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Mathematical Programming Involving B-(H_p, r, α)-Generalized Convex Functions

Xiaoling Liu

Department of Mathematics
Hanshan Normal University, Chaozhou, 521041 China

Dehui Yuan

Department of Mathematics
Hanshan Normal University, Chaozhou, 521041 China

Abstract

In this paper, we introduce new types of generalized convex functions including locally B-(H_p, r, α)-preinvex functions and $\bar{B}$- (H_p, r, α)-invex functions based on H_p-invex set. Some properties of these new classes of functions and sets are established. We also present the optimality conditions for mathematical programming problems in which the functions considered belong to the classes of functions introduced in this paper.

Keywords: H_p-invex set, B-(H_p, r, α)-preinvex functions, $\bar{B}$- (H_p, r, α)-invex functions

1 Introduction

Convexity plays a central role in many aspects of mathematical programming including analysis of stability, sufficient optimality conditions and duality. Based on convexity assumptions, nonlinear programming problems can be solved efficiently. There have been many attempts to weaken the convexity assumptions in order to treat many practical problems. Therefore, many concepts of generalized convex functions have been introduced and applied to mathematical programming problems in the literature [1, 2, 3]. One of these concepts, invexity, was introduced by Hanson in [4]. Hanson has shown that invexity has a common property in mathematical programming with convexity that Karush Kuhn Tucker conditions are sufficient for global optimality of nonlinear programming under the invexity assumptions. Ben-Israel and Mond [5] introduced the concept of pre-invex functions which is a special case of invexity.

Recently, Antczak [6] introduced new definitions of p-invex sets and (p, r)-invex functions which can be seen as generalization of invex functions. He also discussed nonlinear programming problems involving the (p, r)-invexity-type functions in [2, 7]. On the other hand, Kaul et al. [8] introduced the classes of locally connected sets which generalizes the arcwise connected sets [9] and locally star-shaped sets [10]. Yuan et al. introduced the definition of a new class of sets, locally H_p-invex sets, and definitions of classes of generalized convex functions called locally (H_p, r, α)-preinvex functions in [11]. Basing on locally H_p-invex sets, we discussed the programming involving locally differentiable (H_p, r)-invex functions[12].

In this paper, motivated by [13], we present new classes of generalized convex functions including B-(H_p, r, α)-preinvex functions and $\bar{B}$-(H_p, r, α)-invex functions. Based on these definitions of classes of generalized convex functions, we have managed to deal with nonlinear programming problems under some assumptions. The rest of the paper is organized as follows: In Section 2, we discuss concepts and properties regarding locally B-(H_p, r, α)-preinvex functions. In Section 3, we give the definition of locally $\bar{B}$-(H_p, r, α)-invex function, discuss the properties regarding this type of functions. In Section 4, we present the optimality conditions for mathematical programming problems involving B-(H_p, r, α)-preinvex functions and $\bar{B}$-(H_p, r, α)-invex functions, respectively.

2 Locally B-(H_p, r, α)-Preinvex Functions

Let R^n be the n-dimensional Euclidean space, $R^n_+ = \{x \in R^n | x \geq 0\}$ and $\dot{R}^n_+ = \{x \in R^n | x > 0\}$. In this section, we give definitions of locally H_p-invex set. As we mentioned before, the notion of p-invex set was introduced by Antczak in [6] and the notion of a locally connected set was introduced by Kaul et al. in [8]. Basing on these concepts, we introduced the following concept of a locally H_p-invex set in [11].

Definition 2.1. *[11] Let p be real number. The set $S \subset R^n$ is a locally H_p-invex set if and only if, for any x, $u \in S$, there exist a maximum positive number $a(x, u) \leq 1$ and a vector function $H_p : S \times S \times [0, 1] \to R^n$, such that*

$$H_p(x, u; 0) = \mathbf{e}^u, \;\; H_p(x, u; \lambda) \in \dot{R}^n_+, \ln\left(H_p(x, u; \lambda)\right) \in S, \;\; \forall\; 0 < \lambda < a(x, u).$$

and $H_p(x,u;\lambda)$ is continuous on the interval $(0, a(x,u))$, where the logarithm and the exponentials appearing in the relation are understood to be taken componentwise.

With the aid of locally H_p-invex set and the notation $M_r(a,b;\lambda)$(see [6]), we give below a new class of functions, locally B-(H_p, r, α)-preinvex functions, in this section.

