} - E^1_{k,t}} \frac{\left(u^1 - k\right)^p}{(s-t)^p}\ dx \\ &+ \int_{E^1_{k,s}} a(x) (|\nabla u + \nabla \varphi| + |\nabla u| + 1)^{q_1} + b(x) (|u + \varphi| + |u| + 1)^{q_2}\ dx \end{aligned} \tag{3.5}$$

Now let's estimate the following term

$$\int_{E^1_{k,s}} a(x) (|\nabla u + \nabla \varphi| + |\nabla u| + 1)^{q_1} + b(x) (|u + \varphi| + |u| + 1)^{q_2}\ dx$$

since $q_1 < p$ and $q_2 < p^*$, using Hölder's inequality, we obtain

$$\begin{aligned}
&\int_{E^1_{k,s}} a(x)\left(|\nabla u + \nabla\varphi| + |\nabla u| + 1\right)^{q_1} + b(x)\left(|u+\varphi| + |u| + 1\right)^{q_2}\, dx \\
&\leq \int_{E^1_{k,s}} a(x)\left(|\nabla\varphi| + 2|\nabla u| + 1\right)^{q_1} + b(x)\left(|\varphi| + 2|u| + 1\right)^{q_2}\, dx \\
&\leq \left(\int_{E^1_{k,s}} (a(x))^{\frac{p}{p-q_1}}\, dx\right)^{\frac{p-q_1}{p}} \left(\int_{E^1_{k,s}} \left(|\nabla\varphi| + 2|\nabla u| + 1\right)^{p}\, dx\right)^{\frac{q_1}{p}} \\
&+ \left(\int_{E^1_{k,s}} (b(x))^{\frac{p^*}{p^*-q_2}}\, dx\right)^{\frac{p^*-q_2}{p^*}} \left(\int_{E^1_{k,s}} \left(|\varphi| + 2|u| + 1\right)^{p^*}\, dx\right)^{\frac{q_2}{p^*}}
\end{aligned} \tag{3.6}$$

moreover, since $\frac{p}{p-q_1} < \sigma_1$and $\frac{p^*}{p^*-q_2} < \sigma_2$, then using Hölder's inequality we get

$$\begin{aligned}
&\int_{E^1_{k,s}} a(x)\left(|\nabla u + \nabla\varphi| + |\nabla u| + 1\right)^{q_1} + b(x)\left(|u+\varphi| + |u| + 1\right)^{q_2}\, dx \\
&\leq \left(\left|E^1_{k,s}\right|^{\frac{\sigma_1(p-q_1)-p}{\sigma_1(p-q_1)}} \left(\int_{E^1_{k,s}} (a(x))^{\sigma_1}\, dx\right)^{\frac{p}{\sigma_1(p-q_1)}}\right)^{\frac{p-q_1}{p}} \\
&\cdot \left(\int_{E^1_{k,s}} \left(|\nabla\varphi| + 2|\nabla u| + 1\right)^{p}\, dx\right)^{\frac{q_1}{p}} \\
&+ \left(\left|E^1_{k,s}\right|^{\frac{\sigma_2(p^*-q_2)-p^*}{\sigma_2(p^*-q_2)}} \left(\int_{E^1_{k,s}} (b(x))^{\sigma_2}\, dx\right)^{\frac{p^*}{\sigma_2(p^*-q_2)}}\right)^{\frac{p^*-q_2}{p^*}} \\
&\cdot \left(\int_{E^1_{k,s}} \left(|\varphi| + 2|u| + 1\right)^{p^*}\, dx\right)^{\frac{q_2}{p^*}} \\
&\leq \left(\int_{E^1_{k,s}} \left(|\nabla\varphi| + 2|\nabla u| + 1\right)^{p}\, dx\right)^{\frac{q_1}{p}} \left(\left|E^1_{k,s}\right|^{\frac{\sigma_1(p-q_1)-p}{\sigma_1 p}} \|a\|_{L^{\sigma_1}\left(E^1_{k,s}\right)}\right) \\
&+ \left(\int_{E^1_{k,s}} \left(|\nabla\varphi| + 2|\nabla u| + 1\right)^{p}\, dx\right)^{\frac{q_1}{p}} \left(\left|E^1_{k,s}\right|^{\frac{\sigma_2(p^*-q_2)-p^*}{\sigma_2 p^*}} \|b\|_{L^{\sigma_2}\left(E^1_{k,s}\right)}\right)
\end{aligned} \tag{3.7}$$

