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A New Generalized Definition of Singular Kernel for Two-Dimensional Fredholm Integral Equation

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Abstract

This paper proposes a new generalized definition of singular kernel for Two-Dimensional Fredholm integral equation (T-DFIE). Furthermore, this integral equation can be solved by using two different methods. These methods are Toeplitz matrix method (TMM) and Product Nyström method (PNM). Moreover, numerical examples are considered when the generalized kernel takes the following forms: Carleman function, logarithmic form, and Cauchy kernel. The given numerical examples showed the efficiency and accuracy of the introduced methods.

Keywords: Fredholm integral equation, Carleman function**,** logarithmic**,** Cauchy kernel**,** Toeplitz matrix**,** Product Nyström

1. Introduction

Integral equation is a most important branch of mathematics. Integral equation is useful for many branches of science and arts. In recent years, authors have studied integral equations with continuous and singular kernels in more than one dimension, with their applications. Fallahzadeh, in [1], solved T-DFIE by using the method, which is based on the approximation by Gaussian radial basis functions and triangular nodes and weights. Authors, in [2], discussed an efficient iterative method based on quadrature formula to solve two-dimensional nonlinear Fredholm integral equations. Fattahzadeh, in [3], solved two-dimensional linear and nonlinear Fredholm integral equations of the first kind based on Haar wavelet. Authors, in [4], solved two-dimensional integral equation of the first kind by multi-step method. Alturk, in [5], solved two-dimensional Fredholm integral equations of the first kind using regularization-homotopy method. Al-bugami, in [6], discussed the solution of the Two-dimensional singular equation. Al-Bugami, in [7], discussed and proved two-dimensional Fredholm integral equation with time, Abdou, and others, in [8], introduced proof of convergence of the series solution to a class of nonlinear two-dimensional Hammerstein integral equation with singular kernel. In this work, we will study the Two-dimensional Fredholm integral equation with a generalized singular kernel. Then, the existence of a unique solution of the **T-DFIE** are discussed and proved using Picard method. Then, we solved this equation numerically, using TMM and PNM. Moreover, numerical results are obtained. Consider

$$(\lambda_1+\lambda_2)m(x,y)-(\theta_1+\theta_2)\int_\Omega\int_\Omega n(g(x)-g(u),g(y)-g(v))m(u,v)dudv=f(x,y)=[\delta-f_1(x,y)-f_2(x,y)] \quad (1)$$

under the condition

$$\int_\Omega\int_\Omega m(x,y)dxdy=S<\infty, \quad (S \text{ is a constant}) \quad (2)$$

where $f_i(x,y),\ i=1,2$ are the known functions describing the two surface, Ω is the contact domain between the two surfaces, $m(x,y)$ are the unknown normal stresses between $f_1(x,y)$ and $f_2(x,y)$, $\lambda_1(i=1,\ 2)$ are the coefficients bed of the compressible materials that depend on its geometry and its physical properties, δ is the rigid displacement under the action of a force S, $\theta_i=1-\mu_i^2/\pi E_i,\ i=1,2$ where μ_i are the Poisson's coefficients and E_i are the coefficients of Young.

We can write equation (1) in the form

$$m(x,y)-\lambda\int_\Omega\int_\Omega n(g(x)-g(u),g(y)-g(v))m(u,v)dudv=f(x,y) \quad (3)$$

where

$$\lambda=\frac{\theta_1+\theta_2}{\lambda_1+\lambda_2}, f(x,y)=\frac{\delta-f_1(x,y)-f_2(x,y)}{\lambda_1+\lambda_2}.$$

Equation (3) represents Fredholm integral equation of the second kind in two dimensional with a generalized singular kernel. The function $n(g(x)-g(u),g(y)-g(v))$ is the kernel of integral equation , which has a singular term, $f(x,y)$ is a known continuous function.

If $(\lambda_1+\lambda_2)=0, \theta_1+\theta_2=\frac{1}{2\pi\theta}, \ (\theta=G(1-\upsilon)^{-1}), \ f_2(x,y)=0$, we have the following integral equation

$$\int_\Omega\int_\Omega k(g(x)-g(u),g(y)-g(v))m(u,v)dudv = f^*(x,y), \ \ f^*(x,y)=2\pi\theta(\delta-f_1(x,y) \qquad (4)$$

Where G is the displacement magnitude, υ is Poison's coefficient, $f(x,y)$ is a known function, which describes the shape of stamp base.
We can write integral equation (3) in the form

$$\mu m(x,y)-\pi\int_\Omega\int_\Omega n(g(x)-g(u),g(y)-g(v))m(u,v)dudv = f(x,y) \qquad (5)$$

2. The existence and uniqueness solution

In order to guarantee the existence of a unique solution of equation (5), we assume the following conditions:

