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A Method of Proving the Convergence of the Formal Laurent Series Solutions of Nonlinear Evolution Equations

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Abstract

In this paper, we provide an algorithm to convert the third-order nonlinear evolution equations to regular higher-order partial differential equations near movable singularities. Therefore, the Cauchy-Kowalevski theorem is always applicable. As a result, we always have a routine conceptual proof of the convergence of the Laurent series obtained from the Painlevé test.

Mathematics Subject Classification: 30E15, 34A25, 37K10

Keywords: Regularity; Laurent series solution; convergence

1. Introduction

A partial differential equation is said to have the Painlevé property if all solutions are single-valued near any non-characteristic holomorphic movable singularity manifolds [13]. The Painlevé property is intimately connected with integrability [8, 9]. However, the formal series expansion, widely used in the Painlevé analysis, cannot prove this property if it is not convergent. In this paper, we first present an algorithm to convert the third-order evolution equations to higher-order regular partial differential equations and next, as a consequence, prove the convergence of the Laurent series from the Painlevé test. The idea was indeed inspired by an example of Kruskal. For a long time, Kruskal had the following idea: Suppose a solution u of a potentially integrable equation has poles near movable singularities, then u^{-1} should be an analytic function which satisfies some regular equation near movable singularities of u . Kruskal discovered that if u satisfies the second Painlevé equation, then u^{-1} satisfies a *six*th-order regular equation. This property for Painlevé equations is believed to be shared by all integrable equations. Recently, Yee formulated an algorithm of converting any general third-order ordinary differential equations to regular higher-order equations near movable singularities in [12]. The general property of regularity of differential equations is the key, since the existence theorem of analytic solutions applies in the neighborhood of the movable singularities and the Painlevé property follows immediately. In this paper, we further extended the algorithm to the third-order nonlinear evolution equations of the form $u_t = K(z, u, u_z, u_{z^2}, u_{z^3})$. The higher-order regular equations we get are in the form that we can apply the Cauchy-Kowalevski theorem. Consequently, we have a conceptual proof that the formal Laurent series solutions for the evolution equations are always convergent.

Over the years, various more general Painlevé methods have been evolved [1, 2, 3, 4, 5]. Hu and Yan demonstrated in [6, 7] an algorithmic way of regularizing the principal balances for systems of ordinary differential equations passing the Painlevé test. They further showed that their new method (namely, the mirror method) is equivalent to the Painlevé test. Yee [10, 11] also examined a perturbative extension of the mirror method. In this paper, we also demonstrated the mirror algorithm can also be used to prove the convergence of the formal Laurent series obtained from the Painlevé test.

In Sect. 2, the mathematical formulation of the underlying theory will be presented in detail. The indicial normalization $u = \theta^{-k}$ will be used as the transformation for the conversion. In Sect. 3, we shall demonstrate how to deduce the formal series of the derivatives in the powers of a new variable θ. In Sect. 4, we shall present the algorithm for showing the analyticity of coefficient functions of derivatives. In Sect. 5, we shall make use of the θ-series of the derivatives to convert the target equation to a higher-order regular partial differential equation. In Sect. 6, we shall apply an alternative (mirror) approach to prove the convergence result of PDEs. Conclusion can be found in Sect. 7.

2. Mathematical formulation

2.1. **Regular equations for nonlinear evolution equations.** Let $u(z,t)$ satisfy the third-order evolution equation, in the complex domain, of the form

$$u_t \;=\; K\big(z, u, u_z, u_{z^2}, u_{z^3}\big), \tag{1}$$

where K is analytic in z, rational in u, and is polynomial in all derivatives. Note also that we shall use the notation: $u_{z^s t^r} = \partial^{r+s} u/\partial t^r \partial z^s$. The case of ordinary differential equations is then easily incorporated by taking $u_t = 0$.

Moreover, the case of higher-order form of (1) can be generalized accordingly, with the same reasoning presented here.

Let $\phi(z,t)=0$ be the (singular) manifold on which u is singular. One can easily construct a Laurent expansion $u \sim \sum u_n(z,t)\phi^{n-\alpha}$ for a solution of (1) near an arbitrary non-characteristic singular manifold given by $\phi=0$. In the expansion, ϕ can be regarded as a new variable. However in order that the solution can be constructed in the neighborhood of the singular manifold it is necessary to require $\phi_z \neq 0$. This important requirement corresponds to the assumption that the singularity manifold is non-characteristic. By the implicit function theorem, we can express z as a function of t,

$$\phi(z,t) \;=\; z-\psi(t) \;=\; 0, \tag{2}$$

where ψ is an arbitrary (analytic) function of t.

By assuming that (1) passes the Painlevé test, we wish to convert it into a regular partial differential equation near any non-characteristic, holomorphic movable singular manifold. The regularity of the differential equation allows us to apply the existence theorem of analytic solutions in the neighborhood of the movable singularities and this is an approach for directly proving the Painlevé property (PP) for partial differential equations (PDEs) . In fact we cannot conclude that the PDEs have the Painlevé property, without knowing the convergence of the Painlevé series expansions. By showing that they actually converge, we start to prove (1) for u could be formally converted into a regular analytic equation (in a new variable) near any non-characteristic movable manifold $\phi=0$, and then apply the Cauchy-Kowalevskaya theorem to an initial value problem for the solutions of the converted equation. The theorem actually asserts the existence of a unique analytic solution in a neighborhood of any initial point for the Cauchy problem.

In this paper, we provide a simple, direct yet effective proof that equation of the form (1) can be reduced to a regular partial differential equation and demonstrate the result with a few examples such as the Korteweg-de Vries equation, the modified Korteweg-de Vries equation, and the Burgers' equation. We also introduce an application for showing that the formal Laurent series solutions are always convergent, for the original partial differential equations.

2.2. **Indicial normalization as the transformation.** We convert (1) to a regular equation by a simple transform

$$u \;=\; \theta(z,t)^{-k}, \tag{3}$$

where k is to be determined. We call this transformation the indicial normalization, and here k is indeed a positive integer that could be deduced by observing the dominant behaviour in the neighborhood of a movable singularity of order k. We should emphasize that one should determine all possible choices of k in the dominant balance. At this stage, we only remark that for some equations one may find a number of different k's, depending on the nonlinearities, each will lead to a separate Laurent expansion which must be taken in consideration.

It is straightforward to take the partial derivatives of (3)

$$\begin{aligned} u_z &= (-k)\,\theta^{-k-1}\,\theta_z, \\ u_t &= (-k)\,\theta^{-k-1}\,\theta_t, \\ u_{z^2} &= (-k)(-k-1)\,\theta^{-k-2}\,(\theta_z)^2 + (-k)\,\theta^{-k-1}\,\theta_{z^2}, \\ u_{z^3} &= (-k)(-k-1)(-k-2)\,\theta^{-k-3}\,(\theta_z)^3 \\ &\quad + 3(-k)(-k-1)\,\theta^{-k-2}\,\theta_z\theta_{z^2} + (-k)\,\theta^{-k-1}\,\theta_{z^3}, \end{aligned} \tag{4}$$

where the subscripts denote the differentiation with respect to z (and also t). After substituting the partial derivatives in (4) into (1), we first determine the value(s) of k by observing the dominant behaviour near $\phi = 0$. We then multiply $\theta^{-\lambda}$ (λ being the most negative integral power of θ appears in the substituted equation) on both sides of the equation and the transformation eventually converts (1) to the equation

$$\theta^l \theta_{z^3} \;=\; g\big(z, \theta, \theta_z, \theta_t, \theta_{z^2}\big), \tag{5}$$

where l is an integer and g is analytic in all variables.

The transformed equation here is singular when $\theta = 0$. In view of (4) the most negative power of θ appears in the first term of u_{z^3}, so we must have $l \geqslant 2$. In fact, as long as the nonlinearity balances with the highest order derivative in the dominant equation, the index l must be identical to two, that is the number of non-negative resonances, for third-order partial differential equations, in the Laurent expansion solution possessed in the Painlevé test. In the followings we would like to have this assumption so that $l = 2$ is restricted for the case of third-order evolution equations.

3. Deduction of formal series of derivatives in powers of θ

As we assumed that (1) passes the Painlevé test for partial differential equations, it possesses the Laurent series solution, in the neighborhood of (2),

$$u(z,t) \;=\; \big(z - \psi(t)\big)^{-k} \sum_{m=0}^{\infty} u_m(t)\big(z - \psi(t)\big)^m. \tag{6}$$

Here k is a positive integer determined by the dominant balance and the coefficients u_m are (analytic) functions of t only. The series expansion (6) should contain a sufficient number of arbitrary functions. In view of the third-order equation (1) the resonances are $m = -1$, i, j ($j > i \geq 0$), corresponding to the fact that ψ, $u_i = r(t)$, $u_j = s(t)$ are arbitrary analytic functions, respectively. In the neighborhood of the singularity manifold (2), solutions can therefore be explicitly written in the form

$$
\begin{aligned}
u\ =\ & u_0(t,\psi)\big(z-\psi(t)\big)^{-k} \\
& + u_1(t,\psi,\psi_t)\big(z-\psi(t)\big)^{1-k} + u_2(t,\psi,\psi_t,\psi_{t^2})\big(z-\psi(t)\big)^{2-k} + \cdots \\
& + r(t)\,\big(z-\psi(t)\big)^{i-k} \\
& + u_{i+1}(t,\psi,\cdots,\psi_{t^{i+1}};\, r)\big(z-\psi(t)\big)^{i+1-k} \\
& + u_{i+2}(t,\psi,\cdots,\psi_{t^{i+2}};\, r,r_t)\big(z-\psi(t)\big)^{i+2-k} + \cdots \\
& + s(t)\,\big(z-\psi(t)\big)^{j-k} \\
& + u_{j+1}(t,\psi,\cdots,\psi_{t^{j+1}};\, r,\cdots,r_{t^{j-i}};\, s)\big(z-\psi(t)\big)^{j+1-k} \\
& + u_{j+2}(t,\psi,\cdots,\psi_{t^{j+2}};\, r,\cdots,r_{t^{j+1-i}};\, s,s_t)\big(z-\psi(t)\big)^{j+2-k} + \cdots ,
\end{aligned}
\tag{7}
$$

where u_0, u_1, u_2, $\ldots$ are analytic functions of t. We shall assume that the coefficient function u_0 is nonzero everywhere along $z=\psi(t)$. Henceforth, the case $i>0$ is assumed and we will carry out the subsequent argument for $i>0$.

