International Journal of Mathematical Analysis
Vol. 14, 2020, no. 3, 109 - 115
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ijma.2020.91288

The Lebesgue Integral of Trapezoid Rule Based on Derivative

Atia Rehman

Department of Mathematics and Statistics
University of Agriculture Faisalabad, Pakistan

Khurrem Shehzad

Department of Mathematics and Statistics
University of Agriculture Faisalabad, Pakistan

Fatima Noreen

Department of Mathematics and Statistics
University of Agriculture Faisalabad, Pakistan

Abstract

In this paper the trapezoid rule for lebesgue integral based on derivative estimated. In which the use of derivative based lebesgue integral at the terminals. This category of quadrature formula acquires an exactness of two order along lebesgue integral of trapezoid rule and explores by an inappropriate numbers. At the end, the conception of trapezoid rule for lebesgue integral indicated.

Keywords: Numerical Integration; Lebesgue Integral, Riemann Stieltjes Integral, Quadrature Rule, Trapezoid Rule, Degree of Precision, Error term

1 Introduction

Numerical integration tends to determine the numerical inaccuracy for several rates of S, then $f(x)$ is defined by area A under the curve [8]. In numerical integration the fundamental problem is to estimate the approximate results of an integral $\int_i^j k(r)dr$ provided order of exactness or precision. If the function $f(x)$ is smooth under the very small quantity or numbers of dimensions and domain (integration) is bounded, then there are several techniques or methods to solve this integrals by obtaining the desirable precision or exactness. In the history "Quadrature" is a mathematical term which represents the calculating area. In the mathematical analysis there are many but quadrature problems provided one of an important and significant source ([2], [3], [1]).

In many fields such as engineering, physics and statistics it has many applications. We need integrals in many practical problems for calculation. As the well known integral, for $k = \int_{i_1}^{i_2} t(h)dh$ which is known as definite integral of $t(h)$ on the closed interval $[i_1, i_2]$. The Newton Leibniz formula for this integral is given by

$$\int_{i_1}^{i_2} t(h)dh = T(i_1) - T(i_2)$$

where $T(h)$ is primitive function of integrand $t(h)$. Although, the functions which are not easily obtained is explicit primitive functions $T(h)$. Simply the meaning of explicit word is "to express clearly". For example, $z = e^y + 1$, $z = ay^3 + by$, $z = cos^{-1}y + 2$, $siny^2$, $\frac{(siny)}{y}$, $e^{\pm y^2}$ etc. However, the integrand $t(h)$ is defined at some points $h_i = 1, 2, 3, 4, ..., n$. How to get challenges in the field of mathematics to obtain the formula of numerical integration of high precision? The term precision means precise or exactness or exact. Also we can defined as the capability of a quantification to be replicated constantly. In other words a value which may be measured by the number of a significant digits is reliably ([9], [5], [4]).

What are the interpolation polynomials? Why we use? Firstly, between known data points the estimation of the values is called interpolation and this method is called interpolation polynomial method. Interpolation polynomials are usually based on the method of quadrature which is also written in this form

$$\int_{i_1}^{i_2} t(h)dh \approx \sum_{i=0}^{n} w_i t(h_i)$$

where the integration points are $n+1$ which are distinct at $h_o, h_1, ..., h_n$ on the closed interval $[i_1, i_2]$ and the weights w_i are also $n+1$. On the closed interval the distribution of the integration points are consistent. Where $h_i = h_o + is$ in which $s = \frac{(b-a)}{n}$. These weight w_i can be find out in many different methods. At $n+1$ points $h_o, h_1, ..., h_n$ interpolating the function $t(h)$. To obtain the function firstly avail lagrange polynomials and then integrate above polynomial. Other part of the polynomial is depend on exactness of the Quadrature formula. Here select $w_i, i = 0, 1, 2, ..., n$ then the error term is written as,

$$R_n(t) = \int_{i_1}^{i_2} t(h)dh - \sum_{i=0}^{n} w_i t(h_i)$$

is equal to zero when $t(h) = h^j, j = 0, 1, 2, ..., n$. For weights w_i this polynomial gives the $n + 1$ system of linear equations by using the coefficient of undetermined method. Therefore $1, h, ..., h^n$ are linearly independent monomials which gives the distinctive solution of system of linear equations. The well known and most important numerical integration rule is trapezoidal rule. The classical Riemann integral of trapezoidal rule is given as

$$\int_{i_1}^{i_2} t(h)dh = \frac{i_2 - i_1}{2}(t(i_1) + t(i_2)) - \frac{(i_2 - i_1)^3}{12} t''(\xi),$$