(i) $n(g(x)-g(u),g(y)-g(v))\in C([\Omega]\times C[\Omega])$ and satisfies:

$$[\int_\Omega\int_\Omega |n(g(x)-g(u),g(y)-g(v))|^2 dudv]^{\frac{1}{2}} = A<\infty \qquad (A \text{ is a constant})$$

(ii) $f(x,y)$ is continuous with its derivatives and belongs to $J=C([\Omega]\times[\Omega])$, and its norm is defined as

$$\|f(x,y)\| = \max_{x,y\in J}\int_\Omega[\int_\Omega f^2(x,y)dx]^{\frac{1}{2}}dy = M, \ \ \forall x,y\in\Omega$$

(iii)

$$\|m(x,y)\| = [\int_\Omega\int_\Omega |m(x,y)|^2 dxdy]^{\frac{1}{2}} \le C\,\|m\|_2$$

Now, we prove the existence of a unique solution of the equation (5), by using Picard method.

Theorem 1. The solution of the **T-DFIE** with weakly singular kernels is exist and a unique under the condition

$$|\lambda| < \frac{|\mu|}{A} \tag{6}$$

To prove the Theorem 1, we state the following lemmas.

Lemma 1. Beside the conditions (i)-(iii), the series $\sum_{i=0}^{\infty}\psi_i(x,y)$ is uniformly convergent to a solution function $m(x,y)$

Proof:

We construct the sequence of the functions $m_n(x,y)$ as

$$\mu m_n(x,y) = f(x,y) + \int_\Omega\int_\Omega n(g(x)-g(u), g(y)-g(v))dudv\,, n = 1,2,... \tag{7}$$

with

$$m_0(x,y) = f(x,y), \tag{8}$$

It is convenient to introduce

$$\psi_n(x,y) = m_n(x,y) - m_{n-1}(x,y) \tag{9}$$

where

$$m_n(x,y) = \sum_{i=0}^{n}\psi_i(x,y), n = 1,2,... \quad (\psi_0(x,y) = f(x,y)) \tag{10}$$

The formula (9) takes the form

$$\|\psi_n(x,y)\| \le \left|\frac{\lambda}{\mu}\right| \left\| \int_\Omega\int_\Omega n(g(x)-g(u), g(y)-g(v))(m_{n-1}(u,v) - m_{n-2}(u,v))dudv \right\| \tag{11}$$

With the aid of formula (9), we have

$$\|\psi_n(x,y)\|_{L_2(\Omega)\times L_2(\Omega)} \le \left|\frac{\lambda}{\mu}\right| \|\psi_{n-1}(u,v)\|_{L_2(\Omega)\times L_2(\Omega)} \left\| \int_\Omega\int_\Omega n(g(x)-g(u), g(y)-g(v))dudv \right\|$$

Then,

$$\|\psi_n(x,y)\|_{L_2(\Omega)\times L_2(\Omega)} \le \frac{1}{|\mu|} A|\lambda| \|\psi_{n-1}(u,v)\|_{L_2(\Omega)\times L_2(\Omega)}.$$

Which takes the form

$$\|\psi_n(x,y)\|_{L_2(\Omega)\times L_2(\Omega)} \le \alpha \|\psi_{n-1}\|_{L_2(\Omega)\times L_2(\Omega)}, \ (\alpha = \frac{1}{|\mu|} A|\lambda| < 1) \tag{12}$$

Let, in (11) n=1, after using Cauchy-Schwarz inequality, we get

$$\|\psi_1(x,y)\|_{L_2(\Omega)\times L_2(\Omega)} \le \left|\frac{\lambda}{\mu}\right| \left\| \left(\int_\Omega\int_\Omega n^2(g(x)-g(u), g(y)-g(v))dudv\right)^{\frac{1}{2}} \max_{x,y\in J} \int_\Omega \left(\psi_0^2(u,v)du\right)^{\frac{1}{2}} dv \right\| \tag{13}$$

Using the conditions (i) and (ii), we have

$$\|\psi_1(x,y)\|_{L_2(\Omega)\times L_2(\Omega)} \le \frac{1}{|\mu|} AM\,|\lambda| \le \alpha M .$$

So, by the mathematical induction method, we get

$$\|\psi_n(x,y)\|_{L_2(\Omega)\times L_2(\Omega)} \le \alpha^n M, \quad n = 0,1,2,... \tag{14}$$

Hence the sequence $\{m_n\}$ converges, so we can write

$$m(x,y) = \sum_{i=0}^{\infty} \psi_i(x,y) \tag{15}$$

The infinite series (15) is uniformly convergent since the terms $\psi_i(x,y)$ are dominated by α^i.