Definition 2.2. *Let $S \subset R^n$ be locally H_p-invex set and $b : S \times S \times [0,1] \to R_+$, and let r be real number. A function $f : S \to R$ is said to be locally B-(H_p, r, α)-preinvex on S if, for any $x, u \in S$, there exists a maximum positive number $a(x,u) \le 1$ such that*

$$f\left(\ln\left(H_p(x,u;\lambda)\right)\right) \le \ln\left(M_r\left(\mathbf{e}^{f(x)}, \mathbf{e}^{f(u)}; (\lambda b(x,u;\lambda))^{\alpha}\right)\right),$$
$$0 \le \lambda b(x,u;\lambda) \le 1, \forall\ 0 < \lambda < a(x,u)$$

where the logarithm and the exponentials appearing on the left-hand side of the inequality are understood to be taken componentwise. If u is fixed, then f is said to be locally B-(H_p, r, α)-preinvex at u.

Remark 2.3. *Obviously, locally (H_p, r, α)-preinvex functions are (p,r)-pre-invex[6] if $H_p(x,u;\lambda) = M_p(e^{\eta(x,u)+u}, e^u; \lambda)$ and $a(x,u) = 1$ for all x, u. In general case, there exist locally B-(H_p, r, α)-preinvex functions which are not (p,r)-pre-invex. For example, let S be a locally H_p-invex set given by Remark 1 in [11], and the function f be given by*

$$f(x) = 1, \forall\ x = (x_1, x_2) \in S;$$

$$b(x,u;\lambda) = 1, \forall x, u \in S, \lambda \in [0,1].$$

Certainly, for any real number r, f is a locally B-$(H_p, r, 1)$-preinvex function which is not (p,r)-pre-invex only if $\eta(x,u)$ not always zero for any $x, u \in S$, since S is not p-invex.

Remark 2.4. *If the direction of inequality in Definition 2.2 is changed to the opposite one, we say that a function f is B-(H_p, r, α)-preincave. We say that a function $f : S \to R$ defined on a H_p-invex set $S \subset R^n$ is strictly B-(H_p, r, α)-preinvex(strictly B-(H_p, r, α)-preincave) on S if the inequality in definition of B-(H_p, r, α)-preinvex function(B-(H_p, r, α)-preincave function) is sharp and it holds for all $x \ne u$, $0 \le \lambda b(x,u;\lambda) \le 1$ and any $\lambda \in (0, a(x,u))$.*

Definition 2.5. *[11] A function $f : S \to R$ defined on a locally H_p-invex set $S \subset R^n$ is said to be locally (H_p, r, α)-prequasiinvex on S if, for any $x, u \in S$, there exists a maximum positive number $a(x,u) \le 1$ such that*

$$f\left(\ln\left(H_p(x,u;\lambda)\right)\right) \le \max\{f(x), f(u)\}, \forall\ 0 < \lambda < a(x,u)$$

where the logarithm and the exponentials appearing on the left-hand side of the inequality are understood to be taken componentwise. If u is fixed, then f is said to be locally (H_p, r, α)-prequasiinvex at u.

Remark 2.6. *By the definition of locally (H_p, r, α)-prequasiinvex function, it is easy to show that locally B-(H_p, r, α)-preinvex function is locally (H_p, r, α)-prequasi-invex. However, locally (H_p, r, α)-prequasiinvex functions are not always locally B-(H_p, r, α)-preinvex.*

We now introduce the definition of a locally (H_p, r, B, α)-invex set, which will enable us to give a geometric property of locally B-(H_p, r, α)-preinvex functions defined by Definition 2.2.

Definition 2.7. *Assume that p and r are two given real numbers. Let $X \in R^m$, $Y \in R^n$, $H_p : X \times X \times [0, 1] \to R^m$ be a vector function. Then $X \times Y = \{(x, y) : x \in X, y \in Y\}$ is said to be a locally (H_p, r, B, α)-invex set if, for any $(x^1, y^1), (x^2, y^2) \in X \times Y$, there exist a maximum positive number $a(x^2, x^1) \le 1$ and a function $b : X \times X \times [0, 1] \to R_+$ such that*

$$H_p(x^2, x^1; 0) = \mathbf{e}^{x^1},\ H_p(x^2, x^1; \lambda) \in \dot{R}^m_+,\\ \ln\left(H_p(x^2, x^1; \lambda)\right) \in X,\ \forall\ 0 < \lambda < a(x^2, x^1)$$

and

$$\left(\ln\left(H_p(x^2, x^1; \lambda)\right), \ln\left[M_r\left(\mathbf{e}^{y^2}, \mathbf{e}^{y^1}; (\lambda b(x^2, x^1; \lambda))^{\alpha}\right)\right]\right) \in X \times Y,\\ 0 \le \lambda b(x^2, x^1; \lambda) \le 1, \forall\ 0 < \lambda < a(x^2, x^1).$$