and

$$\int_{E_{k,s}^1} a(x)(|\nabla u+\nabla\varphi|+|\nabla u|+1)^{q_1}+b(x)(|u+\varphi|+|u|+1)^{q_2}\,dx$$
$$\leq\left(\int_{E_{k,s}^1}(|\nabla\varphi|+2|\nabla u|+1)^p\,dx\right)^{\frac{q_1}{p}}\left(\left|E_{k,s}^1\right|^{\frac{\sigma_1(p-q_1)-p}{\sigma_1 p}}\|a\|_{L^{\sigma_1}\left(E_{k,s}^1\right)}\right)$$
$$+\left(\int_{E_{k,s}^1}(|\nabla\varphi|+2|\nabla u|+1)^p\,dx\right)^{\frac{q_1}{p}}\left(\left|E_{k,s}^1\right|^{\frac{\sigma_2(p^*-q_2)-p^*}{\sigma_2 p^*}}\|b\|_{L^{\sigma_2}\left(E_{k,s}^1\right)}\right)$$

where $\Theta_1=1-\frac{\sigma_1q_1+p}{\sigma_1p}$ and $\Theta_2=1-\frac{\sigma_2q_2+p^*}{\sigma_2p^*}$. Since $|\nabla\varphi|\leq p\eta^{p-1}|\nabla\eta|(u^1-k)+\eta^p|\nabla u^1|$ on $E_{k,s}^1$ we get

$$\begin{aligned}
&\left(\int_{E_{k,s}^1}(|\nabla\varphi|+2|\nabla u|+1)^p\,dx\right)^{\frac{q_1}{p}}\left(\left|E_{k,s}^1\right|^{\Theta_1}\|a\|_{L^{\sigma_1}\left(E_{k,s}^1\right)}\right)\\
&\leq\left(\left|E_{k,s}^1\right|^{\Theta_1}\|a\|_{L^{\sigma_1}\left(E_{k,s}^1\right)}\right)\\
&\cdot\left[2^{q_1-1}\left(\int_{E_{k,s}^1}p^p\eta^{p(p-1)}|\nabla\eta|^p\left(u^1-k\right)^p+\eta^{p^2}\left|\nabla u^1\right|^p\,dx\right)^{\frac{q_1}{p}}+2^{q_1-1}\left(\int_{E_{k,s}^1}|\nabla u|^p\,dx\right)^{\frac{q_1}{p}}+2^{q_1-1}\left|E_{k,s}^1\right|^{\frac{q_1}{p}}\right]\\
&\leq 2^{q_1-1}\left(\left|E_{k,s}^1\right|^{\Theta_1}\|a\|_{L^{\sigma_1}\left(E_{k,s}^1\right)}\right)\\
&\cdot\left[\frac{1}{\varepsilon^{\frac{p}{p-q_1}}}+\varepsilon^{\frac{p}{q_1}}\left(\int_{E_{k,s}^1}p^p\eta^{p(p-1)}|\nabla\eta|^p\left(u^1-k\right)^p+\eta^{p^2}\left|\nabla u^1\right|^p\,dx\right)+\left(\int_{E_{k,s}^1}|\nabla u|^p\,dx\right)^{\frac{q_1}{p}}+\left|E_{k,s}^1\right|^{\frac{q_1}{p}}\right]\\
&\leq\frac{2^{q_1-1}\left(\left|E_{k,s}^1\right|^{\Theta_1}\|a\|_{L^{\sigma_1}\left(E_{k,s}^1\right)}\right)}{\varepsilon^{\frac{p}{p-q_1}}}+p^p2^{q_1-1}\left(\left|E_{k,s}^1\right|^{\Theta_1}\|a\|_{L^{\sigma_1}\left(E_{k,s}^1\right)}\right)\varepsilon^{\frac{p}{q_1}}\int_{E_{k,s}^1}p^p\eta^{p(p-1)}|\nabla\eta|^p\left(u^1-k\right)^p\,dx\\
&+2^{q_1-1}\left(\left|E_{k,s}^1\right|^{\Theta_1}\|a\|_{L^{\sigma_1}\left(E_{k,s}^1\right)}\right)\left[\varepsilon^{\frac{p}{q_1}}\int_{E_{k,s}^1}\eta^{p^2}\left|\nabla u^1\right|^p\,dx+\left(\int_{E_{k,s}^1}|\nabla u|^p\,dx\right)^{\frac{q_1}{p}}+\left|E_{k,s}^1\right|^{\frac{q_1}{p}}\right]\\
&\leq\frac{2^{q_1-1}\left(\left|E_{k,s}^1\right|^{\Theta_1}\|a\|_{L^{\sigma_1}\left(E_{k,s}^1\right)}\right)}{\varepsilon^{\frac{p}{p-q_1}}}+2^{q_1-1}\left(|B_r|^{\Theta_1}\|a\|_{L^{\sigma_1}(B_r)}\right)\varepsilon^{\frac{p}{q_1}}\left(\int_{E_{k,s}^1}p^p2^p\frac{\left(u^1-k\right)^p}{(t-s)^p}+\eta^{p^2}\left|\nabla u^1\right|^p\,dx\right)\\
&+2^{q_1-1}\left(\left|E_{k,s}^1\right|^{\Theta_1}\|a\|_{L^{\sigma_1}\left(E_{k,s}^1\right)}\right)\left[\left(\int_{E_{k,s}^1}|\nabla u|^p\,dx\right)^{\frac{q_1}{p}}+\left|E_{k,s}^1\right|^{\frac{q_1}{p}}\right]
\end{aligned}\tag{3.8}$$