Lemma 2. A continuous function $m(x,y)$ represents a unique solution of equation (5).

Proof:

To prove that $m(x,y)$ represents a unique solution of equation (5), we first prove that $m(x,y)$ defined by (15) satisfies equation (5), then we set

$$m(x,y) = m_n(x,y) + q_n(x,y), \quad (q_n(x,y) \to 0 \text{ as } n \to \infty).$$

Then, we get

$$m(x,y) - q_n(x,y) = \frac{1}{\mu} f(x,y) + \frac{\lambda}{\mu} \int_\Omega\int_\Omega n(g(x)-g(u), g(y)-g(v))(m(u,v) - q_{n-1}(u,v))dudv .$$

Then, we have

$$\max_{x,y\in J} \left| m(x,y) - \frac{1}{\mu} f(x,y) - \frac{\lambda}{\mu} \int_\Omega\int_\Omega n(g(x)-g(u), g(y)-g(v))dudv \right| \le |q_n(x,y)|$$

$$+ \frac{|\lambda|}{|\mu|} \int_\Omega\int_\Omega |m(g(x)-g(u), g(y)-g(v))| \max_{x,y\in J} |q_{n-1}(u,v)| dudv .$$

In view of the condition (i), the above inequality takes the form

$$\left| m(x,y) - \frac{1}{\mu} f(x,y) - \frac{\lambda}{\mu} \int_{\Omega}\int_{\Omega} n(g(x)-g(u), g(y)-g(v)) m(u,v)\, du dv \right|_{C(\Omega)\times C(\Omega)} \leq \left| q_n(x,y) \right|_{C(\Omega)\times C(\Omega}$$

$$+\alpha \left\| q_{n-1}(u,v) \right\|_{C(\Omega)\times C(\Omega)}, (\alpha = \frac{1}{|\mu|} A |\lambda|) \tag{16}$$

So, by taking n large enough, the right-hand side of (16) can be made as small as desired. Thus, the function $m(x,y)$ satisfies the integral equation (5).

To show that $m(x,y)$ is the only solution, we assume another one $\tilde{m}(x,y)$ is also a continuous solution of (5), hence we get

$$|m(x,y) - \tilde{m}(x,y)| \leq \left| \frac{\lambda}{\mu} \right| \int_{\Omega}\int_{\Omega} |n(g(x)-g(u), g(y)-g(v))| |m(u,v) - \tilde{m}(u,v)| du dv \tag{17}$$

With the help of conditions (i), equation (17) leads to

$$\| m(x,y) - \tilde{m}(x,y) \|_{C(\Omega)\times C(\Omega)} \leq \alpha \| m(u,v) - \tilde{m}(u,v) \|_{C(\Omega)\times C(\Omega)}, \quad (\alpha = \frac{A}{|\mu|} |\lambda| < 1) \tag{18}$$

Since $\alpha < 1$, then the inequality (18) is true only if $\phi(x,y) = \tilde{\phi}(x,y)$ that is the solution of (5) is a unique.

3. The numerical methods

In this section, we state the numerical methods for solving (5) by using the TMM and PNM.

3.1 The TMM

Consider the linear integral equation (5), and let the domain of integration $\Omega = [-a,a]\times[-a,a]$.

$$\mu m(x,y) - \lambda \int_{-a}^{a}\int_{-a}^{a} n(g(x)-g(u), g(y)-g(v)) m(u,v)\, du dv = f(x,y) \tag{19}$$

the integral term of Eq. (19) can be written as :

$$\int_{-a}^{a}\int_{-a}^{a} n(g(x)-g(u), g(y)-g(v)) m(u,v)\, du dv = \sum_{r=-N}^{N-1} \sum_{s=-M}^{M-1} n(g(x)-g(u), g(y)-g(v)) m(u,v)\, du dv \tag{20}$$

where $h = \frac{a}{N}$, then, we write

$$\int_{rh}^{rh+h}\int_{sh}^{sh+h} n(g(x)-g(u),g(y)-g(v))m(u,v)dudv = A_{r,s}(x,y)m(rh,sh)+B_{r,s}(x,y)m(rh+h,sh+h)+R \quad (21)$$

If the error R is assumed negligible, we can clearly solve this set of equations for $B_{r,s}(x,y)$ and $B_{r,s}(x,y)$, then we obtain

$$A_{r,s}(x,y)=\frac{1}{h}\left[\frac{(rh+h)(sh+h)I}{(rh+sh+h)}-\frac{J}{(rh+sh+h)}\right] \quad (22)$$

and

$$B_{r,s}(x,y)=\frac{1}{h}\left[\frac{J}{(rh+sh+h)}-\frac{(rh)(sh)I}{(rh+sh+h)}\right] \quad (23)$$

