

# Mathematical Survey of the Action Principle

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## Abstract

Defining the principle of extremal action in concise mathematical terms, it is shown that this principle does not hold what it physically promises. Instead, it is shown that Lagrange functions need to be locally integrable (in an open region of space), in order that the Lagrange equations strictly apply. The principle of extremal action therefore reduces to the condition of local integrability of the Lagrange function to a (locally defined) Hamilton-Jacobi function.

**Keywords:** Dynamical systems, Lagrange equations

## 1 (Euclidean) Lagrange Function

In current mathematics, the Lagrangian is defined on a symplectic  $2n$ -dimensional manifold of 2-forms. That conceals from some basic problems by the additional layer of a manifold structure; it seems worth to restrict the problem domain to the simplest possible model, which is the global  $(2n+1)$ -dimensional Euclidean space of time  $t$ , the  $n$  generalized momenta  $p := (p_1, \dots, p_n)$ , and the  $n$  generalized location coordinates  $q := (q_1, \dots, q_n)$ .

(All results can later be taken to tangent spaces in straightforward manner for these manifolds.)

With this, the classical action integral for a path  $\omega : [t_1, t_2] \ni t \mapsto q(t) \in \mathbb{R}^n$  is physically defined as

$$S(\omega) := - \int_{t_0}^{t_1} (H(t, p(t), q(t)) - p(t) \cdot \dot{q}(t)) dt, \quad (1.1)$$

and the principle of extremal action is stated in the form that all dynamically possible paths from  $(t_0, q(t_0))$  to  $(t_1, q(t_1))$  are to be extremals of the action integral.

In order to get this physical definition also mathematically defined, we need a proper topology on the space of time curves  $\omega : [t_0, t_1] \rightarrow \mathbb{R}^n$  that allows for a definition of a derivative of  $S : \omega \mapsto S(\omega)$ . The simplest one would be vector space of all continuously differentiable curves  $\omega : [0, 1] \ni t \mapsto q(t) \in \mathbb{R}^n$ , which is a Banachspace (i.e. complete normed space) with its natural norm  $\|\omega\|_1 := \sup_{0 \leq t \leq 1} (|q(t)| + \left| \frac{d}{dt} q(t) \right|)$ .

Sadly, the topology is a bit too strong to meet the physical requirements of  $q(0)$  and  $q(1)$  being fixed endpoints, only. So things become more complicated:

The space of all continuously differentiable paths has two closed subspaces: one is the space of all closed continuously differentiable paths, and the other one its subspace of all continuously differentiable closed paths that start and end in the origin  $0 \in \mathbb{R}^n$ . We denote this  $2^{\text{nd}}$  subspace as  $\mathcal{C}_0^1(\mathbb{R}^n)$ . Then the set of all continuous differentiable paths from  $[0, 1]$  in  $\mathbb{R}^n$  with fixed endpoints  $q(0)$  and  $q(1)$  is an affine Banachspace  $\omega_0 + \mathcal{C}_0^1(\mathbb{R}^n)$ , where  $\omega_0$  is an arbitrary continuously differentiable path from  $q(0)$  to  $q(1)$ . We can then define the differentiability of  $S : \omega \mapsto S(\omega)$  to be:

**Definition 1.1** (Differentiability). *Let  $U$  be an open subset of  $\omega_0 + \mathcal{C}_0^1(\mathbb{R}^n)$  and  $S : U \rightarrow \mathbb{R}$  be defined and continuous on  $U$ . Then  $S$  is defined to be differentiable at  $\omega \in U$  if and only if there exists a continuous linear operator  $DS(\omega) : \mathcal{C}_0^1(\mathbb{R}^n) \rightarrow \mathbb{R}$  such that for all  $\eta \in \mathcal{C}_0^1(\mathbb{R}^n)$  with  $\omega + \eta \in U$  and  $\|\eta\|_1 \rightarrow 0$ :  $S(\omega + \eta) = S(\omega) + DS(\omega)\eta + o(\|\eta\|_1)$ , where  $o(h)$  denotes the rest term with the asymptotic condition  $\frac{o(h)}{h} \rightarrow 0$  as  $h \rightarrow 0$ .  $DS(\omega)$  is called derivative of  $S$  at  $\omega$ .  $S$  is said to be extremal in  $\omega$ , if  $S$  is differentiable at  $\omega$  and if its derivative is the zero operator.*