and, using the Embendding Sobolev Theorem, we get

$$\begin{aligned}
&\left(\int_{E^1_{k,s}} (|\varphi| + 2|u| + 1)^{p^*}\ dx\right)^{\frac{q_2}{p^*}} \left(\left|E^1_{k,s}\right|^{\Theta_2} \|b\|_{L^{\sigma_2}\left(E^1_{k,s}\right)}\right) \\
&\leq \left(\left|E^1_{k,s}\right|^{\Theta_2} \|b\|_{L^{\sigma_2}\left(E^1_{k,s}\right)}\right) \\
&\cdot \left(2^{p^*-1}\int_{E^1_{k,s}} |\varphi|^{p^*}\ dx + 2^{3p^*-2}\int_{E^1_{k,s}} |u|^{p^*}\ dx + 2^{3p^*-2}\left|E^1_{k,s}\right|\right)^{\frac{q_2}{p^*}} \\
&\leq \left(\left|E^1_{k,s}\right|^{\Theta_2} \|b\|_{L^{\sigma_2}\left(E^1_{k,s}\right)}\right) \\
&\cdot \left(2^{p^*-1}\int_{E^1_{k,s}} |u|^{p^*}\ dx + 2^{3p^*-2}\int_{E^1_{k,s}} |u|^{p^*}\ dx + 2^{3p^*-2}\left|E^1_{k,s}\right|\right)^{\frac{q_2}{p^*}} \\
&\leq \left(\left|E^1_{k,s}\right|^{\Theta_2} \|b\|_{L^{\sigma_2}(B_s)}\right)\left(2^{3p^*} C\|u\|_{W^{1,p}(B_s)} + 2^{3p^*-2}|B_s|\right)^{\frac{q_2}{p^*}}
\end{aligned}$$