Hence, Eq (20) becomes

$$\begin{aligned}\int_{-a}^{a}\int_{-a}^{a} n(g(x)-g(u),g(y)-g(v))m(u,v)dudv &= \sum_{r=-N}^{N-1}\sum_{s=-M}^{M-1}[A_{r,s}(x,y)m(rh,sh)+B_{r,s}(x,y)m(rh+h,sh+h)]\\ &=\sum_{r=-N}^{N\text{-}1}\sum_{s=-M}^{M-1}A_{r,s}(x,y)m(rh,msh)+\sum_{r=-N}^{N}\sum_{s=-M}^{M}B_{(r-1)(s-1)}(x,y)m(rh,sh)\\ &=\sum_{r=-N}^{N}\sum_{s=-M}^{M}D_{r,s}(x,y)m(rh,sh) \quad (24)\end{aligned}$$

where

$$D_{r,s}(x,y)=\begin{cases}A_{-N}(x,y) & r=s=-N\\ A_r(x,y)+B_{s-1}(x,y) & -N<r=s<N\\ B_{N-1}(x,y) & r=s=N\end{cases}$$

Thus, the integral equation (5) becomes:

$$\mu m(x,y)-\lambda\sum_{r=-N}^{N}\sum_{s=-M}^{M}D_{r,s}(x,y)m(rh,sh)=f(x,y)$$

If we put $x=kh,\ y=lh,$ then we get:

$$\mu m_{k,l}-\lambda\sum_{r=-N}^{N}\sum_{s=-M}^{M}D_{r,s}m_{rs}=f_{kl} \qquad -N\le k\le N, -M\le l\le M \quad (25)$$

where

$$D_{r,s}=\begin{cases}A_{-N}(kh,lh) & r=s=-N\\ A_n(kh,lh)+B_{r-1}(kh,lh) & -N<r=s<N\\ B_{N-1}(kh,lh) & r=h=N\end{cases} \quad (26)$$

The matrix $D_{r,s}$ may be written as $D_{r,s}=G_{r,s}-E_{r,s}$, where

$$G_{r,s}=A_r(kh,lh)+B_{r-1}(kh,lh), \qquad -N\le k,l,r,\le N \quad (27)$$

is the Toeplitz matrix of order $2N+1$, and the matrix

$$E_{r,s}=\begin{cases}B_{-N-1}(kh,lh) & \mathrm{r}=s=-N\\ 0 & -N<r=s<N\\ A_N(kh,lh) & \mathrm{r}=s=N\end{cases} \tag{28}$$

represents a matrix of order 2N+1.
However, the solution of the system of equations (25) can be obtained in the form

$$m_{k,l}=[\mu I-\lambda(G_{kln}-E_{kln})]^{-1}f_{kl} \tag{29}$$

where I is the identity matrix and $\left|\mu I-\lambda(G_{kln}-E_{kln})\right|\neq 0.$

3.2 The PNM

Consider the **T-DFIE** of the second kind

$$\mu m(x,y)-\lambda\int_a^b\int_a^b n(g(x)-g(u),g(y)-g(v))m(u,v)dudv=f(x,y) \tag{30}$$

We can often factor out the singularity in k by writing

$$n(g(x)-g(u),g(y)-g(v))=\tilde{n}(g(x)-g(u),g(y)-g(v))p(g(x)-g(u),g(y)-g(v)) \tag{31}$$

where $(g(x)-g(u),g(y)-g(v))$, $\tilde{n}(g(x)-g(u),g(y)-g(v))$ are badly behaved and well behaved functions of their arguments, respectively. Then, we get

$$\mu m(x,y)-\lambda\int_a^b\int_a^b p(g(x)-g(u),g(y)-g(v))\tilde{n}(g(x)-g(u),g(y)-g(v))m(u,v)dudv=f(x,y) \tag{32}$$

$$\int_a^b\int_a^b p(g(x_i)-g(u),g(y_i)-g(v))\tilde{n}(g(x_i)-g(u),g(y_i)-g(v))m(u,v)dudv$$
$$\approx\sum_{j=0}^{N}\sum_{i=0}^{M}w_{ij}w_{il}\tilde{n}(g(x_i)-g(u_j),g(y_i)-g(v_l))m(u_j,v_l) \tag{33}$$