## 2 Time Reversal

**Definition 2.1** (Time Inversion). *Given a time curve  $\omega : [0, 1] \rightarrow \mathbb{R}^n$ , the time inverse  $\mathcal{T}\omega$  is defined as:  $\mathcal{T}\omega : [0, 1] \ni t \mapsto \omega(1 - t) \in \mathbb{R}^n$ .  $S : U \rightarrow \mathbb{R}$  will be called reversible, if for all  $\omega \in U$ :  $S(\mathcal{T}\omega) = -S(\omega)$  and  $S(\mathcal{T}\omega|_{[s_1, s_2]}) = -S(\omega|_{[s_1, s_2]})$ , where  $\omega|_{[s_1, s_2]}$  denotes the restriction of  $\omega$  to the open interval  $[s_1, s_2]$  for  $0 < s_1 < s_2 < 1$  and  $S(\mathcal{T}\omega|_{[s_1, s_2]})$  its time inverse.*

**Proposition 2.2.** *Let  $S : U \rightarrow \mathbb{R}$  be reversible and differentiable at  $\omega \in U$ . Then the derivative is zero, that is:  $\omega$  is extremal.*

*Proof.* Because the derivative  $DS(\omega)$  is a linear, continuous mapping, we can split a small path  $\eta \in \mathcal{C}_0^1(\mathbb{R}^n)$  into the sum  $\eta = \eta_1 + \dots + \eta_n$  of its  $n$  component

projections, and  $DS(\omega)\eta = \sum_k DS(\omega)\eta_k$  is the commuting sum of the path integrals each of the components. Each curve  $\eta_k$  is also closed, starting and ending in the origin. Now,  $DS(\omega)\eta_k$  might not be zero. But because of the reversibility, the differentiability of  $S$  at  $\omega$  implies the differentiability of  $S$  at  $\mathcal{T}\omega$  with inverted derivative  $-DS(\omega)$ , which demands  $\int_0^1 DS(\omega)\eta_k = o(\|\eta_k\|_1)$ .  $\square$

### 3 The Lagrange Equations

Let  $S$  be reversible and differentiable at  $\omega$ . Then we know, it is extremal at  $\omega$ , which is usually written as  $\delta S(\omega) = 0$ , and  $\delta$  is called “virtual” displacement. Can we derive the Lagrange equations from that?

The answer is that this is true in the limit  $\delta \rightarrow 0$ , but for any  $\delta \neq 0$  a rest term  $o(\delta)$  needs to be added:

To get in line with the normal representation within physics, let's replace the momenta  $p$  by the (generalized) velocities  $\dot{q}$  and write  $L(q(t), \dot{q}(t), t) := -H(t, p(t), q(t)) + p(t) \cdot \dot{q}(t)$ . Then the condition of extremality at a path  $\omega$  is usually written as:  $\delta \int_\omega L(q(t), \dot{q}(t), t) d\omega = \delta \int_0^1 L(q(t), \dot{q}(t), t) dt = 0$ . In the first step the differential  $\delta$  is commuted with the time integral:

$$\delta \int_0^1 L dt = \int_0^1 \delta L dt.$$

Next,  $\delta L$  is written out as a sum of partial differentials:  $\delta L = \frac{\partial L}{\partial q} \cdot \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \frac{\partial L}{\partial t} \delta t$ , and then it is assumed that - given explicit time independence of  $L$ :

$$\frac{\partial L}{\partial \dot{q}} \cdot \delta \dot{q} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \cdot \delta q \right) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \cdot \delta q.$$

And, indeed, if  $L$  and therefore the Hamiltonian function are explicitly time independant,  $\delta t$  and  $\delta q$  commute, that is: a differential variation of  $\delta t$  followed by a differential  $\delta q$  equals  $\delta q$  followed by  $\delta t$ . (Note, however, that under non-conserved conditions  $\delta t$  and  $\delta q$  will generally not commute.)

But there is one major problem in the last step: The integration is along the curve  $\omega$ , where  $\delta$  is zero. For any slight perturbation  $\omega + \eta$  with  $\eta \neq 0$ , the Lagrange equations will be off by a rest term of order  $o(\eta)$ . To derive these without rest term, however one needs

$$\int_0^1 \left( \frac{\partial L}{\partial q} \cdot \delta q - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \cdot \delta q \right) dt = 0$$

within a small open environment of  $\omega$ . This mandates that for each small  $s > 0$  there must exist an  $\epsilon > 0$ , such that  $\int_s^{1-s} L dt$  is to be path independant

for all paths  $\omega|_{[s,1-s]} + \eta$  with  $\eta \in \mathcal{C}_0^1(\mathbb{R}^n)$  and  $\|\eta\|_1 < \epsilon$ . The union of ranges of all the paths  $\omega|_{[s,1-s]} + \eta$  contains the  $\epsilon$ -balls in  $\mathbb{R}^n$  around each  $\omega(t) \in \mathbb{R}^n$ , ( $t \in [s, 1-s]$ ). And, since these  $\epsilon$ -balls are convex, the conclusion is that within these  $\epsilon$ -balls  $L$  is integrable to a function, i.e.: it is locally integrable. Whether  $L$  also is globally integrable to a function in the whole environment of the range of the paths  $\omega|_{[s,1-s]} + \eta$ , that depends on whether or not this environment is globally convex or at least simply connected (as is well-known: see [?]).