Theorem 2.8. *Let S be a locally H_p-invex set, then $f : S \to R$ is locally B-(H_p, r, α)-preinvex function if and only its epigraph $E(f) = \{(x, y) : x \in S, f(x) \le y\}$ is a locally (H_p, r, B, α)-invex set.*

Proof. "if" part. Obviously, $(x, f(x))$, $(u, f(u)) \in \mathrm{E}(f)$. Therefore, by the definition of locally (H_p, r, B, α)-invex set, there exists a maximum positive number $a(x, u) \le 1$ such that

$$H_p(x, u; 0) = \mathbf{e}^u,\ H_p(x, u; \lambda) \in \dot{R}^m_+,\ \ln\left(H_p(x, u; \lambda)\right) \in S,\ \forall\ 0 < \lambda < a(x, u)$$

and

$$\left(\ln\left(H_p(x, u; \lambda)\right), \ln\left[M_r\left(\mathbf{e}^{f(x)}, \mathbf{e}^{f(u)}; (\lambda b(x, u; \lambda))^{\alpha}\right)\right]\right) \in \mathrm{E}(f),\\ 0 \le \lambda b(x, u; \lambda) \le 1, \forall\ 0 < \lambda < a(x, u).$$

By the definition of $\mathrm{E}(f)$, the last relation means

$$f\left(\ln\left(H_p(x, u; \lambda)\right)\right) \le \ln\left[M_r\left(\mathbf{e}^{f(x)}, \mathbf{e}^{f(u)}; (\lambda b(x, u; \lambda))^{\alpha}\right)\right],\\ 0 \le \lambda b(x, u; \lambda) \le 1, \forall\ 0 < \lambda < a(x, u).$$

Therefore, f is a locally B-(H_p, r, α)-preinvex function.

Moreover the above steps are invertible, hence the “only if ” part is true. Therefore, the result follows.

Theorem 2.9. *Let S be a locally H_p-invex set. If $f : S \to R$ is B-(H_p, r, α)-preinvex function, then the level set $L_\beta = \{x \in S : f(x) \leq \beta\}$ is a locally H_p-invex for every $\beta \in R$.*

Proof. By Corollary 3 in [14] and Remark 2.6, we get the desired result.

Theorem 2.10. *Let $S \subset R^n$ be a H_p-invex set and $f : S \to R$ be a B-(H_p, r, α)-preinvex function on S. For any given pair $x, u \in S$, assume that $H_p(x, u; \lambda)$ is continuous on the interval $[0, a(x, u))$ and is not constant in any subinterval contained in $[0, a(x, u))$. Then each point of a local minimum of the function f is its point of global minimum, and the set of points which are global minima of f is a H_p-invex set.*

Proof. The theorem will be proved only in the case when $r \neq 0$(the other case when $r = 0$ can be dealt with likewise).

Assume that $u \in S$ is a point of local of f which is not a point of global minimum. Hence, there exists a point$\bar{x} \in S$ such that $f(\bar{x}) < f(u)$. By assumption, $f : S \to R$ is B-(H_p, r, α)-preinvex function on S. Thus by definition, for all $x, u \in S$, there exists positive $a(x, u) \leq 1$ such that

$$f\left(\ln(H_p(x, u; \lambda))\right) \leq \ln\left([\lambda b(x, u; \lambda)]^\alpha \mathbf{e}^{rf(x)} + (1 - [\lambda b(x, u; \lambda)]^\alpha)\mathbf{e}^{rf(u)}\right)^{\frac{1}{r}},$$
$$0 \leq \lambda b(x, u; \lambda) \leq 1, \forall\ 0 \leq \lambda \leq a(x, u).$$

In particular, the above inequality holds also in the case when $x = \bar{x}$. Taking into account the fact that $f(\bar{x}) < f(u)$, we get

$$\begin{aligned} f\left(\ln(H_p(\bar{x}, u; \lambda))\right) &\leq \ln\left([\lambda b(\bar{x}, u; \lambda)]^\alpha \mathbf{e}^{rf(\bar{x})} + (1 - [\lambda b(\bar{x}, u; \lambda)]^\alpha)\mathbf{e}^{rf(u)}\right)^{\frac{1}{r}} \\ &< f(u), \qquad\qquad 0 \leq \lambda b(x, u; \lambda) \leq 1, \forall\ \lambda \in (0, a(\bar{x}, u)). \end{aligned}$$

Thus, we have show that

$$f\left(\ln(H_p(\bar{x}, u; \lambda))\right) < f(u), \forall\ \lambda \in (0, a(\bar{x}, u)).$$

This is a contradiction to the fact that u is a local minimum point.