In addition, we approximate the integral term in (32), we may write

$$\int_a^b\int_a^b p(g(x_i)-g(u),g(y_i)-g(v))\tilde{n}(g(x_i)-g(u),g(y_i)-g(v))m(u,v)dudv$$
$$\approx\sum_{j=0}^{\frac{N-2}{2}}\sum_{i=0}^{\frac{M-2}{2}}\int_a^b\int_a^b p(g(x_i)-g(u),g(y_i)-g(v))\tilde{n}(g(x_i)-g(u),g(y_i)-g(v))dudv$$

where $x_i = u_i = y_i = v_i = a + ih,\ i = 0,1,...,N$ with $h = \dfrac{b-a}{N}$ and N even. Now we approximate the nonsingular part of the integrand over each interval $[u_{2j}, u_{2j+2}]$, $[v_{2l}, v_{2l+2}]$ we find

$$\int_a^b\int_a^b p(g(u_i)-g(u),g(v_i)-g(v))\tilde{n}(g(u_i)-g(u),g(v_i)-g(v))m(u,v)dudv = \sum_{j=0}^{\frac{N-2}{2}}\sum_{l=0}^{\frac{M-2}{2}}\int_{u_{2j}}^{u_{2j+2}}\int_{v_{2l}}^{v_{2l+2}} p(g(u_i)-g(u),g(v_i)-g(v))$$

$$\times\{\frac{(u_{2j+1}-u)(v_{2l+1}-v)(u_{2j+2}-u)(v_{2l+2}-v)}{(2h^2)(2h^2)}\tilde{n}(g(u_i)-g(u_{2j}),g(v_i)-g(v_{2l}))m(u_{2j},v_{2l})$$

$$+\frac{(u-u_{2j})(v-v_{2l})(u_{2j+2}-u)(v_{2l+2}-v)}{(h^2)(h^2)}\tilde{n}(g(u_i)-g(u_{2j+1}),g(v_i)-g(v_{2l+1}))m(u_{2j+1},v_{2l+1})$$

$$+\frac{(u-u_{2j})(v-v_{2l})(u-u_{2j+1})(v-v_{2l+1})}{(2h^2)(2h^2)}\tilde{n}(g(u_i)-g(u_{2j+2}),g(v_i)-g(v_{2l+2}))m(u_{2j+2},v_{2l+2})\}dudv$$

$$=\sum_{j=0}^{N}\sum_{l=0}^{M} w_{ij}w_{il}\tilde{n}(g(u_i)-g(u_j),g(v_i)-g(v_l))m(u_i,v_l) \tag{34}$$

where $u_j = jh,\ u_{j+1} = (j+1)h,\ u_j - u_{j+1} = v_l - v_{l+1} = -h,$

In general, if we define and assume $u = u_{2j-2} + \xi h,\ \mathrm{v} = v_{2l-2} + \delta h,\ 0 \le \xi, \delta \le 2,$ then we get

$$\alpha_{j,l}(u_i,v_i) = \frac{h}{4}\int_0^2\int_0^2 \xi\delta(\xi-1)(\delta-1)p(g(u_i)-(g(u_{2j-2})+\xi g(h)),g(v_i)-(g(v_{2l-2})+\delta g(h)))d\xi d\delta$$

$$\beta_{j,l}(u_i,v_i) = \frac{h}{4}\int_0^2\int_0^2 (\xi-1)(\xi-2)(\delta-1)(\delta-2)p(g(u_i)-(g(u_{2j-2})+\xi g(h)),g(v_i)-(g(v_{2l-2})+\delta g(h)))d\xi d\delta,$$

$$\gamma_{j,l}(u_i,v_i) = \frac{h}{4}\int_0^2\int_0^2 \xi\delta(2-\xi)(2-\delta)p(g(u_i)-(g(u_{2j-2})+\xi g(h)),g(v_i)-(g(v_{2l-2})+\delta g(h)))d\xi d\delta \tag{35}$$

If we define,

$$\psi_k = \int_0^2\int_0^2 \xi^k\delta^k p(g(u_i)-(g(u_{2j-2})+\xi g(h)),g(v_i)-(g(v_{2l-2})+\delta g(h)))d\xi d\delta, k = 0,1,2,$$

we have

$$\psi_k = \int_0^2\int_0^2 \zeta^k\delta^k p((z-\zeta)g(h),(t-\delta)g(h))d\zeta d\delta, k = 0,1,2, z = i-2j+2, t = i-2l+2 \tag{36}$$

Equation (30) transforms into the following system of **LAEs**

$$\mu m(x_i,y_i) - \lambda\sum_{j=0}^{N}\sum_{l=0}^{M} w_{ij}w_{il}\tilde{n}(g(u_i)-g(u_j),g(v_i)-g(v_l))m(u_j,v_l) = f(x_i,y_i),\ i = 0,1,...,N \tag{37}$$

or $\mu\Gamma - \lambda W\tilde{W}\Gamma = F$, which has the solution

$$\Gamma = [\mu I - \lambda W\tilde{W}]^{-1} F,$$

where I is the identity matrix and $\left|\mu I - \lambda\, W\tilde{W}\right| \neq 0$.