where $\xi \in (i_1, i_2)$.
Let us take a differentiable function $g : [i_1, i_2] \to R$ at any point of the interval $[i_1, i_2]$. Is on the interval $[i_1, i_2]$ t' also integrable? The answer of this question is totally depend on the integral which we are use. By the area the Riemann stieltjes integral is usually activate. If a function is bounded as $f : [a, b] \to R$ then the lebesgue integral on the interval $[a, b]$ if and only if f is measurable function. What is lebesgue stieltjes integral? Lebesgue integral is a flexible and powerful concept of integration. Thomas stieltjes was the first mathematician who was the responsible to gave the concept of the integration of one function w.r.t another function. We observed that functions are very simple in nature not Riemann integrable and so one has to consider more general integrals. The success came along in the direction through the pioneer work initiated by lebesgue in 1902 and the notion of the integral introduced by him is called lebesgue integral ([10], [6]).
One chief difference between Riemann and lebesgue integrals that in the definition of Riemann integral we use intervals and their lengths whereas in the case of lebesgue integral we use more general sets namely measurable sets and their measures. The lebesgue integration generalizes classical Riemann integration for real valued functions which is continuous on $[a, b]$ to the case of extended real values measurable functions. The question is that how one can extend notion of Riemann integral to lebesgue integral? The usual concept of Riemann integral is based on dividing the domain of the function f into finer and finer pieces ([7]).

2 Preliminary Notes / Materials and Methods

The trapezoid rule for Lebesgue integral based on derivative and trapezoid rule for Riemann Stieltjes integral based on area are estimated in which we were used two terms based on derivatives and area at the terminals.The error term also investigated.

3 Results and Discussion

These are the main results of the paper.

Theorem 3.1 *If p' and q are the continuous functions on the interval $[v, h]$ and q is an increasing function. The derivative based trapezoidal rule for the lebesgue integral is*

$$F = \int_i^j (G - H)\, p(v) + \int_i^j (E)\, p(h) + \int_i^j (I)\, p'(v) + \int_i^j (J)\, p'(h). \qquad (1)$$

where

$$\frac{12}{(h-v)^3} \int_v^h \int_v^s \int_v^n q(r) dr dn ds - q(v) = H$$

$$\frac{6}{(h-v)^2} \int_v^h \int_v^s q(r) dr ds = G$$

$$\int_i^j \int_v^h p(s) dq = F$$

$$q(h) - \frac{6}{(h-v)^2} \int_v^h \int_v^s q(r) dr ds + \frac{12}{(h-v)^3} \int_v^h \int_v^s \int_v^n q(r) dr dn ds = E$$

$$\frac{2}{h-v} \int_v^h \int_v^s q(r) dr ds - \frac{6}{(h-v)^2} \int_v^h \int_v^s \int_v^n q(r) dr dn ds = I$$

$$\frac{4}{h-v} \int_v^h \int_v^s q(r) dr ds - \frac{6}{(h-v)^2} \int_v^h \int_v^s \int_v^n q(r) dr dn ds - \int_h^v q(s) ds = J$$

Proof: Let us consider,

$$\int_v^h p(s) dq = v_o p(v) + v_1 p(h) + h_o p'(v) + h_1 p'(h). \qquad (2)$$

Put $p(s) = 1, p(s) = s, p(s) = s^2, p(s) = s^3$ in equation (2) we have,

$$\int_v^h 1 dq = v_o + v_1, \qquad (3)$$

$$\int_v^h s dq = v_o v + v_1 h + h_o + h_1, \qquad (4)$$

$$\int_v^h s^2 dq = v_o v^2 + v_1 h^2 + 2h_o v + 2h_1 h, \qquad (5)$$

$$\int_v^h s^3 dq = v_o v^3 + v_1 h^3 + 3h_o v^2 + 3h_1 h^2. \qquad (6)$$

Now integrate equation (3) we have,

$$q(h) - q(v) = v_o + v_1. \tag{7}$$

Integrate equation (4) we get,

$$hq(h) - vq(v) - \int_v^h q(s)ds = v_o v + v_1 h + h_o + h_1. \tag{8}$$

Integrate equation (5) we get,

$$h^2 q(h) - v^2 q(v) - 2h \int_v^h q(s)ds + 2\int_v^h \int_v^s q(r)drds = v_o v^2 + v_1 h^2 + 2h_o v + 2h_1 h \tag{9}$$

Integrate equation (6) we have,

$$Q = v_o v^3 + v_1 h^3 + 3h_o v^2 + 3h_1 h^2. \tag{10}$$

where

$$h^3 q(h) - v^3 q(v) - 3h^2 \int_v^h q(s)ds + 6h \int_v^h \int_v^s q(r)drds - 6\int_v^h \int_v^s \int_v^n q(r)drdnds = Q$$

Solving simultaneously (7), (8), (9) and (10) for v_o, v_1, h_o and h_1 so we obtain,

$$\begin{aligned}
v_o &= \frac{6}{(h-v)^2}\int_v^h \int_v^s q(r)drds - \frac{12}{(h-v)^3}\int_v^h \int_v^s \int_v^n q(r)drdnds - q(v) \\
v_1 &= q(h) - \frac{6}{(h-v)^2}\int_v^h \int_v^s q(r)drds + \frac{12}{(h-v)^3}\int_v^h \int_v^s \int_v^n q(r)drdnds \\
h_o &= \frac{2}{h-v}\int_v^h \int_v^s q(r)drds - \frac{6}{(h-v^2)}\int_v^h \int_v^s \int_v^n q(r)drdnds \\
h_1 &= \frac{4}{(h-v)}\int_v^h \int_v^s q(r)drds - \frac{6}{(h-v)^2}\int_v^h \int_v^s \int_v^n q(r)drdsdn - \int_v^h q(s)ds
\end{aligned}$$

so we have the derivative based trapezoid rule for the lebesgue integral as desired.