The following example shows the necessity of local integrability of the Lagrange function:

## 4 Example of a non-integrable Lagrange function with paths of extremal action integrals

The Hamilton function of a free 2-dimensional mechanical system is given by  $H = \frac{1}{2m}(p_1^2 + p_2^2)$ , so  $L = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2)$ , which - as any free n-dimensional mechanical system - is known to be globally integrable, as  $S : \mathbb{R}^3 \ni (t, q_1, q_2) \mapsto -Et + p_1 q_1 + p_2 q_2 \in \mathbb{R}$  is its (Hamilton-Jacobi) action function, where  $E$  and  $p = (p_1, p_2)$  are its constant energy and momentum. Let's write that Lagrange function in polar coordinates  $(\phi, r)$ , where  $\phi \in [0, 2\pi]$  is the angle and  $r \in [0, \infty)$  the radius, which then is:  $L = \frac{m}{2}(\dot{r}^2 + r^2 \dot{\phi}^2)$ . Now, let's add to it the potential  $V = -r^2 \phi$ , which introduces a curl around the origin, and consider the curve  $\omega$  that goes in a straight line from  $q_1 = -1/2$  to  $q_1 = 1/2$  along the  $q_1$  axis. In polar coordinates,  $\omega$  is the step function  $\omega : [0, 1/2) \ni t \mapsto (\pi, 1/2 - t)$  and  $\omega : [1/2, 1] \ni t \mapsto (0, 1/2 - t)$ . Integration of  $V$  along an arc  $\eta_1$ , say, from  $\phi$  to  $\phi + \Delta\phi$  at a fixed radius  $r_1 > 0$  gives  $(1/2)r_1^2(\Delta\phi)^2$ , therefore integration in the opposite direction  $\eta_2$  from  $\phi + \Delta\phi$  to  $\phi$  at some smaller radius  $r_0 < r_1$  adds  $-(1/2)r_0^2(\Delta\phi)^2$ , while integration along the closing paths  $\eta_3$  and  $\eta_4$  from  $r_0$  to  $r_1$  at  $\phi$  and from  $r_1$  to  $r_0$  at  $\phi + \Delta\phi$  cancel each other. Therefore, the path integral of  $V$  or  $L$  along any piecewise continuously differentiable closed path in  $\mathbb{R}^2$  will generally be unequal zero (unless its interior is empty). Yet, for  $r_1 \rightarrow r_0$ , the value is  $o(r_1 - r_0)$ , and integration along a circle at a distance  $r_1$  around the origin, likewise gives  $2\pi^2 r_1^2$ , which again is  $o(r_1)$ . So, the path integrals of  $L$  and  $V$  are differentiable at the straight line curve  $\omega$ : although the path integration of  $V$  along  $\omega$  and  $\omega + \eta$  with  $\eta \in \mathcal{C}_0^1(\mathbb{R}^2)$  might differ, the difference is  $o(\|\eta\|_1)$  as  $\eta \rightarrow 0$  in  $\mathcal{C}_0^1(\mathbb{R}^n)$ . Hence, according to infinitesimal calculus,  $\delta \int_0^1 L dt$  is zero at  $\omega$ .

## 5 Closedness of Differential Forms

Note that allowing the end points of the curves to vary either, will just simplify the proofs as we can now express the differentiation of the path integrals within the Banachspace of continuous differentiable curves  $\omega : [0, 1] \rightarrow \mathbb{R}^n$ , instead of dealing with affine Banachspaces, but other than that, the results will carry over. Only the symbol of variation  $\delta$  is commonly replaced by the differential symbol  $d$ , and is then called “total differential”.