If we expand each coefficient function u_m by

$$
\begin{aligned}
& u_m(t,\psi,\psi_t,\cdots;\, r,r_t,\cdots;\, s,s_t,\cdots) \\
& \quad = \quad u_m(t,z-(z-\psi),\psi_t,\cdots;\, r,r_t,\cdots;\, s,s_t,\cdots)
\end{aligned}
$$

into a series in powers of $(z-\psi)$, with analytic functions as coefficients, then we get a formal series

$$
\begin{aligned}
u\ =\ & \bar{u}_0(z,t)\big(z-\psi(t)\big)^{-k} \\
& + \bar{u}_1(z,t;\,\psi_t)\big(z-\psi(t)\big)^{1-k} + \bar{u}_2(z,t;\,\psi_t,\psi_{t^2})\big(z-\psi(t)\big)^{2-k} + \cdots \\
& + \big[r(t)+\bar{u}_i(z,t;\,\psi_t,\cdots,\psi_{t^i})\big]\big(z-\psi(t)\big)^{i-k} \\
& + \bar{u}_{i+1}(z,t;\,\psi_t,\cdots,\psi_{t^{i+1}};\, r)\big(z-\psi(t)\big)^{i+1-k} + \cdots \\
& + \big[s(t)+\bar{u}_j(z,t;\,\psi_t,\cdots,\psi_{t^j};\, r,\cdots,r_{t^{j-i-1}})\big]\big(z-\psi(t)\big)^{j-k} \\
& + \bar{u}_{j+1}(z,t;\,\psi_t,\cdots,\psi_{t^{j+1}};\, r,\cdots,r_{t^{j-i}};\, s)\big(z-\psi(t)\big)^{j+1-k} + \cdots ,
\end{aligned}
\tag{8}
$$

where $\bar{u}_0$, $\bar{u}_1$, $\bar{u}_2$, $\ldots$ are analytic functions of (z,t), and $\bar{u}_0 \neq 0$ near $z=\psi(t)$. In case $i>0$, we fix one branch of solution $a_1(z,t)=[\bar{u}_0(z,t)]^{-1/k}\neq 0$ near $z=\psi(t)$ and introduce the indicial normalization (3), so that a formal series expansion can be deduced as

$$
\begin{aligned}
\theta \;&= a_1(z,t)\big(z-\psi(t)\big) + a_2(z,t;\,\psi_t)\big(z-\psi(t)\big)^2 \\
&\quad + \Big[-\frac{r(t)}{k}\,a_1(z,t)^{k+1} + a_{i+1}(z,t;\,\psi_t,\cdots,\psi_{t^i})\Big]\big(z-\psi(t)\big)^{i+1} \\
&\quad + a_{i+2}(z,t;\,\psi_t,\cdots,\psi_{t^{i+1}};\,r)\big(z-\psi(t)\big)^{i+2} + \cdots \\
&\quad + \Big[-\frac{s(t)}{k}\,a_1(z,t)^{k+1} \\
&\quad + a_{j+1}(z,t;\,\psi_t,\cdots,\psi_{t^j};\,r,\cdots,r_{t^{j-i-1}})\Big]\big(z-\psi(t)\big)^{j+1} \\
&\quad + a_{j+2}(z,t;\,\psi_t,\cdots,\psi_{t^{j+1}};\,r,\cdots,r_{t^{j-i}};\,s)\big(z-\psi(t)\big)^{j+2} + \cdots .
\end{aligned} \tag{9}
$$

By the method of undetermined coefficients, the series (9) is formally inverted near $z-\psi(t)=0$, so that we can write $z-\psi(t)$ as a formal series in powers of θ. The procedure can be done by substituting

$$
\begin{aligned}
z-\psi(t) \;&=\; A_1\theta + A_2\theta^2 + \cdots + A_{i+1}\theta^{i+1} + \cdots + A_{j+1}\theta^{j+1} + \cdots , \\
\psi_{t^m} \;&=\; -\big(z-\psi(t)\big)_{t^m}
\end{aligned}
$$

into (9) and then solving the resulting recursive relations we obtain

$$
\begin{aligned}
z-\psi(t) \;&= a_1(z,t)^{-1}\,\theta + Z_2\big(z,t;\,\theta_t\big)\,\theta^2 + Z_3\big(z,t;\,\theta_t,\theta_{t^2}\big)\,\theta^3 + \cdots \\
&\quad + \Big[(\frac{r(t)}{k})\,a_1(z,t)^{k-i-1} + Z_{i+1}\big(z,t;\,\theta_t,\cdots,\theta_{t^i}\big)\Big]\theta^{i+1} \\
&\quad + Z_{i+2}\big(z,t;\,\theta_t,\cdots,\theta_{t^{i+1}};\,r\big)\,\theta^{i+2} + \cdots \\
&\quad + \Big[(\frac{s(t)}{k})\,a_1(z,t)^{k-j-1} \\
&\quad + Z_{j+1}\big(z,t;\,\theta_t,\cdots,\theta_{t^j};\,r,\cdots,r_{t^{j-i-1}}\big)\Big]\theta^{j+1} \\
&\quad + Z_{j+2}\big(z,t;\,\theta_t,\cdots,\theta_{t^{j+1}};\,r,\cdots,r_{t^{j-i}};\,s\big)\,\theta^{j+2} + \cdots .
\end{aligned} \tag{10}
$$

Taking the first-derivatives of (9), we have the formal series

$$
\begin{aligned}
\theta_z \;&= \hat{a}^1_0(z,t) + \hat{a}^1_1(z,t;\,\psi_t)\,\big(z-\psi(t)\big) + \cdots \\
&\quad + \Big[\,r(t)\,(-\frac{i+1}{k})\,a_1(z,t)^{k+1} + \hat{a}^1_i(z,t;\,\psi_t,\cdots,\psi_{t^i})\Big]\big(z-\psi(t)\big)^i \\
&\quad + \hat{a}^1_{i+1}(z,t;\,\psi_t,\cdots,\psi_{t^{i+1}};\,r)\big(z-\psi(t)\big)^{i+1} + \cdots \\
&\quad + \Big[\,s(t)\,(-\frac{j+1}{k})\,a_1(z,t)^{k+1} \\
&\quad + \hat{a}^1_j(z,t;\,\psi_t,\cdots,\psi_{t^j};\,r,\cdots,r_{t^{j-i-1}})\Big]\big(z-\psi(t)\big)^j \\
&\quad + \hat{a}^1_{j+1}(z,t;\,\psi_t,\cdots,\psi_{t^{j+1}};\,r,\cdots,r_{t^{j-i}};\,s)\,\big(z-\psi(t)\big)^{j+1} + \cdots ,
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\theta_t \ &= \tilde{a}_0^0(z,t;\,\psi_t) + \tilde{a}_1^0(z,t;\,\psi_t)\,\big(z-\psi(t)\big) + \cdots \\
&+ \Big[\,r(t)\,\psi_t(\frac{i+1}{k})\,a_1(z,t)^{k+1} + \tilde{a}_i^0(z,t;\,\psi_t,\cdots,\psi_{t^i})\Big]\big(z-\psi(t)\big)^i \\
&+ \tilde{a}_{i+1}^0(z,t;\,\psi_t,\cdots,\psi_{t^{i+1}};\,r,r_t)\big(z-\psi(t)\big)^{i+1} + \cdots \\
&+ \Big[\,s(t)\,\psi_t(\frac{j+1}{k})\,a_1(z,t)^{k+1} \\
&+ \tilde{a}_j^0(z,t;\,\psi_t,\cdots,\psi_{t^j};\,r,\cdots,r_{t^{j-i-1}})\Big]\big(z-\psi(t)\big)^j \\
&+ \tilde{a}_{j+1}^0(z,t;\,\psi_t,\cdots,\psi_{t^{j+1}};\,r,\cdots,r_{t^{j-i}};\,s,s_t)\,\big(z-\psi(t)\big)^{j+1} + \cdots\,.
\end{aligned}
\tag{12}
$$