Now, Denote by E the set of points of global minimum of f. Let x and u be arbitrary points belonging to E. We prove that $\ln(H_P(x, u; \lambda)) \in E$. Since x and u belong to E, then $f(x) = f(u)$. Again, $f : S \to R$ is B-(H_p, r, α)-preinvex function on S. Hence, there exists a maximum positive number $a(x, u)$ such that

$$\begin{aligned} f\left(\ln(H_p(x, u; \lambda))\right) &\leq \ln\left([\lambda b(x, u; \lambda)]^\alpha \mathbf{e}^{rf(x)} + (1 - [\lambda b(x, u; \lambda)]^\alpha)\mathbf{e}^{rf(u)}\right)^{\frac{1}{r}} \\ &\leq f(u),\ 0 \leq \lambda b(x, u; \lambda) \leq 1, \forall\ \lambda \in (0, a(x, u))) \end{aligned}$$

That is

$$f\left(\ln(H_p(x,u;\lambda))\right) = f(u), \forall\ \lambda \in (0, a(x,u))$$

Therefore, $\ln(H_p(x,u;\lambda)) \in E, \forall\ \lambda \in (0, a(x,u))$. By Definition 2.1, E is H_p-invex set.

Corollary 2.11. *Let $S \subset R^n$ be a H_p-invex set and $f : S \to R$ be a B-(H_p, r, α)-preinvex function on S. For any given pair $x, u \in S$, assume that $H_p(x,u;\lambda)$ is differentiable on the interval $[0, a(x,u))$ and its derivative is not zero in any subinterval contained in $[0, a(x,u))$. Then each point of a local minimum of the function f is its point of global minimum, and the set of points which are global minima of f is H_p-invex set.*

Proof. If the assumptions of corollary 1 hold, then the assumptions of Theorem 2.10 is true. By Theorem 2.10, the results follows.

3 $\bar{B}$-(H_p, r, α)-invex functions

Motivated by [13], we present the concept $\bar{B}$-(H_p, r, α)-invex functions in this section. For the convenience, we use the following notations.

Definition 3.1. *Let a, b, r be real number. Then the generalized difference of a and b with respect to r, $\nabla_r(a,b)$, is defined by*

$$\nabla_r(a,b) = \begin{cases} e^{ra} - e^{rb} & \text{for } r \neq 0 \\ a - b & \text{for } r = 0 \end{cases}$$

Definition 3.2. *Let $\xi, x, u \in R^n$ and η: $R^n \times R^n \to R^n$, and let p be an arbitrary real number. Then the p-inner product of ξ and $\eta(x,u)$ denoted $\langle \xi, \eta(x,u)\rangle^*$ can be defined as follows:*

$$\langle \xi, \eta(x,u)\rangle^* = \begin{cases} \frac{1}{p}\ \xi^T\left(e^{p\eta(x,u)} - \mathbf{1}\right), \text{for } p \neq 0 \\ \langle \xi, \eta(x,u)\rangle \qquad\qquad \text{for } p = 0 \end{cases}$$

where $\mathbf{1} = (1, \cdots, 1) \in R^n$, $e^{p\eta(x,u)} = \left(e^{p\eta_1(x,u)}, \cdots, e^{p\eta_n(x,u)}\right)$.

Definition 3.3. *Let p, r, α be real number such that $0 < \alpha \leq 1$, and let $f : R^n \to R$. Then the right super order α-(p,r) differential of f at u with respect to $H_p(x,u;\lambda)$, $d^{\,\alpha}_{(p,r)} f(u; H_p(x,u;0+))$ is defined by*

$$d^{\,\alpha}_{(p,r)} f(u; H_p(x,u;0+)) = \begin{cases} \limsup\limits_{\lambda \downarrow 0} \frac{\nabla_r(f(\ln(H_p(x,u;\lambda))), f(u))}{\lambda^\alpha}, \ r \geq 0 \\ \liminf\limits_{\lambda \downarrow 0} \frac{\nabla_r(f(\ln(H_p(x,u;\lambda))), f(u))}{\lambda^\alpha}, \ r < 0 \end{cases}.$$