Theorem 3.2 *Suppose that $p^{(4)}$ and q' are continuous on the interval $[v, h]$ and q is increasing there. The derivative based trapezoid rule for the Riemann stieltjes integral with the error term is*

$$\int_v^h p(s)dq = (T)\,p(v) + (X)\,p(h) + (Z)\,p'(v) + (B)\,p'(h) + (U)\,p^{(4)}(\xi)q'(\eta) \tag{11}$$

where $\xi, \eta \epsilon (v, h)$

$$\frac{6}{(h-v)^2}\int_v^h \int_v^s q(r)drds - \frac{12}{(h-v)^3}\int_v^h \int_v^s \int_v^n q(r)drdnds - q(v) = T$$

$$q(h)-\frac{6}{(h-v)^2}\int_v^h\int_v^s q(r)drds+\frac{12}{(h-v)^3}\int_v^h\int_v^s\int_v^n q(r)drdnds=X$$
$$\frac{2}{h-v}\int_v^h\int_v^s q(r)drds-\frac{6}{(h-v^2)}\int_v^h\int_v^s\int_v^n q(r)drdnds=Z$$
$$\frac{4}{(h-v)}\int_v^h\int_v^s q(r)drds-\frac{6}{(h-v)^2}\int_v^h\int_v^s\int_v^n q(r)drdsdn-\int_v^h q(s)ds=B$$
$$\frac{(h-v)^2}{12}\int_v^h\int_v^s q(r)drds\int_v^h\int_v^s\int_v^e\int_v^n q(r)drdndeds=U$$

. *And the error term* $R[p]$ *of this method is*

$$W+\int_v^h\int_v^s\int_v^e\int_v^n q(r)drdndeds\times p^{(4)}(\xi)q'(\eta)$$

where

$$\frac{(h-v)^2}{12}\int_v^h\int_v^s q(r)drds-\frac{(h-v)}{12}\int_v^h\int_v^s\int_v^n q(r)drdsdn=W$$

Proof Let $p(s)=s^4/4!$ so,

$$\begin{aligned}\frac{1}{4!}\int_v^h s^4dq &= \frac{1}{24}\left(h^4q(h)-v^4q(v)\right)-\frac{h^3}{6}\int_v^h q(s)ds\\ &+ \frac{h^2}{2}\int_v^h\int_v^s q(r)dr-h\int_v^h\int_v^s\int_v^n q(r)drdnds\\ &- h\int_v^h\int_v^s\int_v^n q(r)drdnds+\int_v^h\int_v^s\int_v^e\int_v^n q(r)drdndeds\end{aligned}$$

By above theorem, we have

$$\begin{aligned}T &= \left(\frac{6}{(h-v)^2}\int_v^h\int_v^s q(r)drds-\frac{12}{(h-v)^3}\int_v^h\int_v^s\int_v^n q(r)drdnds-q(v)\right)\frac{v^4}{24}\\ &+ \left(q(h)-\frac{6}{(h-v)^2}\int_v^h\int_v^s q(r)drds+\frac{12}{(h-v)^3}\int_v^h\int_v^s\int_v^n q(r)drdnds\right)\frac{h^4}{24}\\ &+ \left(\frac{2}{h-v}\int_v^h\int_v^s q(r)drds-\frac{6}{(h-v^2)}\int_v^h\int_v^s\int_v^n q(r)drdnds\right)\frac{v^3}{6}\\ &+ \left(\frac{4}{(h-v)}\int_v^h\int_v^s q(r)drds-\frac{6}{(h-v)^2}\int_v^h\int_v^s\int_v^n q(r)drdsdn-\int_v^h q(s)ds\right)\frac{h^3}{6}\end{aligned}$$

subtracting above two equations we obtain,

$$\begin{aligned}\frac{1}{4!}\int_v^h s^4dq-T &= \frac{(h-v)^2}{12}\int_v^h\int_v^s q(r)q(r)drds-\frac{h-v}{12}\int_v^h\int_v^s\int_v^n q(r)\\ drdnds &+ \int_v^h\int_v^s\int_v^e\int_v^n q(r)drdndeds.\end{aligned}$$

This implies that

$$R[p]=W+\int_v^h\int_v^s\int_v^e\int_v^n q(r)drdndedsp^4(\xi)q'(\eta) \qquad (12)$$

4 Conclusion

The lebesgue integral of trapezoid rule based on derivative and its error terms are estimated.

Acknowledgments. The authors express their sincere gratitude to Asad Ali for useful discussions and invaluable advice.

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Received: March 27, 2020; Published: April 15, 2020