Later on, it is found that we need more higher-order derivatives. Therefore, we generally apply the differential operators $D^{(\alpha)}$, with multi-index α, to the equation (9). In fact, we only need the additional derivatives such as θ_{z^2}, θ_{z^3}, $\ldots$, $\theta_{z^{j+2}}$, $\theta_{z^{j+3}}$ and θ_{zt}, θ_{z^2t}, $\ldots$, $\theta_{z^{j+1}t}$. We note that the above underlined derivative functions such as $\theta_{z^{i+1}}$, $\theta_{z^{i+1}t}$, $\theta_{z^{j+1}}$, $\theta_{z^{j+1}t}$ play an important role in connection with the resonance functions $r(t)$ and $s(t)$. Now we can differentiate (9) with respect to z and t accordingly and write explicitly in the form ($n = 1, 2, \ldots, j+1$),

$$
\begin{aligned}
\theta_{z^n} \ &= \hat{a}_0^n(z,t;\,\psi_t,\cdots,\psi_{t^{n-1}}) + \hat{a}_1^n(z,t;\,\psi_t,\cdots,\psi_{t^n})\,\big(z-\psi(t)\big) + \cdots \\
&+ \Big[(-\frac{r(t)}{k})\frac{(i+1)!}{(i-n+1)!}\,a_1(z,t)^{k+1} \\
&+ \hat{a}_{i-n+1}^n(z,t;\,\psi_t,\cdots,\psi_{t^i})\Big]\big(z-\psi(t)\big)^{i-n+1} \\
&+ \hat{a}_{i-n+2}^n(z,t;\,\psi_t,\cdots,\psi_{t^{i+1}};\,r)\,\big(z-\psi(t)\big)^{i-n+2} + \cdots \\
&+ \Big[(-\frac{s(t)}{k})\frac{(j+1)!}{(j-n+1)!}\,a_1(z,t)^{k+1} \\
&+ \hat{a}_{j-n+1}^n(z,t;\,\psi_t,\cdots,\psi_{t^j};\,r,\cdots,r_{t^{j-i-1}})\Big]\big(z-\psi(t)\big)^{j-n+1} \\
&+ \hat{a}_{j-n+2}^n(z,t;\,\psi_t,\cdots,\psi_{t^{j+1}};\,r,\cdots,r_{t^{j-i}};\,s)\,\big(z-\psi(t)\big)^{j-n+2} + \cdots\,,
\end{aligned}
\tag{13}
$$

$$\begin{aligned}
\theta_{z^n t} \; &= \tilde{a}_0^n(z,t;\, \psi_t, \cdots, \psi_{t^n}) + \tilde{a}_1^n(z,t;\, \psi_t, \cdots, \psi_{t^{n+1}}) \big(z-\psi(t)\big) + \cdots \\
&\quad + \Big[\psi_t(\frac{r(t)}{k})\frac{(i+1)!}{(i-n)!}\, a_1(z,t)^{k+1} \\
&\quad + \tilde{a}_{i-n}^n(z,t;\, \psi_t, \cdots, \psi_{t^i})\Big] \big(z-\psi(t)\big)^{i-n} \\
&\quad + \tilde{a}_{i-n+1}^n(z,t;\, \psi_t, \cdots, \psi_{t^{i+1}};\, r, r_t) \big(z-\psi(t)\big)^{i-n+1} + \cdots \\
&\quad + \Big[\psi_t(\frac{s(t)}{k})\frac{(j+1)!}{(j-n)!}\, a_1(z,t)^{k+1} \\
&\quad + \tilde{a}_{j-n}^n(z,t;\, \psi_t, \cdots, \psi_{t^j};\, r, \cdots, r_{t^{j-i-1}})\Big] \big(z-\psi(t)\big)^{j-n} \\
&\quad + \tilde{a}_{j-n+1}^n(z,t;\, \psi_t, \cdots, \psi_{t^{j+1}};\, r, \cdots, r_{t^{j-i}};\, s, s_t) \big(z-\psi(t)\big)^{j-n+1} \\
&\quad + \cdots,
\end{aligned} \tag{14}$$

$$\begin{aligned}
\theta_{z^{j+2}} \; &= \hat{a}_0^{j+2}(z,t;\, \psi_t, \cdots, \psi_{t^{j+1}};\, r, \cdots, r_{t^{j-i}};\, s) \\
&\quad + \hat{a}_1^{j+2}(z,t;\, \psi_t, \cdots, \psi_{t^{j+2}};\, r, \cdots, r_{t^{j-i+1}};\, s, s_t)\big(z-\psi(t)\big) + \cdots,
\end{aligned} \tag{15}$$

$$\begin{aligned}
\theta_{z^{j+3}} \; &= \hat{a}_0^{j+3}(z,t;\, \psi_t, \cdots, \psi_{t^{j+2}};\, r, \cdots, r_{t^{j-i+1}};\, s, s_t) \\
&\quad + \hat{a}_1^{j+3}(z,t;\, \psi_t, \cdots, \psi_{t^{j+3}};\, r, \cdots, r_{t^{j-i+2}};\, s, s_t, s_{t^2})\big(z-\psi(t)\big) \\
&\quad + \cdots.
\end{aligned} \tag{16}$$

One can see that (13)–(16) are series in powers of $(z-\psi)$. Indeed we can expand them into series in powers of θ. This can be done by substituting (10) into (13)–(16), where $n = 1, 2, \cdots, j+1$,

$$\begin{aligned}
\theta_{z^n} \; &= \hat{\alpha}_0^n(z,t;\, \theta_t, \cdots, \theta_{t^{n-1}}) + \hat{\alpha}_1^n(z,t;\, \theta_t, \cdots, \theta_{t^n})\, \theta + \cdots \\
&\quad + \Big[(-\frac{r(t)}{k})\frac{(i+1)!}{(i-n+1)!}\, a_1(z,t)^{k-i+n} \\
&\quad + \hat{\alpha}_{i-n+1}^n(z,t;\, \theta_t, \cdots, \theta_{t^i})\Big]\theta^{i-n+1} \\
&\quad + \hat{\alpha}_{i-n+2}^n(z,t;\, \theta_t, \cdots, \theta_{t^{i+1}};\, r)\, \theta^{i-n+2} + \cdots \\
&\quad + \Big[(-\frac{s(t)}{k})\frac{(j+1)!}{(j-n+1)!}\, a_1(z,t)^{k-j+n} \\
&\quad + \hat{\alpha}_{j-n+1}^n(z,t;\, \theta_t, \cdots, \theta_{t^j};\, r, \cdots, r_{t^{j-i-1}})\Big]\theta^{j-n+1} \\
&\quad + \hat{\alpha}_{j-n+2}^n(z,t;\, \theta_t, \cdots, \theta_{t^{j+1}};\, r, \cdots, r_{t^{j-i}};\, s)\, \theta^{j-n+2} + \cdots,
\end{aligned} \tag{17}$$

$$\begin{aligned} \theta_{z^n t} &= \tilde{\alpha}_0^n(z,t;\,\theta_t,\cdots,\theta_{t^n}) + \tilde{\alpha}_1^n(z,t;\,\theta_t,\cdots,\theta_{t^{n+1}})\,\theta + \cdots \\ &\quad + \Big[\theta_t\,(\frac{r(t)}{k})\frac{(i+1)!}{(i-n)!}\,a_1(z,t)^{k-i+n+1} \\ &\quad + \tilde{\alpha}_{i-n}^n(z,t;\,\theta_t,\cdots,\theta_{t^i})\Big]\theta^{i-n} \\ &\quad + \tilde{\alpha}_{i-n+1}^n(z,t;\,\theta_t,\cdots,\theta_{t^{i+1}};\,r,r_t)\,\theta^{i-n+1} + \cdots \\ &\quad + \Big[\theta_t\,(\frac{s(t)}{k})\frac{(j+1)!}{(j-n)!}\,a_1(z,t)^{k-j+n+1} \\ &\quad + \tilde{\alpha}_{j-n}^n(z,t;\,\theta_t,\cdots,\theta_{t^j};\,r,\cdots,r_{t^{j-i-1}})\Big]\theta^{j-n} \\ &\quad + \tilde{\alpha}_{j-n+1}^n(z,t;\,\theta_t,\cdots,\theta_{t^{j+1}};\,r,\cdots,r_{t^{j-i}};\,s,s_t)\,\theta^{j-n+1} + \cdots\,, \end{aligned} \tag{18}$$

$$\begin{aligned} \theta_{z^{j+2}} &= \hat{\alpha}_0^{j+2}(z,t;\,\theta_t,\cdots,\theta_{t^{j+1}};\,r,\cdots,r_{t^{j-i}};\,s) \\ &\quad + \hat{\alpha}_1^{j+2}(z,t;\,\theta_t,\cdots,\theta_{t^{j+2}};\,r,\cdots,r_{t^{j-i+1}};\,s,s_t)\,\theta + \cdots\,, \end{aligned} \tag{19}$$

$$\begin{aligned} \theta_{z^{j+3}} &= \hat{\alpha}_0^{j+3}(z,t;\,\theta_t,\cdots,\theta_{t^{j+2}};\,r,\cdots,r_{t^{j-i+1}};\,s,s_t) \\ &\quad + \hat{\alpha}_1^{j+3}(z,t;\,\theta_t,\cdots,\theta_{t^{j+3}};\,r,\cdots,r_{t^{j-i+2}};\,s,s_t,s_{t^2})\,\theta + \cdots\,. \end{aligned} \tag{20}$$

The two derivative series $\theta_{z^{i+1}}$ and $\theta_{z^{j+1}}$ are extracted from (17) and are written explicitly in the following.

$$\begin{aligned} \theta_{z^{i+1}} &= \Big[(-\frac{r(t)}{k})(i+1)!\,a_1(z,t)^{k+1} + \hat{\alpha}_0^{i+1}(z,t;\,\theta_t,\cdots,\theta_{t^i})\Big] \\ &\quad + \hat{\alpha}_1^{i+1}(z,t;\,\theta_t,\cdots,\theta_{t^{i+1}};\,r)\,\theta + \cdots \\ &\quad + \Big[(-\frac{s(t)}{k})\frac{(j+1)!}{(j-i)!}\,a_1(z,t)^{k-j+i+1} \\ &\quad + \hat{\alpha}_{j-i}^{i+1}(z,t;\,\theta_t,\cdots,\theta_{t^j};\,r,\cdots,r_{t^{j-i-1}})\Big]\theta^{j-i} \\ &\quad + \hat{\alpha}_{j-i+1}^{i+1}(z,t;\,\theta_t,\cdots,\theta_{t^{j+1}};\,r,\cdots,r_{t^{j-i}};\,s)\,\theta^{j-i+1} + \cdots\,, \end{aligned} \tag{21}$$

$$\begin{aligned} \theta_{z^{j+1}} &= \Big[(-\frac{s(t)}{k})(j+1)!\,a_1(z,t)^{k+1} \\ &\quad + \hat{\alpha}_0^{j+1}(z,t;\,\theta_t,\cdots,\theta_{t^j};\,r,\cdots,r_{t^{j-i-1}})\Big] \\ &\quad + \hat{\alpha}_1^{j+1}(z,t;\,\theta_t,\cdots,\theta_{t^{j+1}};\,r,\cdots,r_{t^{j-i}};\,s)\,\theta \\ &\quad + \hat{\alpha}_2^{j+1}(z,t;\,\theta_t,\cdots,\theta_{t^{j+2}};\,r,\cdots,r_{t^{j-i+1}};\,s,s_t)\,\theta^2 + \cdots\,. \end{aligned} \tag{22}$$

One can observe that the arbitrary functions $r(t)$ and $s(t)$ first appear in the leading terms of $\theta_{z^{i+1}}$ and $\theta_{z^{j+1}}$ respectively. The derivative series $\theta_{z^{i+1}}$ and $\theta_{z^{j+1}}$ are indeed intimately connected with the resonance functions $r(t)$ and $s(t)$, therefore we often call them as derivative resonances.