Again let $\eta : R^n \times R^n \to R^n$ be a continuous vector function and $H_p(x, u; \lambda) = M_p(e^{\eta(x,u)+u}, e^u; \lambda)$. Then the differentiable $d^{\alpha}_{(p,r)} f(u; H_p(x, u; 0+))$ is denoted by $d^{\alpha}_{(p,r)} f(u; \eta(x, u))$, and the order α-(p, r) subgradient set of f at u, $\partial^{\,\alpha}_{(p,r)} f(u; \eta(x, u))$, can be defined as follows:

$$\partial^{\,\alpha}_{(p\,,\,r)} f(u; \eta(x, u)) = \begin{cases} \{\xi | \langle \xi, \eta(x, u)\rangle^* \leq d^{\,\alpha}_{(p\,,\,r)} f(u; \eta(x,\, u)), \forall\, x \in S\},\ r \geq 0 \\ \{\xi | \langle \xi, \eta(x, u)\rangle^* \geq d^{\,\alpha}_{(p\,,\,r)} f(u; \eta(x,\, u)),\, \forall\, x \in S\},\ r < 0 \end{cases}.$$

Remark 3.4. *Note that $f(x) = e^x$ is differentiable, the right super order α-(p, r) differential, can be rewritten as follows:*

$$d^{\,\alpha}_{(p,r)} f(u; H_p(x, u; 0+)) = \begin{cases} re^{rf(u)} \limsup\limits_{\lambda\downarrow 0} \frac{f(\ln(H_p(x,u;\lambda)))-f(u)}{\lambda^{\alpha}},\ r > 0 \\ \limsup\limits_{\lambda\downarrow 0} \frac{f(\ln(H_p(x,u;\lambda)))-f(u)}{\lambda^{\alpha}}, \qquad r = 0 \\ re^{rf(u)} \liminf\limits_{\lambda\downarrow 0} \frac{f(\ln(H_p(x,u;\lambda)))-f(u)}{\lambda^{\alpha}},\ r < 0 \end{cases}.$$

Definition 3.5. *Let $S \subset R^n$ be a locally H_p-invex set, the function $f : S \to R$ is said to be $\bar{B}$-(H_p, r, α)-invex at $u \in S$ if there exists function $\bar{b} : S \times S \to R$ such thatthe inequalities*

$$\bar{b}^{\alpha}(x, u)\nabla_r\,(f(x), f(u)) \geq d^{\,\alpha}_{(p\,,\,r)} f(u; H_p(x, u; 0+)),\ \ r \geq 0,$$

$$\bar{b}^{\alpha}(x, u)\nabla_r(f(x), f(u)) \leq d^{\,\alpha}_{(p\,,\,r)} f(u; H_p(x, u; 0+)),\ \ r < 0.$$

hold for all $x \in S$. f is said to $\bar{B}$-(H_p, r, α)-invex on S if it is $\bar{B}$-(H_p, r, α)-invex at each $u \in S$.

Theorem 3.6. *Let $S \subset R^n$ be a locally H_p-invex set and $f : S \to R$ be a locally B-(H_p, r, α)-preinvex at u, then, for all $x \in S$, the inequalities*

$$\bar{b}^{\alpha}(x, u)\nabla_r\,(f(x), f(u)) \geq d^{\,\alpha}_{(p\,,\,r)} f(u; H_p(x, u; 0+)),\ \ r \geq 0,$$

$$\bar{b}^{\alpha}(x, u)\nabla_r(f(x), f(u)) \leq d^{\,\alpha}_{(p\,,\,r)} f(u; H_p(x, u; 0+)),\ \ r < 0.$$

where $\bar{b}(x, u) = \limsup\limits_{\lambda\downarrow 0} b(x, u; \lambda)$ when $r \geq 0$ and $\bar{b}(x, u) = \liminf\limits_{\lambda\downarrow 0} b(x, u; \lambda)$ when $r < 0$.