4. Applications and numerical results for the T-DFIE with a generalized singular kernel

In this section, we state some applications and numerical results to discuss the approximate solution. The **TMM** and **PNM** are used to get numerical solution for values of μ=1. We divided the position interval by N=40 units. **In tables (1-2):** *Exact sol.* $\rightarrow$ the exact solution, ***Approx. T.*** $\rightarrow$approximate solution of **TMM**, ***Error. T.*** $\rightarrow$ the absolute error of **TMM, *Approx. N.*** $\rightarrow$approximate solution of **PNM**, ***Error. N.*** $\rightarrow$ the absolute error of **PNM.**

4.1 Applications for a generalized Carleman kernel.

Example 1. Consider:

$$m(x,y) - \lambda \int_{-1}^{1}\int_{-1}^{1} \left|x^4 - u^4\right|^{-\upsilon} \left|y^4 - v^4\right|^{-\upsilon} m(u,v)\,du\,dv = f(x,y)$$

Here the values of μ = 1, with λ = 0.2500, 0.31579, and we divided the interval by N = 40 units, and 0 < v < 1/2, where:

$$\lambda = \frac{E\upsilon}{(1+\upsilon)(1-2\upsilon)}, \mu = \frac{E}{2(1+\upsilon)}$$

Where E is called Young ratio. The exact solution $m(x, y) = x^5 \cdot t^6$

x	*y*	*Exact sol.*	*Approx. T.*	*Error. T.*	*Appro. N.*	*Error. N.*
-1.00	-1.00	-1.0000000	-1.01315423	0.01315423	-1.0660222	0.0660222
-0.60	-0.60	-0.0777600	-0.07878387	0.00102287	-0.0820027	0.00424276
-0.20	-0.20	-0.00032000	-0.00032420	0.420935×10^{-5}	0.00055782	0.00087782
0.20	0.20	0.00032000	0.00032410	0.420935×10^{-5}	0.00114682	0.00082681
0.60	0.60	0.07776000	0.07878287	0.001022873	0.08380305	0.00604305
1.00	1.00	1.00000000	1.01315424	0.01315424	1.06747125	0.06747125

Table 1. The values of exact, approximate and absolute error values by **TMM** and **PNM for** λ = 0.2500, ν = 0.1

x	*y*	*Exact sol.*	*Approx. T.*	*Error. T.*	*Appro. N.*	*Error. N.*
-1.00	-1.00	-1.0000000	-1.01667196	0.01667196	-1.02158416	0.02152416
-0.60	-0.60	-0.0777600	-0.07905641	0.00129641	-0.08122544	0.00346544
-0.20	-0.20	-0.00032000	-0.00032533	0.533503×10^{-5}	-0.00299501	0.00267501
0.20	0.20	0.00032000	0.00032533	0.533502×10^{-5}	0.00452778	0.00420778
0.60	0.60	0.07776000	0.07905641	0.001296411	0.07094392	0.00681607
1.00	1.00	1.00000000	1.01667196	0.01667196	1.01375084	0.01375084

Table 2. The values of exact, approximate and absolute error values by **TMM** and **PNM for** λ = 0.31579, ν = 0.12

Example 2. Consider:

$$m(x,y)-\lambda\int_{-1}^{1}\int_{-1}^{1}\left|e^{x}-e^{u}\right|^{-\upsilon}\left|e^{y}-e^{v}\right|^{-\upsilon}m(u,v)\,dudv=f(x,y)$$

Here the values of μ = 1, with λ = 0.111111, 0.13636, and we divided the interval by N = 40 units. The exact solution is $m(x,y)=(e^{x}.y^{3})/3$

x	*y*	*Exact sol.*	*Approx. T.*	*Error. T.*	*Appro. N.*	*Error. N.*
-1.00	-1.00	0.04087549	0.16481416	0.20568966	0.15990196	0.11902647
-0.60	-0.60	0.06097907	0.06674974	0.00577067	0.06458070	0.00360163
-0.20	-0.20	0.09097008	0.09559025	0.00462016	0.09292056	0.00195048
0.20	0.20	0.13571141	0.12221233	0.01349907	0.11735921	0.01835219
0.60	0.60	0.20245764	0.11898300	0.08347463	0.11087051	0.09158712
1.00	1.00	0.30203131	0.16481416	0.20568966	0.16189304	0.14013826

Table 3. The values of exact, approximate and absolute error values by **TMM** and **PNM for** λ = 0.111111, ν = 0.05

x	*y*	*Exact sol.*	*Approx. T.*	*Error. T.*	*Appro. N.*	*Error. N.*
-1.00	-1.00	0.04087549	0.23517676	0.27605225	0.23026456	0.18938907
-0.60	-0.60	0.06097907	0.06830464	0.00732556	0.06613560	0.00515653
-0.20	-0.20	0.09097008	0.09680555	0.00583546	0.09413586	0.00316578
0.20	0.20	0.13571141	0.11701311	0.01869829	0.11215999	0.02355141
0.60	0.60	0.20245764	0.08746605	0.11499158	0.07935356	0.12310407
1.00	1.00	0.30203131	0.23517676	0.27615726	0.23225564	0.06977566