Moreover, we may invert the series (21) into

$$
\begin{aligned}
r(t) \;=\; & \Big[- \frac{k}{(i+1)!}\, a_1(z,t)^{-k-1}\, \theta_{z^{i+1}} + \hat\beta_0(z,t;\, \theta_t, \cdots, \theta_{t^i}) \Big] \\
& + \hat\beta_1(z,t;\, \theta_t, \cdots, \theta_{t^{i+1}};\, \theta_{z^{i+1}})\, \theta \\
& + \hat\beta_2(z,t;\, \theta_t, \cdots, \theta_{t^{i+2}};\, \theta_{z^{i+1}}, \theta_{z^{i+1}t})\, \theta^2 + \cdots \\
& + \Big[- s(t)\, \frac{(j+1)!}{(j-i)!(i+1)!}\, a_1(z,t)^{i-j} \\
& + \hat\beta_{j-i}(z,t;\, \theta_t, \cdots, \theta_{t^j};\, \theta_{z^{i+1}}, \theta_{z^{i+1}t}, \cdots, \theta_{z^{i+1}t^{j-i-1}}) \Big] \theta^{j-i} \\
& + \hat\beta_{j-i+1}(z,t;\, \theta_t, \cdots, \theta_{t^{j+1}};\, \theta_{z^{i+1}}, \theta_{z^{i+1}t}, \cdots, \theta_{z^{i+1}t^{j-i}};\, s)\, \theta^{j-i+1} \\
& + \cdots .
\end{aligned}
\tag{23}
$$

The above series contains another resonance function $s(t)$ and we next substitute that into (22) and invert it into

$$
\begin{aligned}
s(t) \;=\; & \Big[- \frac{k}{(j+1)!}\, a_1(z,t)^{-k-1}\, \theta_{z^{j+1}} \\
& + \hat\gamma_0(z,t;\, \theta_t, \cdots, \theta_{t^j};\, \theta_{z^{i+1}}, \theta_{z^{i+1}t}, \cdots, \theta_{z^{i+1}t^{j-i-1}}) \Big] \\
& + \hat\gamma_1(z,t;\, \theta_t, \cdots, \theta_{t^{j+1}};\, \theta_{z^{i+1}}, \theta_{z^{i+1}t}, \cdots, \theta_{z^{i+1}t^{j-i}};\, \theta_{z^{j+1}})\, \theta \\
& + \hat\gamma_2(z,t;\, \theta_t, \cdots, \theta_{t^{j+2}};\, \theta_{z^{i+1}}, \theta_{z^{i+1}t}, \cdots, \\
& \quad \theta_{z^{i+1}t^{j-i+1}};\, \theta_{z^{j+1}}, \theta_{z^{j+1}t})\, \theta^2 + \cdots .
\end{aligned}
\tag{24}
$$

Now we substitute (23)–(24) back into (17)–(20) and again expand them into series in powers of θ. By observing that more derivatives, including $\theta_{z^{i+1}t^2}$, $\theta_{z^{i+1}t^3}$, $\cdots$, $\theta_{z^{j+1}t^2}$, $\theta_{z^{j+1}t^3}$, $\cdots$, are generated in (23)–(24), we expect to get the series in terms of the (resonance) derivatives $\theta_{z^{i+1}t^q}$ and $\theta_{z^{j+1}t^q}$, instead of $r(t)$

and $s(t)$. For $n = 1, 2, \ldots, j \quad (n \neq i+1)$,

$$
\begin{aligned}
\theta_{z^n} &= \hat{\mathcal{A}}_0^n(z, t;\, \theta_t, \cdots, \theta_{t^{n-1}}) + \hat{\mathcal{A}}_1^n(z, t;\, \theta_t, \cdots, \theta_{t^n})\, \theta + \cdots \\
&\quad + \Big[\frac{1}{(i-n+1)!}\, a_1(z,t)^{n-i-1}\, \theta_{z^{i+1}} \\
&\quad + \hat{\mathcal{A}}_{i-n+1}^n(z, t;\, \theta_t, \cdots, \theta_{t^i}) \Big] \theta^{i-n+1} \\
&\quad + \hat{\mathcal{A}}_{i-n+2}^n(z, t;\, \theta_t, \cdots, \theta_{t^{i+1}};\, \theta_{z^{i+1}})\, \theta^{i-n+2} + \cdots \\
&\quad + \Big[\frac{1}{(j-n+1)!}\, a_1(z,t)^{n-j-1}\, \theta_{z^{j+1}} \\
&\quad + \hat{\mathcal{A}}_{j-n+1}^n(z, t;\, \theta_t, \cdots, \theta_{t^j};\, \theta_{z^{i+1}}, \cdots, \theta_{z^{i+1}t^{j-i-1}}) \Big] \theta^{j-n+1} \\
&\quad + \hat{\mathcal{A}}_{j-n+2}^n(z, t;\, \theta_t, \cdots, \theta_{t^{j+1}};\, \theta_{z^{i+1}}, \cdots, \theta_{z^{i+1}t^{j-i}};\, \theta_{z^{j+1}})\, \theta^{j-n+2} \\
&\quad + \cdots,
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
\theta_{z^n t} &= \tilde{\mathcal{A}}_0^n(z, t;\, \theta_t, \cdots, \theta_{t^n}) + \tilde{\mathcal{A}}_1^n(z, t;\, \theta_t, \cdots, \theta_{t^{n+1}})\, \theta + \cdots \\
&\quad + \Big[- \frac{\theta_t}{(i-n)!}\, a_1(z,t)^{n-i}\, \theta_{z^{i+1}} \\
&\quad + \tilde{\mathcal{A}}_{i-n}^n(z, t;\, \theta_t, \cdots, \theta_{t^i}) \Big] \theta^{i-n} \\
&\quad + \tilde{\mathcal{A}}_{i-n+1}^n(z, t;\, \theta_t, \cdots, \theta_{t^{i+1}};\, \theta_{z^{i+1}}, \theta_{z^{i+1}t})\, \theta^{i-n+1} + \cdots \\
&\quad + \Big[- \frac{\theta_t}{(j-n)!}\, a_1(z,t)^{n-j}\, \theta_{z^{j+1}} \\
&\quad + \tilde{\mathcal{A}}_{j-n}^n(z, t;\, \theta_t, \cdots, \theta_{t^j};\, \theta_{z^{i+1}}, \cdots, \theta_{z^{i+1}t^{j-i-1}} \Big] \theta^{j-n} \\
&\quad + \tilde{\mathcal{A}}_{j-n+1}^n(z, t;\, \theta_t, \cdots, \theta_{t^{j+1}};\, \theta_{z^{i+1}}, \cdots, \\
&\qquad \theta_{z^{i+1}t^{j-i}};\, \theta_{z^{j+1}}, \theta_{z^{j+1}t})\, \theta^{j-n+1} + \cdots,
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
\theta_{z^{j+2}} &= \hat{\mathcal{A}}_0^{j+2}(z, t;\, \theta_t, \cdots, \theta_{t^{j+1}};\, \theta_{z^{i+1}}, \cdots, \theta_{z^{i+1}t^{j-i}};\, \theta_{z^{j+1}}) \\
&\quad + \hat{\mathcal{A}}_1^{j+2}(z, t;\, \theta_t, \cdots, \theta_{t^{j+2}};\, \theta_{z^{i+1}}, \cdots, \\
&\qquad \theta_{z^{i+1}t^{j-i+1}};\, \theta_{z^{j+1}}, \theta_{z^{j+1}t})\, \theta + \cdots,
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
\theta_{z^{j+3}} &= \hat{\mathcal{A}}_0^{j+3}(z, t;\, \theta_t, \cdots, \theta_{t^{j+2}};\, \theta_{z^{i+1}}, \cdots, \theta_{z^{i+1}t^{j-i+1}};\, \theta_{z^{j+1}}, \theta_{z^{j+1}t}) \\
&\quad + \hat{\mathcal{A}}_1^{j+3}(z, t;\, \theta_t, \cdots, \theta_{t^{j+3}};\, \theta_{z^{i+1}}, \cdots, \\
&\qquad \theta_{z^{i+1}t^{j-i+2}};\, \theta_{z^{j+1}}, \theta_{z^{j+1}t}, \theta_{z^{j+1}t^2})\, \theta + \cdots.
\end{aligned} \tag{28}
$$

4. Algorithm to show analyticity of coefficient functions of derivatives

In view of (25)–(28), they are expressed as asymptotic series in powers of θ and can be written in the form

$$D^{(\alpha)}\theta \;=\; \mathcal{A}_0 + \mathcal{A}_1\theta + \mathcal{A}_2\theta^2 + \mathcal{A}_3\theta^3 + \cdots . \tag{29}$$

Our main task is to show the analyticity of the coefficient functions $\mathcal{A}_n$ in (29). In case of the third-order partial differential equation (1), this can be done by writing (29) into $D^{(\alpha)}\theta = \mathcal{A}_0 + \mathcal{A}_1\theta + \mathcal{B}_2\theta^2$, where $\mathcal{B}_2 = \mathcal{A}_2 + \mathcal{A}_3\theta + \cdots$, and showing the corresponding functions $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{B}_2$ of derivatives are analytic in all variables.