Proof. Let $f\colon S \to R$ be defined on a locally H_p-invex $S \subset R^n$. Moreover, we assume that f is a locally B-(H_p, r, α)-preinvex function at u and $r \geq 0$ (the proof in the case when $r < 0$ is analogous). Hence, we have

$$\begin{cases} f\left(\ln\left(H_p(x, u; \lambda)\right)\right) \leq \ln\left(M_r(e^{f(x)}, e^{f(u)}; (\lambda b(x, u; \lambda))^{\alpha})\right),\ p \neq 0 \\ f\left(\left(H_p(x, u; \lambda)\right)\right) \leq \ln\left(M_r(e^{f(x)}, e^{f(u)}; (\lambda b(x, u; \lambda))^{\alpha})\right),\ p = 0 \end{cases}$$

for any $0<\lambda<a(x,u)$. That is

$$\begin{cases} e^{rf(\ln(H_p(x,u;\lambda)))}-e^{rf(u)}\leq(\lambda b(x,u;\lambda))^{\alpha}\left(e^{rf(x)}-e^{rf(u)}\right), & r>0; \\ f\left(\ln\left(H_p(x,u;\lambda)\right)\right)-f(u)\leq(\lambda b(x,u;\lambda))^{\alpha}\left(f(x)-f(u)\right), & r=0 \end{cases}$$

when $p\neq 0$ and

$$\begin{cases} e^{rf(H_p(x,u;\lambda))}-e^{rf(u)}\leq(\lambda b(x,u;\lambda))^{\alpha}\left(e^{rf(x)}-e^{rf(u)}\right), & r>0 \\ f\left(H_p(x,u;\lambda)\right)-f(u)\leq(\lambda b(x,u;\lambda))^{\alpha}\left(f(x)-f(u)\right), & r=0 \end{cases}$$

when $p=0$, for any $0<\lambda<a(x,u)$. Therefore,

$$\begin{cases} \frac{e^{rf\left(\ln\left(H_p(x,u;\lambda)\right)\right)}-e^{rf(u)}}{\lambda^{\alpha}}\leq b^{\alpha}(x,u,\lambda)\left(e^{rf(x)}-e^{rf(u)}\right), \text{for } r>0; \\ \frac{f(\ln(H_p(x,u;\lambda)))-f(u)}{\lambda^{\alpha}}\leq b^{\alpha}(x,u,\lambda)(f(x)-f(u)), \text{for } r=0, \end{cases}$$

when $p\neq 0$ and

$$\begin{cases} \frac{e^{rf\left(H_p(x,u;\lambda)\right)}-e^{rf(u)}}{\lambda^{\alpha}}\leq b^{\alpha}(x,u,\lambda)\left(e^{rf(x)}-e^{rf(u)}\right), \text{for } r>0; \\ \frac{f(H_p(x,u;\lambda))-f(u)}{\lambda^{\alpha}}\leq b^{\alpha}(x,u,\lambda)(f(x)-f(u)), \text{for } r=0 \end{cases}$$

when $p=0$, for any $0<\lambda<a(x,u)$. By the Definition 3.1 and Definition 3.3, we deduce that,

$$\bar{b}^{\alpha}(x,u)\nabla_r\left(f(x),f(u)\right)\geq d^{\,\alpha}_{(p,r)}f(u;H_p(x,u;0+)),\ r\geq 0,$$

for any $x\in S$.

Remark 3.7. *By Theorem 3.6, we know that locally B-(H_p,r,α)-preinvex function is $\bar{B}$-(H_p,r,α)-invex.*

Corollary 3.8. *Let $S\subset R^n$ be a locally H_p-invex set and $f:S\to R$ be a locally B-(H_p,r,α)-preinvex at u, and let $H_p(x,u;\lambda)=M_p(e^{\eta(x,u)+u},e^u;\lambda)$, then, for all $x\in S$*

$$\bar{b}^{\alpha}(x,u)\nabla_r\left(f(x),f(u)\right)\geq\langle\xi,\eta(x,u)\rangle^*,\ \forall\,\xi\in\partial^{\,\alpha}_{(p,r)}f(u;\eta(x,u)),\ r\geq 0,$$

$$\bar{b}^{\alpha}(x,u)\nabla_r(f(x),f(u))\leq\langle\xi,\eta(x,u)\rangle^*,\ \forall\,\xi\in\partial^{\,\alpha}_{(p,r)}f(u;\eta(x,u)),\ r<0.$$

Proof. Using the Definition 3.3 and Theorem 3.6, we can derive the result.

4 Optimality Conditions

In this section, we present the optimality conditions for mathematical programming problems in which the functions considered belong to the classes of functions introduced earlier in this paper.

Consider the following form of optimization problem

$$(P) \qquad \begin{array}{cc} \min & f(x) \\ & g(x) \leq 0, \; x \in S, \end{array}$$

where $S \subset R^n$, $f : S \to R$, $g : S \to R^m$.

Let us denote by E the set of feasible solutions of (P), i.e., the set of the form $E := \{x \in S | g(x) \leq 0\}$.

