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Semi-Invariant Submanifold of an Indefinite Lorentzian Para-Sasakian Manifold

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Abstract

In the present paper we have obtained some properties of semi-invariant submanifold of an indefinite Lorentzian para-Sasakian manifold. The integrability condition of the distribution on semi-invariant submanifolds have also been discussed.

Keywords: Definite Lorentzian para-Sasakian manifold, semi-invariant submanifolds

Introduction

An n-dimensional differentiable manifold is called indefinite Lorentzian para Sasakian manifold if the following condition hold,

$$\phi^2 X = X + \eta(X)\xi \tag{1.1}$$

$$\eta \, o \, \phi = 0, \quad \phi\xi = 0 \tag{1.2}$$

$$\eta(\xi) = 1 \tag{1.3}$$

$\tilde{g}\ (\phi X, \phi Y) = \ \tilde{g}(X,Y) - \in \eta(X)\eta(Y)$ (1.4)

$\tilde{g}(X,\xi) = \in \eta(X)$ (1.5)

for all vector fields X, Y on $\widetilde{M}$ [3] and where $\in$ is 1 or -1 according to ξ is space like or like vector field.

An indefinite almost metric structure $(\phi, \xi, \eta, \tilde{g})$ is called an indefinite Lorentzian para Sasakian manifold if

$\left(\widetilde{\nabla_X}\phi\right)Y = g(X,Y)\xi + \in \eta(Y) + 2 \in \eta(X)\eta(Y)\xi$ (1.6)

where $\widetilde{\nabla}$ is the Lexi-Civita connection for a semi-Riemannian metric $\tilde{g}$. Also we have

$\widetilde{\nabla_X}\xi = \in\ \phi\, X$ (1.7)

where $X\ \in T\widetilde{M}$

Definition (1.1) The submanifold M of an indefinite Lorentzian para Sasakian manifold is said to be semi-invariant if it is endowed with the pair of orthogonal distribution $(D, D^{\perp})$ satisfying the conditions.

(a) $TM = D \oplus D^{\perp} \oplus \{\xi\}$

(b) the distribution D is invariant under ϕ, that is $\phi\ D_x = \ D_x\ ,\ for\ each\ x \in M$

(c) the distribution $D^{\perp}$ is anti-invariant under ϕ, that is

$\phi {D_x}^{\perp}\ \subset T_x M^{\perp}$, for each $x \in M$

The distribution D (respectively $D^{\perp}$) is the horizontal (resp. vertical) distribution.

A semi-invariant submanifold M is said to be invariant (respectively anti-invariant) submanifold if we have ${D_x}^{\perp} = \{0\}$ respectively $(D_x = 0)$ for each $x \in M$. We say that M is a proper semi-invariant submanifold if it is a semi-invariant submanifold, which is neither an invariant nor an anti-invariant submanifold. The projection morphism of TM to D and $D^{\perp}$ are denoted by P and Q respectively.

For any $X\ \in\ \lceil(TM)$ and $N \in\ \lceil(TM)^{\perp}$, we have

$X = PX + QX +\ \eta(X)\xi$ (1.8)

$\phi N = BN + CN$ (1.9)

where BN (respectively CN) denotes the tangential (respectively normal) component of ϕN.

The Gauss and Weingarten formulae are as follows.

$$\widetilde{\nabla_X}\, Y = \nabla_X Y + h(X, Y) \tag{1.10}$$

$$\widetilde{\nabla_X}\, N = -A_N X + {\nabla_X}^{\perp} N \tag{1.11}$$

for any $X, Y \in \lceil(TM)$ and $N \in \lceil(TM^{\perp})$, where ∇ is the Lexi-Civita connection on M, $\nabla^{\perp}$ is the connection on the normal bundle $TM^{\perp}$, h is the second fundamental form and A_N is the Weingarten map associated with N. Also we have

$$g(h(X, Y), N) = g(A_N\, X, Y) \tag{1.12}$$

for any $X, Y \in \lceil(TM), N \lceil(TM^{\perp})$.

2. Decomposition of basic equations along horizontal and vertical projection

For $X, Y \in \lceil(TM)$, we put

$$u(X, Y) = \nabla_X \phi PY - A_{\phi QY} X \tag{2.1}$$

Lemma (2.1) Let M be a semi-invariant submanifold of an indefinite Lorentzian para-Sasakian manifold $\widetilde{M}$. Then we have

$$\nabla_X \xi = \in \phi PX, h(X, \xi) = 0 \; for\; any\; X \in \lceil(D) \qquad (2.1)(a)$$

$$\nabla_Y \xi = 0, h(Y, \xi) = \in \phi QX \;\; for\; any\; Y \in \lceil(D^{\perp}) \tag{2.2}$$

$$\nabla_\xi \xi = o\,, h(\xi, \xi) = 0 \tag{2.3}$$

Proof. We have

$$\widetilde{\nabla_X} \xi = \nabla_X \xi + h(X, \xi)$$

which with the help of (1.7) and (1.8) gives

$$\nabla_X \xi + h(X, \xi) = \in \phi PX + \in \phi QX, for\; X \in TM \tag{2.4}$$

Now (2.1), (2.2) and (2.3) follow from (2.4).