By substituting (4) into the original equation (1), we obtain

$$\begin{aligned}\theta^{-k-1}\theta_t \;=\; \bar{K}\big(z,\, \theta^{-k},\, \theta^{-k-1}\theta_z,\, \theta^{-k-2}\theta_z^2 \\ + \alpha\theta^{-k-1}\theta_{z^2},\, \theta^{-k-3}\theta_z^3 \\ + \beta\theta^{-k-2}\theta_z\theta_{z^2} + \gamma\theta^{-k-1}\theta_{z^3}\big),\end{aligned} \tag{30}$$

where $\bar{K}$ is an expression in powers of θ, and α, β, γ are only constants. By observing (30) near $\theta = 0$, for a fixed value of k, one can see that the term with the most negative power of θ appears in $\bar{K}$ may depend on z and θ_z only. Therefore, we prefer to rewrite the singular equation (30) into

$$\tilde{g}(z,\theta_z) + \theta\,\tilde{h}(z,\theta,\theta_z,\theta_t,\theta_{z^2}) - \theta^2\,\theta_{z^3} \;=\; 0, \tag{31}$$

where $\tilde{g}$ and $\tilde{h}$ are analytic functions and $\tilde{g}$ is not identically zero. In fact, (31) can be simplified as a condition on θ_z, $\tilde{g}(z,\theta_z) = O_{A,3}(\theta)$. Here we shall use the notation throughout the context: $O_{A,m}(\theta^n)$ means an expression of the form $\theta^n\,\mathcal{R}\big(z,\theta,\cdots,D^{(\alpha')}\theta\big)$, where $D^{(\alpha')}$ is a differential operator with multi-index $|\alpha'| \leqslant m$, and $\mathcal{R}$ is a function analytic in all variables. It determines the leading behaviour of θ_z near $\theta = 0$. Since $\theta_z = O_{A,3}(\theta)$ violates the non-characteristic assumption, the condition yields that

$$\theta_z \;=\; \alpha_0^{\{1,0\}}(z) + O_{A,3}(\theta), \tag{32}$$

where $\alpha_0^{\{1,0\}}$ is a nonzero analytic function of z.

Next we differentiate (31) with respect to z and t and get respectively

$$\tilde{g}_{1,0}\big(z,\theta_z,\theta_t,\theta_{z^2}\big) + \theta\,\tilde{h}_{1,0}\big(z,\theta,\theta_z,\theta_t,\theta_{z^2},\theta_{zt},\theta_{z^3}\big) \;=\; \theta^2\,\theta_{z^4} \tag{33}$$

and

$$\tilde{g}_{0,1}\big(z,\theta_z,\theta_t,\theta_{z^2},\theta_{zt}\big) + \theta\,\tilde{h}_{0,1}\big(z,\theta,\theta_z,\theta_t,\theta_{z^2},\theta_{zt},\theta_{t^2},\theta_{z^3},\theta_{z^2t}\big) \;=\; \theta^2\,\theta_{z^3t}. \tag{34}$$

At this stage, we aim to determine the leading behaviours of θ-derivatives near $\theta = 0$. Therefore, after substituting (32) into (33)–(34), we can obtain

$$\hat{g}_{1,0}\big(z,\theta_t,\theta_{z^2}\big) \;=\; O_{A,4}(\theta), \tag{35}$$

$$\hat{g}_{0,1}\big(z,\theta_t,\theta_{z^2},\theta_{zt}\big) \;=\; O_{A,4}(\theta), \tag{36}$$

respectively. Here (35)–(36) can be viewed as corresponding conditions on θ_{z^2} and θ_{zt}. Indeed, (35) helps to determine the leading behaviour of θ_{z^2}, and substitute this expression into (36) to get the condition on θ_{zt}. The analyticity of $\hat{g}_{1,0}$ and $\hat{g}_{0,1}$ guarantees the solvability of θ_{z^2} and θ_{zt} in terms of analytic functions. Hence we can write

$$\theta_{z^2} = \alpha_0^{\{2,0\}}(z,\theta_t) + O_{A,4}(\theta), \tag{37}$$

$$\theta_{zt} = \alpha_0^{\{1,1\}}(z,\theta_t) + O_{A,4}(\theta), \tag{38}$$

where $\alpha_0^{\{2,0\}}$ and $\alpha_0^{\{1,1\}}$ are some (possibly zero) analytic functions of z and θ_t.

We continuously differentiate (33) with respect to z and t, and (34) with respect to t, we get respectively

$$\tilde{g}_{2,0}\big(z,\theta_z,\theta_t,\theta_{z^2},\theta_{zt},\theta_{z^3}\big) + \theta\,\tilde{h}_{2,0} = \theta^2\,\theta_{z^5}, \tag{39}$$

$$\tilde{g}_{1,1}\big(z,\theta_z,\theta_t,\theta_{z^2},\theta_{zt},\theta_{t^2},\theta_{z^3},\theta_{z^2t}\big) + \theta\,\tilde{h}_{1,1} = \theta^2\,\theta_{z^4t}, \tag{40}$$

$$\tilde{g}_{0,2}\big(z,\theta_z,\theta_t,\theta_{z^2},\theta_{zt},\theta_{t^2},\theta_{z^3},\theta_{z^2t},\theta_{zt^2}\big) + \theta\,\tilde{h}_{0,2} = \theta^2\,\theta_{z^3t^2}. \tag{41}$$

By substituting the leading-order behaviours of (32) and (37)–(38) into (39)–(41), we can obtain the respective conditions on θ_{z^3}, θ_{z^2t} and θ_{zt^2}. They are written as

$$\hat{g}_{2,0}\big(z,\theta_t,\theta_{z^3}\big) = O_{A,5}(\theta), \tag{42}$$

$$\hat{g}_{1,1}\big(z,\theta_t,\theta_{t^2},\theta_{z^3},\theta_{z^2t}\big) = O_{A,5}(\theta), \tag{43}$$

$$\hat{g}_{0,2}\big(z,\theta_t,\theta_{t^2},\theta_{z^3},\theta_{z^2t},\theta_{zt^2}\big) = O_{A,5}(\theta). \tag{44}$$

By solving (42)–(44) recursively, it follows immediately that

$$\theta_{z^3} = \alpha_0^{\{3,0\}}\big(z,\theta_t\big) + O_{A,5}(\theta), \tag{45}$$

$$\theta_{z^2t} = \alpha_0^{\{2,1\}}\big(z,\theta_t,\theta_{t^2}\big) + O_{A,5}(\theta), \tag{46}$$

$$\theta_{zt^2} = \alpha_0^{\{1,2\}}\big(z,\theta_t,\theta_{t^2}\big) + O_{A,5}(\theta), \tag{47}$$

where $\alpha_0^{\{3,0\}}$, $\alpha_0^{\{2,1\}}$ and $\alpha_0^{\{1,2\}}$ are analytic functions.

Following this idea, we generally apply the differential operators $D^{(\alpha)}$ to (31) where $|\alpha| \leqslant j$. Here j is the index at which the coefficient function u_j in the Laurent expansion (6) is to be indeterminate, the value of j being called the largest resonance. We then obtain respectively

$$\tilde{g}(z,\theta_z) + \theta\,\tilde{h} - \theta^2\,\theta_{z^3} = 0, \tag{48}$$

$$\tilde{g}_{1,0}(z,\theta_z,\theta_t,\theta_{z^2}) + \theta\,\tilde{h}_{1,0} - \theta^2\,\theta_{z^4} = 0, \tag{49}$$

$$\tilde{g}_{0,1}(z,\theta_z,\theta_t,\theta_{z^2},\theta_{zt}) + \theta\,\tilde{h}_{0,1} - \theta^2\,\theta_{z^3t} = 0, \tag{50}$$

$$\tilde{g}_{2,0}(z,\theta_z,\theta_t,\theta_{z^2},\theta_{zt},\theta_{z^3}) + \theta\,\tilde{h}_{2,0} - \theta^2\,\theta_{z^5} = 0, \tag{51}$$

$$\tilde{g}_{1,1}(z,\theta_z,\theta_t,\theta_{z^2},\theta_{zt},\theta_{t^2},\theta_{z^3},\theta_{z^2t}) + \theta\,\tilde{h}_{1,1} - \theta^2\,\theta_{z^4t} = 0, \tag{52}$$

$$\tilde{g}_{0,2}(z,\theta_z,\theta_t,\theta_{z^2},\theta_{zt},\theta_{t^2},\theta_{z^3},\theta_{z^2t},\theta_{zt^2}) + \theta\,\tilde{h}_{0,2} - \theta^2\,\theta_{z^3t^2} = 0, \tag{53}$$

and so on,

$$\begin{aligned} \tilde{g}_{j,0}(z,\theta_z,\theta_t,\theta_{z^2},\theta_{zt},\cdots,\theta_{z^q},\theta_{z^{q-1}t},\cdots,\theta_{z^j},\theta_{z^{j-1}t},\theta_{z^{j+1}}) \\ +\,\theta\,\tilde{h}_{j,0}-\theta^2\,\theta_{z^{j+3}} \;=\; 0, \end{aligned} \tag{54}$$