Albeit  $L$  is just a function, it can be rewritten as a differential 1-form  $\alpha$ , namely by the use of the Legendre “transformation”.  $L = p \cdot \dot{q} - H(t, p, q)$  by taking its differentials:  $\alpha := p \cdot dq - H(t, p, q)dt$ . That way, if that form is well-defined on a simply connected, or perhaps even convex open set  $\Omega \subset \mathbb{R}^n$  of location coordinates, and is integrable in there, which means that path integration along piecewise continuously differentiable, closed paths  $\omega : [0, 1] \rightarrow \Omega$  all vanish, then  $\alpha$  is the exterior differential  $dS$  of an action function  $S : [0, 1] \times \Omega \ni (t, q) \mapsto S(t, q) \in \mathbb{R}$ . And then  $S$  also is unique up to an additive constant as well as eventually time and space dilatations  $t \mapsto t - t_0$ ,  $q \mapsto q - q_0$  of the time and space coordinates (in case of conserved energy and momentum). In case  $\Omega$  is just open and not simply connected, then at least we can find such a function within the open  $\epsilon$ -balls around each  $q \in \Omega$ .

Now, a differential form  $\beta$ , say, is said to be “closed”, if and only if its exterior derivative  $d\beta$  exists and vanishes, i.e.:  $d\beta \equiv 0$ . And conversely, if  $\beta$  is the exterior differential of another differential form  $\theta$ , say, then  $\beta$  is called “exact”.

This conjures up a peculiarity of the definition of closed forms: The principle theorem of differential forms is Poincaré’s lemma, which states that a differential form  $\beta$  on a simply connected (open) region  $U \subset \mathbb{R}^n$ , for which the exterior derivative is well-defined and continuous on  $U$ , is exact, if and only if  $\beta$  is closed (see: [?]).

The fundamental difference between the derivative of the action integral  $DS(\omega)$  and the external differential is the domain of definition:  $S$  is a function of paths, while the 1-form is defined through an  $(n+1)$ -dimensional vector field  $(t, x) \mapsto (p(t), -H(t, x))$ , on which the derivatives are taken. In the above example,  $S$  is extremal for every non-intersecting continuously differentiable curve  $\omega$ , hence the Lagrange equations hold on all these paths up to a small rest term. But no non-empty open subset of  $\mathbb{R}^2$  exists, on which  $p \cdot dx - Hdt$  is closed.

If the Hamilton function is explicitly independent of time and the location coordinates within the set  $U_\epsilon$  of all  $(t, \eta(t)) \in \mathbb{R}^{n+1}$  with  $\eta \in \Omega_\epsilon$ , then local integrability of  $p \cdot dq - Hdt$  follows. (Because then the directional derivatives of the Hamilton function  $H$  in time and the location coordinates vanish on  $U_\epsilon$ .) That means, in case the Hamilton function is well-defined and continuously

differentiable on  $U_\epsilon$ , conservation of momentum and energy on  $U_\epsilon$  imply local integrability of the system. Now, the conservation of energy and momentum for dynamical systems are mostly expressed globally, not only locally. In these cases, the action integral can be continued  $U_\epsilon$  onto the region  $U \subset \mathbb{R}^{n+1}$  on which Lagrange function  $L$  and/or Hamilton function  $H$  are well-defined, in other words: the differential 1-form  $p \cdot dq - Hdt$  becomes globally integrable.

## 6 Summary

From a mathematical standpoint, the principle of extremal action should be replaced by the more pristine principle of closedness of the exterior 1-form  $p \cdot dq - Hdt$ , to which the principle of extremal action runs up to, if only one wants to derive the Lagrange equations from.

It allows to express the Lagrange equation in simpler terms:

Given any open ball  $B \subset \mathbb{R}^n$  on which  $p \cdot dq - Hdt$  is closed (and therefore exact, i.e. integrable), then it integrates to an action function  $S : B \rightarrow \mathbb{R}$ , for which

- (i)  $p(t, q) = \frac{\partial S(t, q)}{\partial q}$  for all  $t \in [0, 1]$  and  $q \in B$ ,
- (ii)  $E(t, q) = -\frac{\partial S(t, q)}{\partial t}$  for all  $t \in [0, 1]$  and  $q \in B$ , and
- (iii)  $\dot{p}(t, q) = -F(t, q)$ , where  $F := \frac{\partial E(t, q)}{\partial q}$  for all  $t \in [0, 1]$  and  $q \in B$ ,

where the last statement follows from  $\frac{\partial^2 S}{\partial t \partial q} = \frac{\partial^2 S}{\partial q \partial t}$ .

Above all, the closedness of  $p \cdot dq - Hdt$  is equivalent to the closedness of  $\pm p \cdot dq \pm Hdt$ , and it's likewise irrelevant physically, because of the symmetry of time and space inversion (parity). As such, the form  $Hdt + p \cdot dq$  will suffice.

## References

- [1] H. Cartan, *Differential Forms*, Herman Kershaw, 1971.

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