Theorem 4.1. *Suppose that the following three statements*

(a) *E is a H_p-invex set;*

(b) *f is strictly B-(H_p, r, α)-preincave on E;*

(c) *for any $y \in \mathrm{int}E$, there exists point $x \in E$, $x \neq y$, and $\bar{\lambda} \in (0, a(x, y))$ such that*

$$\ln\left(H_p(x, y; \bar{\lambda})\right) \in E \tag{1}$$

Then there are no interior points of E which are solutions of (P)*; i.e., if u is a solution of* (P)*, then u is a boundary point of E.*

Proof. If the feasible solution set E of (P) is empty, or int E is empty, the proof is obvious. Assume that u is a solution of (P), and $u \in$ int E. By (c), there exist $x, \in E$, $x \neq u$, and $\bar{\lambda} \in (0, a(x, u))$ such that (1) holds. Hence, by (b), we have

(i) in the case $r \neq 0$:

$$f(u) = f\left(\ln\left(H_p(x, u; \bar{\lambda})\right)\right) > \ln\left((\bar{\lambda}b)^\alpha \mathbf{e}^{rf(x)} + (1 - (\bar{\lambda}b)^\alpha)\mathbf{e}^{rf(u)}\right)^{\frac{1}{r}} \geq f(u)$$

(ii) in the case $r = 0$:

$$f(u) = f\left(\ln\left(H_p(x, u; \bar{\lambda})\right)\right) > (\bar{\lambda}b)^\alpha f(x) + (1 - (\bar{\lambda}b)^\alpha)f(u) \geq f(u).$$

This contradiction leads us to the conclusion that u is not a solution of (P). The proof is complete.

Theorem 4.2. *Suppose that the following four statements*

(a) *E is a H_p-invex set;*

(b) *f is strictly B-(H_p, r, α)-preinvex on E;*

(c) *$u \in E$ is a local minimum of (P);*

(d) *for any positive real number ε and any point $x \in E$, there exists $\bar{\lambda} \in (0, a(x, u))$ such that*

$$\ln\left(H_p(x, u; \bar{\lambda})\right) \in B(u, \varepsilon).$$

Then u is a strict global minimum of (P).

Proof. By assumption (a), the set E is H_p-invex set and therefore, for any $x \in E$, there exists a maximum positive number $a(x,u) \leq 1$ such that

$$\ln\left(H_p(x,u;\lambda)\right) \in E.$$

Since u is a local minimum of (P), there exists $\bar{\varepsilon} > 0$ such that the inequality $f(x) \geq f(u)$ holds for any $x \in B(u,\varepsilon) \cap E$. Now, let x be a point of E such that $x \neq u$. By assumption (d) and (b), with $\varepsilon = \bar{\varepsilon}$, we get

$$f(u) \leq f\left(\ln\left(H_p(x,u;\bar{\lambda})\right)\right)$$

is true for some $\bar{\lambda} \in (0, a(x,u))$. By (c) and [6, Lemma 40], we have

(i) in the case of $r \neq 0$:

$$\begin{aligned} f(u) &\leq f\left(\ln\left(H_p(x,u;\bar{\lambda})\right)\right) \\ &< \ln\left((\bar{\lambda}b)^{\alpha}\mathbf{e}^{rf(x)} + (1-\left(\bar{\lambda}b)^{\alpha}\right)\mathbf{e}^{rf(u)}\right)^{\frac{1}{r}} \leq \max\{f(x), f(u)\} \end{aligned}$$

(ii) in the case of $f = 0$:

$$\begin{aligned} f(u) &\leq f\left(\ln\left(H_p(x,u;\bar{\lambda})\right)\right) \\ &< (\bar{\lambda}b)^{\alpha}f(x) + \left(1-(\bar{\lambda}b)^{\alpha}\right)f(u) \leq \max\{f(x), f(u)\} \end{aligned}$$

Obviously, $\max\{f(x), f(u)\} \neq f(u)$(otherwise, we obtain the contradict inequality $f(u) < f(u)$). Therefore, $f(u) < f(x)$. Since x is an arbitrary point of E, the proof of the theorem is complete.

Corollary 4.3. *Suppose that the following four statements*

(a) *E is a p-invex set;*

(b) *f is strictly B-(H_p, r, α)-preinvex on E;*

(c) *$u \in E$ is a local minimum of* (P)*;*

Then u is a strict global minimum of (P).

Proof. By Remark 2.3, Remark 2.4, Lemma42 in[6] and Theorem 4.2, we get the desired result. The corollary is Theorem 43 of [6]. From now on, we consider the mathematical programming problem (P) with .