$$\begin{aligned} \tilde{g}_{j-1,1}\big(z,\theta_z,\theta_t,\theta_{z^2},\theta_{zt},\theta_{t^2},\cdots,\theta_{z^q},\theta_{z^{q-1}t},\theta_{z^{q-2}t^2},\cdots, \\ \theta_{z^j},\theta_{z^{j-1}t},\theta_{z^{j-2}t^2},\theta_{z^{j+1}},\theta_{z^jt}\big)+\theta\,\tilde{h}_{j-1,1}-\theta^2\,\theta_{z^{j+2}t} \;=\; 0, \end{aligned} \tag{55}$$

and so on,

$$\begin{aligned} \tilde{g}_{0,j}\big(z,\theta_z,\theta_t,\theta_{z^2},\theta_{zt},\theta_{t^2},\theta_{z^3},\theta_{z^2t},\theta_{zt^2},\theta_{t^3},\cdots, \\ \theta_{z^3t^{q-3}},\theta_{z^2t^{q-2}},\theta_{zt^{q-1}},\theta_{t^q},\cdots, \\ \theta_{z^3t^{j-3}},\theta_{z^2t^{j-2}},\theta_{zt^{j-1}},\theta_{t^j},\theta_{z^3t^{j-2}},\theta_{z^2t^{j-1}},\theta_{zt^j}\big)+\theta\,\tilde{h}_{0,j}-\theta^2\,\theta_{z^3t^j} \;=\; 0, \end{aligned} \tag{56}$$

where $\tilde{g}_{m,n}$ and $\tilde{h}_{m,n}$ are some functions analytic in all variables, where the indexes (m,n) have all combinations such that $1 \leqslant m+n \leqslant j$.

Generally, (48)–(56) give algebraic conditions on the derivatives θ_z, θ_{z^2}, θ_{zt}, θ_{z^3}, θ_{z^2t}, θ_{zt^2}, θ_{z^4}, θ_{z^3t}, $\theta_{z^2t^2}$, $\theta_{zt^3}, \cdots, \theta_{z^{j+1}}$, $\theta_{z^jt}, \cdots, \theta_{zt^j}$. By solving (48)–(56) recursively, we can deduce the leading-order behaviours of the derivatives

$$\theta_{z^pt^q} \;=\; \alpha_0^{\{p,q\}}\big(z,\theta_t,\theta_{t^2},\ldots,\theta_{t^{p+q-1}}\big)+O_{A,p+q+2}(\theta), \tag{57}$$

where $1 \leqslant p \leqslant j+1$, $0 \leqslant q \leqslant j$, $1 \leqslant p+q \leqslant j+1$, and $\alpha_0^{\{p,q\}}$ are some analytic functions. But we should remark that the formulae for some derivatives cannot be found because of the existence of resonances. In the case of third-order equations, we simply skip the formulae for $\theta_{z^{i+1}}$, $\theta_{z^{j+1}}$ and $\theta_{z^{i+1}t^q}$ (the values of i and j being the two resonances, $j > i$), which correspond to the resonance terms appear and consequently, they cannot be determined.

Now we need to work out the second round calculations in order to refine the formulae (57) up to $O_{A,j+3}(\theta^2)$. We start to put the past formulae of derivatives θ_z and θ_{z^2} into (48) and solve the equation of the next order, we can determine

$$\theta_z \;=\; \alpha_0^{\{1,0\}}(z)+\alpha_1^{\{1,0\}}(z,\theta_t)\,\theta+O_{A,4}(\theta^2), \tag{58}$$

where $\alpha_1^{\{1,0\}}$ is analytic. In order to find the refined formula of θ_{z^2}, we put the past formulae of θ_{z^2} and θ_{z^3} with the refined formula (58) of θ_z into (49). Then after solving the equation we can determine the next order formula

$$\theta_{z^2} \;=\; \alpha_0^{\{2,0\}}(z,\theta_t)+\alpha_1^{\{2,0\}}(z,\theta_t,\theta_{t^2})\,\theta+O_{A,5}(\theta^2), \tag{59}$$

where $\alpha_1^{\{2,0\}}$ is analytic.

Following the same procedure with the equations (48)–(56), we can eventually obtain a refinement of all expansion formulae of the form

$$\begin{aligned} \theta_{z^pt^q} \;&=\alpha_0^{\{p,q\}}\big(z,\theta_t,\theta_{t^2},\cdots,\theta_{t^{p+q-1}}\big) \\ &\quad+\alpha_1^{\{p,q\}}\big(z,\theta_t,\theta_{t^2},\cdots,\theta_{t^{p+q}}\big)\,\theta+O_{A,k_{p,q}}(\theta^2), \end{aligned} \tag{60}$$

where $1 \leqslant p \leqslant j+3$, $0 \leqslant q \leqslant j$, $1 \leqslant p+q \leqslant j+1$, $3 \leqslant k_{p,q} \leqslant j+3$, and the coefficient functions $\alpha_m^{\{p,q\}}$ are analytic in all variables.

By observing (60) with (25)–(28), we showed the analyticity of all coefficient functions of derivatives, for $n = 1, 2, \ldots, j$ $(n \neq i+1)$,

$$\begin{aligned} \theta_{z^n} = {} & \hat{\mathcal{A}}_0^n\big(z,t;\,\theta_t,\cdots,\theta_{t^{n-1}}\big) + \hat{\mathcal{A}}_1^n\big(z,t;\,\theta_t,\cdots,\theta_{t^n}\big)\,\theta + \cdots \\ & + \Big[\frac{1}{(i-n+1)!}\,a_1(z,t)^{n-i-1}\,\theta_{z^{i+1}} \\ & + \hat{\mathcal{A}}_{i-n+1}^n\big(z,t;\,\theta_t,\cdots,\theta_{t^i}\big)\Big]\theta^{i-n+1} \\ & + \hat{\mathcal{A}}_{i-n+2}^n\big(z,t;\,\theta_t,\cdots,\theta_{t^{i+1}};\,\theta_{z^{i+1}}\big)\,\theta^{i-n+2} + \cdots \\ & + \Big[\frac{1}{(j-n+1)!}\,a_1(z,t)^{n-j-1}\,\theta_{z^{j+1}} \\ & + \hat{\mathcal{A}}_{j-n+1}^n\big(z,t;\,\theta_t,\cdots,\theta_{t^j};\,\theta_{z^{i+1}},\cdots,\theta_{z^{i+1}t^{j-i-1}}\big)\Big]\theta^{j-n+1} \\ & + \hat{\mathcal{A}}_{j-n+2}^n\big(z,t;\,\theta_t,\cdots,\theta_{t^{j+1}};\,\theta_{z^{i+1}},\cdots,\theta_{z^{i+1}t^{j-i}};\,\theta_{z^{j+1}}\big)\,\theta^{j-n+2} \\ & + \cdots + O_{A,k_{n0}}(\theta^2), \end{aligned} \tag{61}$$

$$\begin{aligned} \theta_{z^n t} = {} & \tilde{\mathcal{A}}_0^n\big(z,t;\,\theta_t,\cdots,\theta_{t^n}\big) + \tilde{\mathcal{A}}_1^n\big(z,t;\,\theta_t,\cdots,\theta_{t^{n+1}}\big)\,\theta + \cdots \\ & + \Big[-\frac{\theta_t}{(i-n)!}\,a_1(z,t)^{n-i}\,\theta_{z^{i+1}} + \tilde{\mathcal{A}}_{i-n}^n\big(z,t;\,\theta_t,\cdots,\theta_{t^i}\big)\Big]\theta^{i-n} \\ & + \tilde{\mathcal{A}}_{i-n+1}^n\big(z,t;\,\theta_t,\cdots,\theta_{t^{i+1}};\,\theta_{z^{i+1}},\theta_{z^{i+1}t}\big)\,\theta^{i-n+1} + \cdots \\ & + \Big[-\frac{\theta_t}{(j-n)!}\,a_1(z,t)^{n-j}\,\theta_{z^{j+1}} \\ & + \tilde{\mathcal{A}}_{j-n}^n\big(z,t;\,\theta_t,\cdots,\theta_{t^j};\,\theta_{z^{i+1}},\cdots,\theta_{z^{i+1}t^{j-i-1}}\big)\Big]\theta^{j-n} \\ & + \tilde{\mathcal{A}}_{j-n+1}^n\big(z,t;\,\theta_t,\cdots,\theta_{t^{j+1}};\,\theta_{z^{i+1}},\cdots, \\ & \quad \theta_{z^{i+1}t^{j-i}};\,\theta_{z^{j+1}},\theta_{z^{j+1}t}\big)\,\theta^{j-n+1} + \cdots + O_{A,k_{n1}}(\theta^2), \end{aligned} \tag{62}$$

$$\begin{aligned} \theta_{z^{j+2}} = {} & \hat{\mathcal{A}}_0^{j+2}\big(z,t;\,\theta_t,\cdots,\theta_{t^{j+1}};\,\theta_{z^{i+1}},\cdots,\theta_{z^{i+1}t^{j-i}};\,\theta_{z^{j+1}}\big) \\ & + \hat{\mathcal{A}}_1^{j+2}\big(z,t;\,\theta_t,\cdots,\theta_{t^{j+2}};\,\theta_{z^{i+1}},\cdots, \\ & \quad \theta_{z^{i+1}t^{j-i+1}};\,\theta_{z^{j+1}},\theta_{z^{j+1}t}\big)\,\theta + O_{A,k_{j+2}}(\theta^2), \end{aligned} \tag{63}$$

$$\begin{aligned} \theta_{z^{j+3}} = {} & \hat{\mathcal{A}}_0^{j+3}\big(z,t;\,\theta_t,\cdots,\theta_{t^{j+2}};\,\theta_{z^{i+1}},\cdots, \\ & \quad \theta_{z^{i+1}t^{j-i+1}};\,\theta_{z^{j+1}},\theta_{z^{j+1}t}\big) \\ & + \hat{\mathcal{A}}_1^{j+3}\big(z,t;\,\theta_t,\cdots,\theta_{t^{j+3}};\,\theta_{z^{i+1}},\cdots, \\ & \quad \theta_{z^{i+1}t^{j-i+2}};\,\theta_{z^{j+1}},\theta_{z^{j+1}t},\theta_{z^{j+1}t^2}\big)\,\theta + O_{A,k_{j+3}}(\theta^2), \end{aligned} \tag{64}$$

where $O_{A,k_m}(\theta^n)$ means an expression of the form $\theta^n\,\mathcal{R}(z,\theta,\cdots,\mathrm{D}^{(\beta)}\theta)$, with $|\beta| \leqslant k_m$, $3 \leqslant k_m \leqslant j+3$, and $\mathcal{R}$ is analytic in all variables.