then it follows

$$\begin{aligned}
&\int_{E^1_{k,s}} a(x)\left(|\nabla u + \nabla\varphi| + |\nabla u| + 1\right)^{q_1} + b(x)\left(|u+\varphi| + |u| + 1\right)^{q_2}\ dx \\
&\leq 2^{q_1-1}\left(|B_r|^{\Theta_1}\|a\|_{L^{\sigma_1}(B_r)}\right)\varepsilon^{\frac{p}{q_1}}\left(\int_{E^1_{k,s}} p^p 2^p \frac{\left(u^1-k\right)^p}{(t-s)^p} + \eta^{p^2}\left|\nabla u^1\right|^p\ dx\right) \\
&+2^{q_1-1}\left(\left|E^1_{k,s}\right|^{\Theta_1}\|a\|_{L^{\sigma_1}(B_r)}\right)\left[\frac{1}{\varepsilon^{\frac{p}{p-q_1}}} + \left(\int_{B_r}|\nabla u|^p\ dx\right)^{\frac{q_1}{p}} + |B_r|^{\frac{q_1}{p}}\right] \\
&+\left(\left|E^1_{k,s}\right|^{\Theta_2}\|b\|_{L^{\sigma_2}(B_s)}\right)\left(2^{3p^*}C_{SN}\|u\|_{W^{1,p}(B_s)} + 2^{3p^*-2}|B_s|\right)^{\frac{q_2}{p^*}} \\
&\leq D_{1,\Sigma}\varepsilon^{\frac{p}{q_1}}\left(\int_{E^1_{k,s}} \eta^p\left|\nabla u^1\right|^p\ dx\right) + D_{2,\varepsilon,\Sigma}\int_{E^1_{k,s}} \frac{\left(u^1-k\right)^p}{(t-s)^p}\,dx \\
&+D_{3,\varepsilon,\Sigma}\left|E^1_{k,s}\right|^{\Theta_1} + D_{4,\varepsilon,\Sigma}\left|E^1_{k,s}\right|^{\Theta_2}
\end{aligned}$$

where

$$D_{1,\Sigma} = 2^{q_1-1}\left(|\Sigma|^{\Theta_1}\|a\|_{L^{\sigma_1}(\Sigma)}\right)$$

$$D_{2,\varepsilon,\Sigma} = 2^{q_1-1}p^p2^p\left(|\Sigma|^{\Theta_1}\|a\|_{L^{\sigma_1}(\Sigma)}\right)\varepsilon^{\frac{p}{q_1}}$$

$$D_{3,\varepsilon,\Sigma} = 2^{q_1-1}\|a\|_{L^{\sigma_1}(\Sigma)}\left[\frac{1}{\varepsilon^{\frac{p}{p-q_1}}} + \left(\int_{\Sigma}|\nabla u|^p\ dx\right)^{\frac{q_1}{p}} + |\Sigma|^{\frac{q_1}{p}}\right]$$

and

$$D_{4,\varepsilon,\Sigma} = \|b\|_{L^{\sigma_2}(\Sigma)}\left(2^{3p^*}C_{SN}\|u\|_{W^{1,p}(\Sigma)} + 2^{3p^*-2}|\Sigma|\right)^{\frac{q_2}{p^*}}$$

Usinig (3.12) and (3.16) we get

$$\int_{E^1_{k,s}} |\nabla u^1|^p \, dx$$
$$\leq 2^{p-1} \int_{E^1_{k,s}} (1-\eta^p) |\nabla u^1|^p \, dx + 2^{2p-1} p^p \int_{E^1_{k,s}-E^1_{k,t}} \frac{(u^1-k)^p}{(s-t)^p} \, dx$$
$$+ D_{1,\Sigma} \varepsilon^{\frac{p}{q_1}} \left(\int_{E^1_{k,s}} \eta^p |\nabla u^1|^p \, dx \right) + D_{2,\varepsilon,\Sigma} \int_{E^1_{k,s}} \frac{(u^1-k)^p}{(t-s)^p} \, dx + D_{3,\varepsilon,\Sigma} \left| E^1_{k,s} \right|^{\Theta_1} + D_{4,\varepsilon,\Sigma} \left| E^1_{k,s} \right|^{\Theta_2}$$

Fix $\varepsilon = \left(\frac{1}{2D_{1,\Sigma}} \right)^{\frac{q_1}{p}}$ it follows

$$\frac{1}{2} \int_{E^1_{k,s}} |\nabla u^1|^p \, dx$$
$$\leq 2^{p-1} \int_{E^1_{k,s}} (1-\eta^p) |\nabla u^1|^p \, dx + (2^{2p} p^p + D_{2,\Sigma}) \int_{E^1_{k,s}} \frac{(u^1-k)^p}{(s-t)^p} \, dx \tag{3.9}$$
$$+ D_{3,\Sigma} \left| E^1_{k,s} \right|^{\Theta_1} + D_{4,\Sigma} \left| E^1_{k,s} \right|^{\Theta_2}$$

