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Applications of Scalar Type Operators to Some Cauchy Problems

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Abstract

In this paper, we investigate some questions related to some Cauchy equations. Our interest is to apply the $(\alpha, \alpha+1)$ type $\mathbb{R}$ operators to analyze such equations.

Keywords: Integrated semigroup, $(\alpha, \alpha+1)$ type $\mathbb{R}$ operators

1 Introduction

Let X be a Banach space and $B(X)$ denotes a bounded operator on X. Also let H be a closed densely defined operator on a Banach space X with $\sigma(H) \subseteq \mathbb{R}$ and whose resolvent $\| (z-H)^{-1} \|$ is bounded for all $z \notin \mathbb{R}$ and that it satisfy the hypothesis below

$$\| (z-H)^{-1} \| \leq c \mid Iz \mid^{-1} \left(\frac{\langle z \rangle}{\mid Iz \mid} \right)^{\alpha} \tag{1}$$

for some $\alpha \geq 0$ and $c > 0$ then H is of $(\alpha, \alpha+1)$ type $\mathbb{R}$.
Here $\langle z \rangle := (1+ \mid z \mid^2)^{\frac{1}{2}}$ and Iz is the imaginary part of z. The hypothesis above appears in [1] which we can state is important in application of the $\mathcal{U}$ functional Calculus for $(\alpha, \alpha+1)$ type $\mathbb{R}$ operators [2]. The $\mathcal{U}$ functional calculus for an operator H of $(\alpha, \alpha+1)$ type $\mathbb{R}$ is defined via the formula

$$f(H) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} (z-H)^{-1} dx dy \tag{2}$$

for $f \in \mathcal{U}$ and $\tilde{f}$ is an analytic extension of f and $\mathcal{U}$ denotes the space of smooth functions. This definition is due to Helffer and Sjostrand [6]. We now consider the general abstract Cauchy equation given by;

$$\begin{cases} \text{u'(t)=-Hu(t)}, & t \geq 0; \\ \text{u(0)=x}, & x \in X. \end{cases} \tag{3}$$

It is well known that a function $u(.) : [0, \infty) \to D(H)$(Domain of H) with $u(.) \in ([0, \infty); X)$ and $u(0) = x$ and satisfy (3) is a solution of (3). In studying (3), the notion of integrated semigroups comes in handy. This class comprises of the one parameter semi-group and the cosine families. It is also important to note that some classes of abstract Cauchy equations exist where the elements of e^{-tH} is not bounded operators, for example the Schrodinger operators acting on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ for $p \neq 2$. To deal with such problems, one needs to find larger sets of functions f giving rise to bounded operators in form of $e^{-tH} f(H)$ such that the solution of (3) exist. In [6] it had been realized that (2) can be used to study Schrodinger operators on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ for $p \neq 2$, in which case the general solution $u(t) = e^{-itH}$ of the Schrodinger equation is unbounded. This means that in (3) one must look for a suitable functional calculus involving H and e^{-tH}, and so the notion of $\mathcal{U}$ functional calculus comes in handy. In our study therefore, we shall apply the $\mathcal{U}$ functional calculus for $(\alpha, \alpha+1)$ type $\mathbb{R}$ operator H satisfying (1) to study abstract Cauchy equations of the form given by (3) where the solution denoted by $u(t)$ is unbounded. An operator H on a Banach space X is the generator of k times integrated semigroup (where $k \in \mathbb{N}_o$) if there exist $w \geq 0$ and $S(.) : [0, \infty) \to B(H)$ a strongly continuous group such that (w, ∞) is contained in the resolvent set of H and

$$(\lambda I - H)^{-1} = \lambda^k \int_0^\infty e^{-\lambda t} S(t) x dt \tag{4}$$

for all $x \in X$ and $\lambda > w$. The function $S(.)$ is called k-times integrated semigroup. It follows from the Hille Yosida theorem that one can characterize the operators H satisfying (1) for which (3) admits a unique solution given by a strongly continuous C_o semi group of $(\alpha, \alpha+1)$ type $\mathbb{R}$ operators acting on the Banach space X. The solution of (3) is given by $u(t, x) = T(t)x$ where $T(t) = e^{-tH}$ for $t \geq 0$ and $x \in X$. It follows that H is the infinitesimal generator of $u(t)$. We now consider the abstract Cauchy equation given by (3). If a closed densely defined linear operator H has a resolvent in the half right plane and if $u(.)$ is an exponential bounded solution of (3) with $u(0) = x$, then the resolvent $R(\lambda, H)x$ is the Laplace transform of $u(.)$ that is;

$$R(\lambda, H)x = \int_0^\infty e^{-\lambda t} u(t) x dt \tag{5}$$

We now state some definitions and Theorems necessary in proving our results.