We remark that $\theta_{z^{i+1}t^q}$, $\theta_{z^{j+1}t^q}$ are resonance derivatives and cannot be determined.

5. Conversion of higher-order regular partial differential equation

In order to find the regular partial differential equation, we first differentiate (5) with respect to z and get

$$\begin{aligned} 2\theta\theta_z\,\theta_{z^3} + \theta^2\,\theta_{z^4} &= g_1(z, \theta, \theta_z, \theta_t,\, \theta_{z^2}, \theta_{zt},\, \theta_{z^3}), \\ \theta^2\,\theta_{z^4} &= \bar{g}_1(z, \theta, \theta_z, \theta_t,\, \theta_{z^2}, \theta_{zt},\, \theta_{z^3}), \end{aligned}$$

where g_1 (and hence $\bar{g}_1$) is analytic in all variables. Using the same idea, we differentiate (5) with respect to z by $j+1$ times and eventually get

$$\theta^2\,\theta_{z^{j+4}} \;=\; G\big(z, \theta, \theta_z, \theta_t,\, \theta_{z^2}, \theta_{zt}, \cdots, \theta_{z^{j+2}}, \theta_{z^{j+1}t},\, \theta_{z^{j+3}}\big), \tag{65}$$

where G is analytic in all variables. The analytic function G in (65) contains derivative functions (except the resonances $\theta_{z^{i+1}}$, $\theta_{z^{i+1}t}$, $\theta_{z^{j+1}}$, $\theta_{z^{j+1}t}$) which can be replaced by (61)–(64), we then have

$$\begin{aligned} \theta^2\,\theta_{z^{j+4}} = \hat{G}\big(&z, \theta;\, \theta_t, \theta_{t^2}, \cdots, \theta_{t^{j+3}};\, \theta_{z^{i+1}}, \theta_{z^{i+1}t}, \cdots, \\ &\theta_{z^{i+1}t^{j-i+2}};\, \theta_{z^{j+1}}, \theta_{z^{j+1}t}, \theta_{z^{j+1}t^2}\big) + O_{A,j+3}(\theta^2), \end{aligned}$$

where $\theta_{z^{i+1}t^q}$, $\theta_{z^{j+1}t^q}$ are viewed as arbitrary functions, and $\hat{G}$ is analytic in all variables. By dividing θ^2 on both sides, we rewrite the above equation into

$$\begin{aligned} \theta_{z^{j+4}} = \bar{G}\big(&z, \theta;\, \theta_t, \theta_{t^2}, \cdots, \theta_{t^{j+3}};\, \theta_{z^{i+1}}, \theta_{z^{i+1}t}, \cdots, \\ &\theta_{z^{i+1}t^{j-i+2}};\, \theta_{z^{j+1}}, \theta_{z^{j+1}t}, \theta_{z^{j+1}t^2}\big) + O_{A,j+3}(1), \end{aligned} \tag{66}$$

where $\bar{G}$ is analytic in z and meromorphic in θ. Since $\bar{G}$ is meromorphic in θ, we have the following Laurent series near $\theta = 0$

$$\begin{aligned} \bar{G} = {} & H\big(z, \theta;\, \theta_t, \theta_{t^2}, \cdots, \theta_{t^{j+3}};\, \theta_{z^{i+1}}, \theta_{z^{i+1}t}, \cdots, \\ & \quad \theta_{z^{i+1}t^{j-i+2}};\, \theta_{z^{j+1}}, \theta_{z^{j+1}t}, \theta_{z^{j+1}t^2}\big) \\ & + \theta^{-1}\,\bar{G}_0\big(z;\, \theta_t, \theta_{t^2}, \cdots, \theta_{t^{j+3}};\, \theta_{z^{i+1}}, \theta_{z^{i+1}t}, \cdots, \\ & \quad \theta_{z^{i+1}t^{j-i+2}};\, \theta_{z^{j+1}}, \theta_{z^{j+1}t}, \theta_{z^{j+1}t^2}\big) \\ & + \theta^{-2}\,\bar{G}_1\big(z;\, \theta_t, \theta_{t^2}, \cdots, \theta_{t^{j+3}};\, \theta_{z^{i+1}}, \theta_{z^{i+1}t}, \cdots, \\ & \quad \theta_{z^{i+1}t^{j-i+2}};\, \theta_{z^{j+1}}, \theta_{z^{j+1}t}, \theta_{z^{j+1}t^2}\big), \end{aligned} \tag{67}$$

where $\bar{G}_0$, $\bar{G}_1$ and H are all analytic. As formal solution, the series (9) must satisfy (66) with $\bar{G}$ given by (67). Therefore, we substitute the series into the equation and expand both sides into Laurent series of powers of $z - \psi(t)$. One can see that the left side is formally analytic. In the limit $z \to \psi$, we have

$$\begin{aligned} 0 = \bar{G}_0\big(&\psi;\, \theta_t, \theta_{t^2}, \cdots, \theta_{t^{j+3}};\, \theta_{z^{i+1}}, \theta_{z^{i+1}t}, \cdots, \\ &\theta_{z^{i+1}t^{j-i+2}};\, \theta_{z^{j+1}}, \theta_{z^{j+1}t}, \theta_{z^{j+1}t^2}\big)\, a_1(\psi)^{-1}, \end{aligned} \tag{68}$$

$$
\begin{aligned}
0 \;=\; & \bar{G}_1\big(\psi;\, \theta_t, \theta_{t^2}, \cdots, \theta_{t^{j+3}};\, \theta_{z^{i+1}}, \theta_{z^{i+1}t}, \cdots, \\
& \theta_{z^{i+1}t^{j-i+2}};\, \theta_{z^{j+1}}, \theta_{z^{j+1}t}, \theta_{z^{j+1}t^2}\big)\, a_1(\psi)^{-2},
\end{aligned} \tag{69}
$$

respectively. The derivative functions appear in (68)–(69) are functions of r and s. Thus when $a_1(\psi) \neq 0$, we must have

$$
\begin{aligned}
& \bar{G}_{0,1}\big(\psi;\, \theta_t, \theta_{t^2}, \cdots, \theta_{t^{j+3}};\, \theta_{z^{i+1}}, \theta_{z^{i+1}t}, \cdots, \\
& \theta_{z^{i+1}t^{j-i+2}};\, \theta_{z^{j+1}}, \theta_{z^{j+1}t}, \theta_{z^{j+1}t^2}\big) \;=\; 0
\end{aligned}
$$

for all ψ, r and s. Since ψ, r and s, and so all resonance derivatives, are all arbitrary, we find that $\bar{G}_0 \equiv \bar{G}_1 \equiv 0$ and therefore

$$
\bar{G} \;=\; H \quad \text{is analytic.} \tag{70}
$$

The equation (66) is now reduced to

$$
\theta_{z^{j+4}} \;=\; H + O_{A,j+3}(1).
$$

Now we can conclude that θ satisfies a $(j+4)$-order regular partial differential equation where j is the largest resonance in the Painlevé test. For a n-order partial differential equation, we can expect to get a regular differential equation of order $(j+n+1)$.

We offer several examples below to list the results of the converted regular differential equations near non-characteristic, movable singularity manifolds.

The Korteweg-de Vries equation

For the KdV equation

$$
u_t + 6uu_x + u_{xxx} \;=\; 0, \tag{71}
$$

we substitute the dominant behaviour $u \sim u_0\phi^k$ near a non-characteristic, movable singular manifold $\phi = 0$ and solve the dominant equation to have $k = -2$. As a result, we proved that the function θ, defined by

$$
u(x,t) \;=\; \frac{1}{\theta(x,t)^2}, \tag{72}
$$

satisfies a 10th-order regular partial differential equation near $\theta = 0$.

The modified Korteweg-de Vries equation

For the mKdV equation

$$
u_t - 6u^2u_x + u_{xxx} \;=\; 0, \tag{73}
$$

we substitute the dominant behaviour $u \sim u_0\phi^k$ near a non-characteristic, movable singular manifold $\phi = 0$ and solve the dominant equation to have $k = -1$. As a result, we proved that the function θ, defined by

$$
u(x,t) \;=\; \frac{1}{\theta(x,t)}, \tag{74}
$$

satisfies a 8th-order regular partial differential equation near $\theta = 0$.

The Burgers' equation
For the Burgers' equation

$$u_t + uu_x + u_{xx} \;=\; 0, \tag{75}$$

we substitute the dominant behaviour $u \sim u_0\phi^k$ near a non-characteristic, movable singular manifold $\phi = 0$ and solve the dominant equation to have $k = -1$. As a result, we proved that the function θ, defined by

$$u(x,t) \;=\; \frac{1}{\theta(x,t)}, \tag{76}$$

satisfies a 5th-order regular partial differential equation near $\theta = 0$.