Lemma 2.2 Let M be a semi-invariant submanifold of an indefinite Lorentzian para-sasakian manifold $\widetilde{M}$. Then we have

$$P\big(u(X,Y)\big) = \phi P\nabla_X Y + \in \eta(Y)PX + 2 \in \eta(X)\,\eta(Y)P\xi + g(X,Y)P\,\xi \tag{2.5}$$

$$Q\big(u(X,Y)\big) = Bh(X,Y) + \in \eta(Y)QX + 2 \in \eta(X)\,\eta(Y)Q\xi + g(X,Y)Q\xi \tag{2.6}$$

$$h(X,\phi PY) + \nabla_X{}^{\perp}\phi QY = \phi Q\nabla_X + ch(X,Y) \tag{2.7}$$

$$\eta\big(u(X,Y)\big) = g(X,Y)\eta(X) + \in \eta(Y)\eta(X) \tag{2.8}$$

for all $X, Y \in TM$

Proof. From (1.8), we see that

$$Y = PY + QY + \eta(Y)\xi \tag{2.9}$$

Differentiating (2.9) covariantly along X and using (1.6), (1.8), (1.10) and (1.11), we get

$$\phi P\nabla_X Y + \phi Q\nabla_X Y + Bh(X,Y) + Ch(X,Y) + g(X,Y)P\,\xi + g(X,Y)Q\xi + g(X,Y)\eta(X)\xi + \in \eta(Y)PX + \in \eta(Y)QX + \in \eta(Y)\eta(X)\xi + 2 \in \eta(X)\eta(Y)P\xi + 2 \in \eta(X)\eta(Y)Q\xi$$

$$= P\nabla_X\phi PY + Q\nabla_X\phi PY + \eta(\nabla_X\phi PY)\xi + h(X,\phi PY) - PA_{\phi QY}X - QA_{\phi QY}X - \eta\big(A_{\phi QY}X\big)\xi + \nabla_X{}^{\perp}\phi QY$$

Equating tangent and normal parts we get (2.5), (2.6), (2.7) and (2.8).

Lemma (2.3) Let M be a semi-invariant submanifold of an indefinite Lorentzian para Sasakian manifold, Then we have

$$A_{\phi X}Y = -A_{\phi Y}X, \ for\ all\ X, Y \in \lceil(D^{\perp}) \tag{2.10}$$

Proof. With the help of (1.6), (1.10) and (1.12), gives

$$g\big(A_{\phi X}Y, Z\big) = g(h(Y,Z), \phi X)$$

$= g(\widetilde{\nabla}_Z Y, \phi X)$
$= g\big(\phi \widetilde{\nabla}_Z Y, X\big)$
$= g\big(\widetilde{\nabla}_Z \phi Y, X\big) - g\big(\widetilde{\nabla}_Z \phi Y, X\big)$
$= -g\big(\phi Y, \widetilde{\nabla}_Z X\big)$
$= -g\big(A_{\phi Y} X, Z\big)$
for all $X, Y \ \in D^{\perp}\ and\ Z \in TM$ which implies (2.10).

3. Integrability of distribution on a semi-invariant submanifold of indefinite Lorentzian para-Sasakian manifold

Theorem (3.1) Let M be a semi-invariant submanifold of indefinite Lorentzian para Sasakian manifold $\widetilde{M}$. Then the distribution D is integrable if and only if

$$h(X, \phi Y) = h(Y, \phi X) \tag{3.1}$$

Proof. By using 2.1(a), we have

$g([X, Y], \xi) = g(\nabla_X Y - \nabla_Y X, \xi)$
$= -g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)$
$= -g(\phi X, Y) + g(\phi Y, X)$
$= 0, \ \ for\ all\ X, Y\ \in \lceil(D)$

In consequence of (2.7), we find.

$h(X, \phi Y) - h(Y, \phi X) = \ \phi Q[X, Y]$

which proves the theorem.

Corollary (3.1) The distribution $D \oplus \{\xi\}$ is integrable if and only if $h(X, \phi Y) = h(Y, \phi X)$ is satisfied.

The proof follows from (3.1)

Theorem (3.2) Let M be a semi-invariant submanifold of an indefinite Lorentzian para Sasakian manifold $\widetilde{M}$. Then the distribution $D^{\perp}$ is never integrable.

Proof.

For $X, Y \in D^{\perp}$, (2.1) give

$$u(X,Y) = -A_{\phi Y} X$$

Applying ϕ to (2.5) and using (1.1), we get

$$P\nabla_X Y = -\phi P\left(A_{\phi Y} X\right) \; for\ any\ X, Y \in D^{\perp}$$

which with the help of Lemma (2.3), gives

$$P([X,Y]) = \phi P\left(-A_{\phi Y} X + A_{\phi X} Y\right)$$
$$= -2\phi P(A_{\phi Y} X)$$

Showing the non integrability of $D^{\perp}$.

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