Theorem 4.4. *Let S be a H_p-invex set. Assume that $\bar{x} \in S$ is feasible for problem* (P)*, and there exists $\mu = (\mu_1, \cdots, \mu_m) \geq 0$ such that*

$$\begin{cases} d^{\,\alpha}_{(p,r)}f(\bar{x}; H_p(x,\bar{x};0+)) + \sum\limits_{i=1}^{m} \mu_i d^{\,\alpha}_{(p,r)}g_i(\bar{x}; H_p(x,\bar{x};0+)) \geq 0, r \geq 0 \\ d^{\,\alpha}_{(p,r)}f(\bar{x}; H_p(x,\bar{x};0+)) + \sum\limits_{i=1}^{m} \mu_i d^{\,\alpha}_{(p,r)}g_i(\bar{x}; H_p(x,\bar{x};0+)) \geq 0, r < 0 \end{cases} \tag{2}$$

$$\sum_{i=1}^{m} \mu_i g_i(\bar{x}) = 0. \tag{3}$$

If f, $g_i(i = 1, \cdots, m)$ are $\bar{B}$-(H_p, r, α)-invex at $\bar{x}$ on S, then $\bar{x}$ is a global minimum point in problem (P).

Proof. Here we prove only the cases when $r \geq 0$(the proof of the case when $r < 0$ is similar; the only changes arise from the form of inequalities defining the class of (H_p, r)-invex functions). Assume that x is an arbitrary feasible point for problem (P).

By hypothesis, f and $g_i (i = 1, \cdots, m)$ are $\bar{B}$-(H_p, r, α)-invex at $\bar{x}$ on S; therefore, for all $x \in S$, the inequalities

$$\bar{b}^{\alpha}(x, \bar{x}) \nabla_r \left(f(x), f(\bar{x})\right) \geq d^{\,\alpha}_{(p,r)} f(\bar{x}; H_p(x, \bar{x}; 0+)), \tag{4}$$

$$\bar{b}^{\alpha}(x, \bar{x}) \nabla_r \left(g_i(x), g_i(\bar{x})\right) \geq d^{\,\alpha}_{(p,r)} g_i(\bar{x}; H_p(x, \bar{x}; 0+)), \ i = 1, \cdots, m, \tag{5}$$

are true. Denote $I(\bar{x}) = \{i | \mu_i > 0, i = 1, \cdots, m\}$. By (3), we have $g_i(\bar{x}) = 0$ if $i \in I(\bar{x})$ and $\mu_i = 0$ if $g_i(\bar{x}) \neq 0$, thus $g_i(x) \leq g_i(\bar{x})$ for $i \in I(\bar{x})$. Therefore, from (5), we have

$$d^{\,\alpha}_{(p,r)} g_i(\bar{x}; H_p(x, \bar{x}; 0+)) \leq 0, \ i \in I(\bar{x}). \tag{6}$$

Multiplying (6) with $\mu_i (i \in I(\bar{x}))$, respectively, we deduce that

$$\sum\nolimits_{i \in I(\bar{x})} \mu_i d^{\,\alpha}_{(p,r)} g_i(\bar{x}; H_p(x, \bar{x}; 0+)) \leq 0.$$

hence,

$$\sum\nolimits_{i=1}^{m} \mu_i d^{\,\alpha}_{(p,r)} g_i(\bar{x}; H_p(x, \bar{x}; 0+)) \leq 0.$$

This, together with (4), follows

$$\begin{aligned} \bar{b}^{\alpha}(x, \bar{x}) \nabla_r \left(f(x), f(\bar{x})\right) \geq\ & d^{\,\alpha}_{(p,r)} f(\bar{x}; H_p(x, \bar{x}; 0+)) \\ & + \sum\nolimits_{i=1}^{m} \mu_i d^{\,\alpha}_{(p,r)} g_i(\bar{x}; H_p(x, \bar{x}; 0+)) \end{aligned} \tag{7}$$

By (3) and (7), we derive $\nabla_r \left(f(x), f(\bar{x})\right) \geq 0$. That is $f(x) \geq f(\bar{x})$, which means that $\bar{x}$ is an optimal point in problem (P).

The assumption on functions in Theorem 4.4 could also be given in another form. It is enough to assume that the Lagrange function $f + \sum_{i=1}^{m} \mu_i g_i$ is a B-(H_p, r, α)-invex function. And so, the following theorem is true. Its proof is on the same line as Theorem 4.4, therefore we delete it here.

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