2 Definitions and theorems

Definition 2.1 Let $A \in B(X)$, then there exist a constant $C \geq 1$ and $\gamma \geq 0$ such that

$$\| e^{tA} \| \leq Ce^{t\gamma} \tag{6}$$

for all $t \geq 0$.
Theorem 2.2 Let H be a bounded operator with $\sigma(H) \subseteq \mathbb{R}$ and $T_t = e^{iHt}$ such that

$$\| T_t \| \leq C(1+ \mid t \mid)^{\alpha} \tag{7}$$

where α is non-negative integer. Then H is of $(\alpha, \alpha+1)$-type $\mathbb{R}$
Proof: See[5]
The following two theorems are consistent with the $(\alpha, \alpha+1)$ type $\mathbb{R}$ operators and whose proofs can be found in [3].
Theorem 2.3 Let H be a linear operator on a Banach space X. If there exist constants w and C such that the resolvent $R(\lambda, H)$ exist and satisfy

$$\mid R(\lambda, H) \mid \leq C(1+ \mid u \mid)^{k} \tag{8}$$

for some $-1 \leq k$ and for all $u \in \mathbb{C}$ with $R(\lambda) > w$ ($R(\lambda)$ denotes Real part of λ), then (3) has a unique solution $u(.)$ for every $x \in D(H)$ such that $\mid u(t) \mid \leq Ce^{pt} \parallel x \parallel$ for $p > w$.
Theorem 2.4 Let H be a linear operator with nonempty resolvent set. If (3) has a solution $u(.)$, with $u(0) = x$ such that $\mid u(t) \mid \leq Ce^{pt}$ for some constants C, p then for every $\lambda \in \rho(H)$ with $R(\lambda) > p$ we've

$$R(\lambda, H)x = \int_0^{\infty} e^{\lambda t} u(t) x dt \tag{9}$$

The following is an immediate consequence of Theorem 2.3
Corollary 2.5 If H is of $(0, 1)$ type $\mathbb{R}$ with $C = 1$, then (8) reduces to

$$\mid R(\lambda, H) \mid \leq 1 \tag{10}$$

for $k = 0$ and (3) has a unique solution $u(t)$ which is bounded above by 1 for $\parallel x \parallel = 1$. In this case

$$R(\lambda, H)x = \int_0^{\infty} e^{-\lambda t} u(t) x dt \leq \int_0^{\infty} e^{-\lambda t} dt \leq \frac{1}{\lambda} \tag{11}$$

In particular, if $u(t)$ is a contraction then the solution $u(t)$ satisfying (3) is bounded above by 1.
Definition 2.6 The Schwartz space $S(\mathbb{R}^n)$ of rapidly decreasing smooth functions consists of all $f \in C^{\infty}(\mathbb{R}^n)$ satisfying

$$lim_{|x| \to \infty} P(x) \frac{\partial^{\alpha_1 + \ldots, + \alpha_n}}{\partial x_1^{\alpha_1} \ldots, \partial x_n^{\alpha_n}}(x) = 0$$

for each polynomial P and each partial derivative as indicated above and $(\alpha_1, ..., \alpha_n \in \{0, 1, 2, ...\})$

Remark 2.7 $C_c^\infty(\mathbb{R}) \subset S(\mathbb{R}^n)$. Here, $f \in C_c^\infty(\mathbb{R})$ if and only if $f \in C^\infty(\mathbb{R})$ and f has compact support. Also $C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R}^n)$, $1 \le p < \infty$, and in $C_o(\mathbb{R}^n)$, the continuous functions on $\mathbb{R}^n$ that vanish at infinity; hence $S(\mathbb{R}^n)$ is also dense in these spaces. We now define the following family $S(\beta)$, $\beta \in \mathbb{R}$ found in [7] as follows:

Definition 2.8 $f \in S(\beta)$ if $f \in C_c^\infty(\mathbb{R})$ and $f(\lambda)$ has an asymptotic expansion in λ^{-1} as $\lambda \to \infty$ in the following sense. For any $N > 0$

$$f(\lambda) = \sum_{K=0}^{N} a_k \lambda^{-\beta-k} + \gamma_N(\lambda) \tag{12}$$

$\lambda \ge 1$ and where $\gamma_N(\lambda)$ satisfy

$$\mid \left(\frac{d}{d\lambda} \gamma_N(\lambda) \right) \mid \le C_{N_k} (1+ \mid \lambda \mid)^{-\beta-N-1} \tag{13}$$

for all $\lambda \ge 1$ and $k = 0, 1, 2...,$

If $\beta = 0$ then (12) reduces to

$$f(\lambda) = \sum_{K=0}^{N} a_k \lambda^{-k} + \gamma_N(\lambda) \tag{14}$$

and $\gamma_N(\lambda)$ satisfy

$$\mid \left(\frac{d}{d\lambda} \gamma_N(\lambda) \right) \mid \le C_{N_k} (1+ \mid \lambda \mid)^{-N-1} \tag{15}$$

for all $\lambda \ge 1$ and $k = 0, 1, 2...,$

We now state the following theorems whose proofs can be found in [7].