where

$$D_{2,\Sigma} = \frac{2^{q_1-1} p^p 2^p \left(|\Sigma|^{\Theta_1} \|a\|_{L^{\sigma_1}(\Sigma)} \right)}{2D_{1,\Sigma}}$$

$$D_{3,\Sigma} = 2^{q_1-1} \|a\|_{L^{\sigma_1}(\Sigma)} \left[(2D_{1,\Sigma})^{\frac{q_1}{p-q_1}} + \left(\int_{\Sigma} |\nabla u|^p \, dx \right)^{\frac{q_1}{p}} + |\Sigma|^{\frac{q_1}{p}} \right]$$

and

$$D_{4,\Sigma} = \|b\|_{L^{\sigma_2}(\Sigma)} \left(2^{3p^*} C_{SN} \|u\|_{W^{1,p}(\Sigma)} + 2^{3p^*-2} |\Sigma| \right)^{\frac{q_2}{p^*}}.$$

It is easy to observe that the constants $D_{2,\Sigma}$, $D_{3,\Sigma}$ and $D_{4,\Sigma}$ are independent of the point x_0 and that they depend only on the initial data.

Using (3.9), we get

$$\int_{E^1_{k,s}} |\nabla u^1|^p \, dx$$
$$\leq \frac{2^p}{1+2^p} \int_{E^1_{k,s}} |\nabla u^1|^p \, dx + \left(\frac{2^{2p} p^p + D_{2,\Sigma}}{1+2^p} \right) \int_{E^1_{k,s}} \frac{(u^1-k)^p}{(s-t)^p} \, dx \tag{3.10}$$
$$+ \frac{2D_{3,\Sigma}}{1+2^p} \left| E^1_{k,s} \right|^{\Theta_1} + \frac{2D_{4,\Sigma}}{1+2^p} \left| E^1_{k,s} \right|^{\Theta_2}$$

Now, using Lemma 3 we get

$$\int_{A^1_{k,\varrho}} |\nabla u^1|^p \, dx \leq \frac{C_{C,1}}{(R-\varrho)^p} \int_{A^1_{k,R}} (u^1-k)^p \, dx + C_{C,2} \left| A^1_{k,R} \right|^{\Theta_1} + C_{C,3} \left| A^1_{k,R} \right|^{\Theta_2} \tag{3.11}$$

Since $-u$ is a local minimizer of the following integral functional

$$\tilde{J}(v,\Omega)=\int_{\Omega}\sum_{\alpha=1}^{n}|\nabla v^{\alpha}(x)|^{p}+\tilde{G}(x,v(x),\nabla v(x))\,dx$$

where $\tilde{G}(x,v(x),\nabla v(x))=G(x,-v(x),-\nabla v(x))$ then we get

$$\int_{B^1_{k,\varrho}}|\nabla u^1|^p\,dx\le\frac{C_{C,1}}{(R-\varrho)^p}\int_{B^1_{k,R}}(k-u^1)^p\,dx+C_{C,2}\left|B^1_{k,R}\right|^{\Theta_1}+C_{C,3}\left|B^1_{k,R}\right|^{\Theta_2}\tag{3.12}$$

Similarly we can proceed for u^{α} with $\alpha=2,...,m$. Since $\Theta_1,\Theta_2>1-\frac{p}{n}+\epsilon$ then it follows

$$\int_{A^1_{k,\varrho}}|\nabla u^{\alpha}|^p\,dx\le\frac{C_{C,1}}{(R-\varrho)^p}\int_{A^1_{k,R}}(u^{\alpha}-k)^p\,dx+(C_{C,2}+C_{C,3})\left|A^1_{k,R}\right|^{1-\frac{p}{n}+\epsilon}$$

and

$$\int_{B^1_{k,\varrho}}|\nabla u^{\alpha}|^p\,dx\le\frac{C_{C,1}}{(R-\varrho)^p}\int_{B^1_{k,R}}(k-u^{\alpha})^p\,dx+(C_{C,2}+C_{C,3})\left|B^1_{k,R}\right|^{1-\frac{p}{n}+\epsilon}$$

for every $\alpha=1,...,m$.

4. Proof of Theorem 1

The proof follows by applying Theorem 2 and Theorem 8.

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