6. Convergence result for PDEs

In general, if n is the order of the equation, and $\theta_{z^{j+1}}$ is the highest-order resonance derivative, then we would expect a regular $(n+j+1)$-order regular equation for θ. Note that the higher-order equation we get for the KdV equations and the Burgers' equation are in the form that we can directly apply the Cauchy-Kowalevski theorem. Consequently, we have provided a direct method to prove that the formal Laurent series solutions for these equations are always convergent. However, one obvious drawback of this method is its complexity. As a matter of fact, we can hardly write explicitly the whole regular equation. To overcome this technical issue, in the following, we also demonstrated an alternative algorithm that the target equation can be converted to a regular mirror system. The Cauchy-Kowalevski theorem is always applicable in this case. As a result, we always have a routine conceptual proof of the convergence of the Laurent series obtained from the Painlevé test.

We shall use the Burgers' equation as an example for illustration. The system (75) has the associated space-evolution equations

$$u_x \;=\; v, \qquad v_x \;=\; -u_t - uv \tag{77}$$

which passes the Painlevé test and has the principal balance given by

$$u \;=\; \frac{2}{x-\psi(t)} + \psi'(t) + r_2(t)\big(x-\psi(t)\big) - \frac{\psi''(t)}{4}\big(x-\psi(t)\big)^2 + \cdots\,, \tag{78}$$

where ψ and r_2 are arbitrary functions of t. Next, we shall construct the mirror system for (77). By introducing the indicial normalization

$$u \;=\; \theta^{-1}(x,t), \qquad v \;=\; v_2(x,t)\,\theta^{-2}(x,t),$$

one can find the Laurent θ-series

$$\begin{aligned}
\theta_x \;&=\; \frac{1}{2} + 2\theta_t\,\theta - \bar{r}_2\,\theta^2 + \big(4\bar{r}_2\theta_t - 8\theta_{tt}\big)\,\theta^3 + \cdots\,,\\
v \;&=\; -\frac{1}{2\theta^2} - \frac{2\theta_t}{\theta} + \bar{r}_2 + \big(-4\bar{r}_2\,\theta_t + 8\theta_{tt}\big)\,\theta + \cdots\,,
\end{aligned}$$

where $\bar{r}_2$ is an arbitrary (resonance) function. We then truncate the θ-series of v at θ^0 (where the resonance parameter $\bar{r}_2$ first appears) by introducing a new variable η to have the transformation $(u, v) \longleftrightarrow (\theta, \eta)$:

$$u = \frac{1}{\theta}, \qquad v = -\frac{1}{2\theta^2} - \frac{2\theta_t}{\theta} + \eta. \tag{79}$$

By substituting (79) into (77), we obtain a regular (mirror) system of the form

$$\theta_x = \frac{1}{2} + 2\theta_t\,\theta - \eta\,\theta^2, \qquad \eta_x = -2\eta\,\theta_t + 4\theta_{tt} - 2\eta_t\,\theta,$$

in which the time-derivatives of θ and η appear in right hand side of the mirror system.

In fact, the mirror system for Burgers' equation is not suitable for applying the Cauchy-Kowalevski theorem, because of the second-order derivative θ_{tt} on the right side. By introducing a new variable $\alpha = \theta_t$, we may extend the mirror system to

$$\begin{cases} \theta_x = \dfrac{1}{2} + 2\alpha\,\theta - \eta\,\theta^2, \\ \eta_x = -2\alpha\,\eta + 4\alpha_t - 2\eta_t\,\theta, \\ \alpha_x = 2\alpha^2 + 2\big(\alpha_t - \alpha\eta\big)\,\theta - \eta_t\,\theta^2, \end{cases} \tag{80}$$

in which we use $\alpha_x = (\theta_x)_t$ to find the third equation. The extended mirror system is now suitable for applying the Cauchy-Kowalevski theorem.

The next thing we should do is to convert the series (78) into an equivalent initial value condition for (80) along the singularity manifold $x = \psi(t)$. From (78) and $\theta = u^{-1}$, we obtain

$$\theta = \frac{1}{2}\big(x - \psi(t)\big) - \frac{\psi'}{4}\big(x - \psi(t)\big)^2 + \Big(\frac{\psi'^2}{8} - \frac{r_2}{4}\Big)(x - \psi(t))^3 + \cdots. \tag{81}$$

Taking derivative of this with respect to t, we have

$$\alpha = -\frac{\psi'}{2} + \frac{\psi'^2}{2}\big(x - \psi(t)\big) + \Big(\frac{3r_2\,\psi'}{4} - \frac{3\psi'^3}{8} - \frac{\psi''}{4}\Big)\big(x - \psi(t)\big)^2 + \cdots.$$

Substituting the series for θ and α into the first equation in (80) and solving for η, we have

$$\eta = 3r_2 + \frac{3}{2}\,\psi'^2 + \Big(3r_2\,\psi' - \frac{\psi'^3}{2} - 2\psi''\Big)\big(x - \psi(t)\big) + \cdots.$$

From these power series, we find the following initial data for the mirror system along the singularity

$$\theta = 0, \qquad \eta = 3r_2 + \frac{3}{2}\,\psi'^2, \qquad \alpha = -\frac{\psi'}{2}, \qquad \text{at} \quad x = \psi(t). \tag{82}$$

Now, the convergence of (78) follows immediately. By the Cauchy-Kowalevski theorem, the extended mirror system (80) with the initial data (82) has a unique analytic solution $\big(\theta(x,t), \eta(x,t), \alpha(x,t)\big)$ near $x = \psi(t)$. Then $u = \theta^{-1}$ is a solution of Burgers' equation near $x = \psi(t)$. Moreover, from the usual power series method, we find the expansion for θ is indeed (81). Then an easy calculation reveals that the Laurent series of $u = \theta^{-1}$ is exactly (78).

7. Conclusion

An algorithm of converting any third-order evolution equations to regular higher-order partial differential equations was presented. As an immediate consequence, the regularity of the higher-order equations enables us to use the Cauchy-Kowalevski theorem to prove the convergence of the Laurent series obtained from the Painlevé test. An alternative algorithm using the mirror method was also presented. We used the Burgers' equation as an example to illustrate the steps: Find the mirror system and if necessary, extend the system so that the Cauchy-Kowalevski theorem is applicable; Use the given resonance functions to find an initial condition for the (extended) mirror system; Use the standard power series method to find the series solution of the initial value problem (with all resonance functions appear); Verify that the inverse indicial normalization converts the series obtained from the mirror system into the Laurent series expected from the Painlevé test. For ODEs, the right side of the mirror system involves no derivatives. Therefore, the Cauchy-Kowalevski theorem is always applicable. As a result, we always have a routine conceptual proof of the convergence of the series from the Painlevé test.

References

[1] M. J. Ablowitz, A. Ramani and H. Segur, A connection between nonlinear evolution equations and ordinary differential equations of P-type. I; II, *J. Math. Phys.*, **21** (1980), 715–721; 1006–1015. https://doi.org/10.1063/1.524491, https://doi.org/10.1063/1.524548

[2] M. Adler and P. van Moerbeke, The complex geometry of the Kowalevski-Painlevé analysis, *Invent. Math.*, **97** (1989), 3–51. https://doi.org/10.1007/bf01850654

[3] F. J. Bureau, Differential Equations with fixed critical points, *Annali di Mat. pura ed applicata* **LXVI**, (1964), 1–116. https://doi.org/10.1007/bf02412437

[4] R. Conte, A. P. Fordy and A. Pickering, A perturbative Painlevé approach to nonlinear differential equations, *Physica D*, **69** (1993), 33–58. https://doi.org/10.1016/0167-2789(93)90179-5

[5] R. Conte, The Painlevé approach to nonlinear ordinary differential equations, *The Painlevé property, one century later*, 77–180, CRM series in mathematical physics, Springer, Berlin (1999). https://doi.org/10.1007/978-1-4612-1532-5_3

[6] J. Hu and M. Yan, Singularity analysis for integrable systems by their mirrors, *Nonlinearity*, **12** (1999), 1531–1543. https://doi.org/10.1088/0951-7715/12/6/306

[7] J. Hu and M. Yan, The mirror systems of integrable equations, *Stud. Appl. Math.*, **104** (2000), 67–90. https://doi.org/10.1111/1467-9590.00131

[8] S. V. Kovalevskaya, Sur le problème de la rotation d'un corps solide autour d'un point fixe, *Acta Math.*, **12** (1889), 177–232. https://doi.org/10.1007/bf02592182

[9] P. Painlevé, *Leçons sur la théorie analytique des équations différentielles* (Leçons de Stockholm, delivered in 1895) (Hermann, Paris, 1897). Reprinted, *Œuvres de Paul Painlevé*, vol. **I** (Éditions du CNRS, Paris, 1973).

[10] T. L. Yee, A New Perturbative Approach in Nonlinear Singularity Analysis, *Journal of Mathematics and Statistics*, **7** (2011), 249–254. https://doi.org/10.3844/jmssp.2011.249.254

[11] T. L. Yee, The Painlevé Test for a System of Coupled Equations from Nonlinear Birefringence, *Applied Mathematical Sciences*, **12** (2018), 27–36. https://doi.org/10.12988/ams.2018.712357

[12] T. L. Yee, An algorithm to convert integrable third-order ODEs to regular higher-order equations near any movable singularities, *Nonlinear Analysis and Differential Equations*, **8** (2020), 129–144. https://doi.org/10.12988/nade.2020.91130

[13] J. Weiss, M. Tabor and G. Carnevale, The Painlevé property for partial differential equations, *J. Math. Phys.*, **24** (1983), 522–526. https://doi.org/10.1063/1.525721

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