Theorem 2.9 Let $1 \le p \le \infty$ and let $f \in S(\infty)$. Then $e^{-itH} f(H)$ is bounded in $L^p(\mathbb{R}^N)$ for $t \in \mathbb{R}$. Moreover, for $\beta > N \mid 1/p - 1/2 \mid$,

$$\parallel e^{-itH} f(H) \parallel \le C(1+ \mid t \mid)^{\beta}, \, t \in \mathbb{R} \tag{16}$$

Theorem 2.10 Suppose $N \le 3$ and let $1 \le p \le \infty$. If $f \in S(\beta)$ for some $\beta > 2 + N/4$ then

$$\parallel e^{-itH} f(H) \parallel \le C(1+ \mid t \mid)^{N|1/p-1/2|}, \, t \in \mathbb{R} \tag{17}$$

Theorem 2.11 Let H be a schrodinger operator on $L^p(\mathbb{R}^N)$ then H is of $(\alpha, \alpha+1)'$ type $\mathbb{R}$ for $\alpha := N \mid 1/p - 1/2 \mid$.

Remark 2.12 Theorem 2.11 holds whenever we replace $\langle\rangle$ by $||$ in (1) and it is stronger than (1) since $\mid z \mid \le < z >$ for all $z \in \mathbb{C}$. Therefore $(\alpha, \alpha+1)'$ type $\mathbb{R}$ implies $(\alpha, \alpha+1)$ type $\mathbb{R}$.

3 Main Results

Consider the Cauchy equation given by;

$$\begin{cases} u'(t) = -iHu(t), & t \geq 0; \\ u(0) = x, & x \in C_c^\infty(\mathbb{R}). \end{cases} \tag{18}$$

and H satisfy (1), then our first result is given by the following theorem.
Theorem 2.13 Let H be $(\alpha, \alpha+1)$ type $\mathbb{R}$ operator, then $u(t) \in C_c^\infty(\mathbb{R})$is a solution of (18) provided that $u(t)$ satisfies Theorem 2.2

Proof. Let $u(t) \in C_c^\infty(\mathbb{R})$ such that $u(t) = e^{-iHt}$ for $t \in \mathbb{R}$ and H is of $(\alpha, \alpha+1)$ type $\mathbb{R}$, then $u(t)$ satisfy Theorem 2.2. It follows that $u'(t) = -iHu(t)$ satisfy (18) and $u(0)x = x$ for each $x \in \mathbb{R}$ and therefore, $u(t)$ is a solution of (18). Now since H has a resolvent lying on the right half plane, and u(t) is a solution of (18) with $u(0) = x$ and $u(t)$ satisfying Theorem 2.2, we have that

$$\begin{aligned} R(\lambda, -iH)x &= \int_0^\infty e^{\lambda t} u(t) x dt \\ &= \int_0^\infty e^{\lambda t} e^{-iHt} x dt \\ &\leq C(1+ \mid t \mid)^\alpha \end{aligned}$$

for all $t \in \mathbb{R}$ and some $\alpha \geq 0$ and some. It follows that u(t) is the unique solution of (18) and that $R(\lambda, -iH)$ is the inverse Laplace transform of $u(t)$. □

Theorem 2.14 Let H be of $(\alpha, \alpha+1)$ type $\mathbb{R}$ operator, and $u(t) \in C_c^\infty(\mathbb{R})$ be a convergent solution of (18) then $u(t)f(H)$ is also a convergent solution of (18) for every $f \in C_c^\infty(\mathbb{R})$

Proof. Suppose Theorem 2.9 holds and $f \in C_c^\infty(\mathbb{R})$, then $f(H)$ can be extended to a bounded operator $L^p(\mathbb{R})$ for $1 \leq p \leq \infty$. Letting $\alpha = 0$, it is also shown in [7], that if $1 \leq p \leq \infty$ and $\alpha \to \infty$ then $u(t)f(H)$ is bounded in $L^p(\mathbb{R})$ for $t \in \mathbb{R}$ and satisfy inequality in theorem 2.2. In particular, if $\alpha = N \mid 1/p - 1/2 \mid$,then it follows from Theorem 2.10 that

$$\parallel u(t)f(H) \parallel \leq C(1+ \mid t \mid)^{N|1/p-1/2|}, t \in \mathbb{R} \tag{19}$$

Since (19) and inequality of Theorem 2.2 have the same bound, and $u(t)$ is a convergent solution of (18), it follows that $u(t)f(H)$ is also a convergent a solution of (18) and this completes our proof. □

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