Table 4. The values of exact, approximate and absolute error values by **TMM** and **PNM for** $\lambda = 0.13636,\ \nu = 0.06$

4.2 Applications for a generalized logarithmic kernel.

Example 1. Consider:

$$m(x,y) - \lambda \int_{-1}^{1}\int_{-1}^{1} \ln\left|x^4 - u^4\right| \ln\left|y^4 - v^4\right| m(u,v)\,du\,dv = f(x,y)$$

Here values of $\mu = 1$, $\lambda = 0.25$, 0.6666666667, and, $N = 40$. The exact solution $m(x, y) = x^5 \cdot t^6$

x	*y*	*Exact sol.*	*Approx. T.*	*Error. T.*	*Appro. N.*	*Error. N.*
-1.00	-1.00	-1.0000000	-0.97925541	0.02074458	-1.06659353	0.06659353
-0.60	-0.60	-0.0777600	-0.19001761	0.11225761	-0.08288072	0.00512072
-0.20	-0.20	-0.00032000	-0.00828671	0.00796671	-0.00030394	0.000016055
0.20	0.20	0.00032000	-0.00174109	0.00206109	0.00035858	0.000038580
0.60	0.60	0.07776000	0.12397784	0.04621784	0.08294157	0.005181574
1.00	1.00	1.00000000	0.985227487	0.01477251	1.06665348	0.06665348

Table 5. The values of exact, approximate and absolute error values by **TMM** and **PNM for** $\lambda = 0.25$

x	*y*	*Exact sol.*	*Approx. T.*	*Error. T.*	*Appro. N.*	*Error. N.*
-1.00	-1.00	-1.0000000	-0.97806096	0.021939036	-1.19975439	0.19975439
-0.60	-0.60	-0.0777600	-0.74672047	0.668960477	-0.093098597	0.01533859
-0.20	-0.20	-0.00032000	-0.04306515	0.042745152	-0.000257864	0.00006213
0.20	0.20	0.00032000	0.014754856	0.014434856	0.000442169	0.00012216
0.60	0.60	0.07776000	0.377978276	0.300218276	0.093303879	0.01554387
1.00	1.00	1.00000000	0.962595232	0.037404767	1.199956684	0.19995668

Table 6. The values of exact, approximate and absolute error values by **TMM** and **PNM for** $\lambda = 0.6666666667$

Example 2. Consider:

$$m(x,y)-\lambda\int_{-1}^{1}\int_{-1}^{1}\ln\left|e^{x}-e^{u}\right|\ln\left|e^{y}-e^{v}\right|m(u,v)\,dudv=f(x,y)$$

Here the values of $\mu = 1$, $\lambda = 0.001$, 0.01, and $N = 40$. The exact solution is

$$m(x,y)=e^{x}.y^{3}$$

x	*y*	*Exact sol.*	*Approx. T.*	*Error. T.*	*Appro. N.*	*Error. N.*
-1.00	-1.00	0.36787944	0.36975141	0.00187197	0.3648392	0.00304023
-0.60	-0.60	0.54881163	0.54905583	0.00024420	0.5468867	0.00192483
-0.20	-0.20	0.81873075	0.81885667	0.00012592	0.8161869	0.00254376
0.20	0.20	1.22140275	1.22204674	0.00064398	1.2171936	0.00420912
0.60	0.60	1.82211880	1.82338961	0.00127081	1.8527712	0.00684167
1.00	1.00	2.71828182	2.67302896	0.04525285	2.0670107	0.04817397

Table 7. The values of exact, approximate and absolute error values by **TMM** and **PNM for** $\lambda = 0.001$

x	*y*	*Exact sol.*	*Approx. T.*	*Error. T.*	*Appro. N.*	*Error. N.*
-1.00	-1.00	0.36787944	0.38928557	0.02140613	0.38437337	0.01649393
-0.60	-0.60	0.54881163	0.55104552	0.00223388	0.54887648	0.00006485
-0.20	-0.20	0.81873075	0.81974409	0.00101333	0.81707440	0.00165634
0.20	0.20	1.22140275	1.22742566	0.00602290	1.22257254	0.00116979
0.60	0.60	1.82211880	1.83091133	0.00879253	1.82279884	0.00068004
1.00	1.00	2.71828182	2.31783895	0.40044287	2.31491783	0.40336398

Table 8. The values of exact, approximate and absolute error values by **TMM** and **PNM for** $\lambda = 0.01$

4.3 Applications for a generalized Cauchy kernel.

Example 1: Consider:

$$m(x,y)-\lambda\int_{-1}^{1}\int_{-1}^{1}\frac{1}{x^2-u^2}\cdot\frac{1}{y^2-v^2}\,m(u,v)\,du\,dv = f(x,y)$$

Here the values of $\mu = 1$, $\lambda = 0.6666666667, 1.5$ and $N = 40$. The exact solution $m(x, y) = x^{5}\cdot t^{6}$

x	*y*	*Exact sol.*	*Approx. T.*	*Error. T.*	*Appro. N.*	*Error. N.*
-1.00	-1.00	-1.0000000	-1.03585959	0.03585959	-1.0407717	0.04077179
-0.60	-0.60	-0.0777600	-0.0805483	0.002788300	-0.0827173	0.00495733
-0.20	-0.20	-0.00032000	-0.0003296	0.960000×10^{-5}	-0.0029992	0.00267928
0.20	0.20	0.00032000	0.0003313	0.000011300	0.00452181	0.00420181
0.60	0.60	0.07776000	0.0805483	0.002788300	0.07243581	0.00532418
1.00	1.00	1.00000000	1.0358595	0.03585957	1.0329383	0.03293838

Table 9. The values of exact, approximate and absolute error values by **TMM** and **PNM for** $\lambda = 0.6666666667$

x	*y*	*Exact sol.*	*Approx. T.*	*Error. T.*	*Appro. N.*	*Error. N.*
-1.00 -0.60 -0.20 0.20 0.60 1.00	-1.00 -0.60 -0.20 0.20 0.60 1.00	-1.0000000 -0.0777600 - 0.00032000 0.00032000 0.07776000 1.00000000	- 1.08434088 -0.0843158 -0.0003469 0.0003469 0.08443158 1.0843088	0.08430883 0.00655585 0.00002698 0.00026980 0.00655585 0.08430883	-1.0892530 - 0.08648483 - 0.00301658 0.00450621 0.07631909 1.08138768	0.08925308 0.00872483 0.00269658 0.00418621 0.00144090 0.08138768

Table 10. The values of exact, approximate and absolute error values by **TMM** and **PNM for** λ =1.5

Example 2: Consider:

$$m(x,y)-\lambda\int_{-1}^{1}\int_{-1}^{1}\frac{1}{e^{x}-e^{u}}.\frac{1}{e^{y}-e^{v}}m(u,v)dudv=f(x,y)$$

Here the values of μ = 1, with λ = 0.0001, 0.001, and we divided the interval by N = 41 units.
Exact solution $= e^{x}.t^{3}$

x	*y*	*Exact sol.*	*Approx. T.*	*Error. T.*	*Appro. N.*	*Error. N.*
-1.00	-1.00	0.36787944	0.35943344	0.00844599	0.3545212	0.0133582
-0.60	-0.60	0.54881163	0.54867367	0.00105573	0.5465046	0.0023069
-0.20	-0.20	0.81873075	0.81979248	0.00106173	0.8171227	0.0016079
0.20	0.20	1.22140275	1.22162755	0.00022479	1.2167744	0.0046283
0.60	0.60	1.82211880	1.81923080	0.00288799	1.8111183	0.0110004
1.00	1.00	2.71828182	2.80597325	0.08769142	2.8030521	0.0847703

Table 11. The values of exact, approximate and absolute error values by **TMM** and **PNM for** λ = 0.0001

x	*y*	*Exact sol.*	*Approx. T.*	*Error. T.*	*Appro. N.*	*Error. N.*
-1.00	-1.00	0.36787944	0.28963690	0.07824253	0.2847247	0.0831547
-0.60	-0.60	0.54881163	0.55938832	0.01057668	0.5572192	0.0084076
-0.20	-0.20	0.81873075	0.82961080	0.01088005	0.8269411	0.0082103
0.20	0.20	1.22140275	1.22427599	0.00287323	1.2194228	0.0019798
0.60	0.60	1.82211880	1.79562206	0.02649673	1.7875095	0.0346092
1.00	1.00	2.71828182	3.70246043	0.98417861	3.6995323	0.9812574

Table 12. The values of exact, approximate and absolute error values by **TMM** and **PNM for** $\lambda = 0.001$

5. Conclusion

This paper introduced the new definition of singular kernel in the generalized form for T-DFIE. Also, is proposed effective numerical methods to obtain the solution of this equation. Error analysis and some numerical examples are presented to illustrate the effectiveness and accuracy of the methods. From the previous examples, we note that the error is increasing when the values of λ and υ are increasing. The error using the TMM is less than the error using the PNM.

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