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L^∞-Norm Estimates of the Solution for the Incompressible Navier-Stokes Equations for Non-Decaying Initial Data

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Abstract

In this paper, we reprove the principal result of a paper by H-O Kreiss and Jens Lorenz from a "different approach" than the method proposed in their paper. More precisely, we consider the Cauchy problem for the incompressible Navier-Stokes equations in $\mathbb{R}^n$ for $n \geq 3$ with non-decaying initial data and derive a priori estimates of the maximum norm of all derivatives of the solution in terms of the maximum norm of the initial data. This paper is also an extension of their paper to higher dimension.

Mathematics Subject Classification: 35G25, 35Q30; 76D03, 76D05

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1 Introduction

We consider the Cauchy problem of the Navier-Stokes equations in $\mathbb{R}^n$ for $n \geq 3$:

$$u_t + u \cdot \nabla u + \nabla p = \triangle u, \quad \nabla \cdot u = 0, \tag{1.1}$$

with initial condition

$$u(x,0) = f(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $u = u(x,t) = (u_1(x,t), \cdots u_n(x,t))$ and $p = p(x,t)$ stand for the unknown velocity vector field of the fluid and its pressure, while $f = f(x) = (f_1(x), \cdots f_n(x))$ is the given initial velocity vector field. In what follows, we will use the same notations for the space of vector-valued and scalar functions for convenience in writing.

There is a large literature on the existence and uniqueness of solution of the Navier-Stokes equations in $\mathbb{R}^n$. For given initial data solutions of (1.1) and (1.2) have been constructed in various function spaces. For example, if $f \in L^r$ for some r with $3 \leq r < \infty$, then it is well known that there is a unique classical solution in some maximum interval of time $0 \leq t < T_f$ where $0 < T_f \leq \infty$. But for the uniqueness of the pressure one requires $|p(x,t)| \to 0$ as $|x| \to \infty$. See [6] and [9] for $r = 3$ and [1] for $3 < r < \infty$.

If $f \in L^\infty$ then existence of a regular solution follows from [2]. The solution is only unique if one puts some growth restrictions on the pressure as $|x| \to \infty$. A simple example of non-uniqueness is demonstrated in [7] where the velocity u is bounded but $|p(x,t)| \leq C|x|$. In addition, an estimate $|p(x,t)| \leq C(1 + |x|^\sigma)$ with $\sigma < 1$ (see [3]) implies uniqueness. Also, the assumption $p \in L^1_{loc}(0,T; BMO)$ (see [4]) implies uniqueness.

In this paper we are interested in reproving the results of a paper by H-O Kreiss and J. Lorenz (see [8]) for the initial data $f \in L^\infty(\mathbb{R}^n)$ for $n \geq 3$ using different approach than theirs in terms of dealing with the pressure term in the Navier-Stokes equations. The approach in this paper, to prove the principal result of the Kreiss and Lorenz paper, is more "functional analytic" approach in which the role of "the Leray projector" is being implemented to get rid of the pressure term from the Navier-Stokes equations. As a consequence of that, the details and techniques in obtaining some significantly complicated results related to the pressure part in the Kreiss and Lorenz paper are being avoided which makes this paper different and simpler in that sense. At the same time, this paper is also an extension of the work by Kreiss and Lorenz to the higher space dimension whereas such generalization, in the Kreiss and Lorenz paper by their approach, seems complicated because of the non-local nature of the pressure term in the Navier-Stokes equations. Since the main source of this paper is the Kreiss and Lorenz paper, it is appropriate to give some insight of their work in this paper as well. Before we start outlining some key aspects of their paper, we introduce the following notations and will be using them throughout this paper.

$$|f|_\infty = \sup_x |f(x)| \quad \text{with} \quad |f(x)|^2 = \sum_i f_i^2(x),$$

and $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, D_i = \partial/\partial x_i$ for a multiindex $\alpha = (\alpha_1, \cdots, \alpha_n)$. In what follows, if $|\alpha| = j$, for any $j = 0, 1, \cdots$, then we will denote $D^\alpha = D_1^\alpha \cdots D_n^{\alpha_n}$ by D^j. We also set

$$|\mathcal{D}^j u(t)|_\infty := |\mathcal{D}^j u(\cdot, t)|_\infty = \max_{|\alpha|=j} |D^\alpha u(\cdot, t)|_\infty.$$

Clearly, $|\mathcal{D}^j u(t)|_\infty$ measures all space derivatives of order j in maximum norm.

Following theorem is the main result of the paper by Kreiss and Lorenz [8] for $n = 3$ which is also the principal result of this paper for $n \geq 3$.

Theorem 1.1. *Consider the Cauchy problem for the Navier-Stokes equations (1.1), (1.2), where $f \in L^\infty(\mathbb{R}^n)$ and $\nabla \cdot f = 0$ is understood in the sense of distribution. There is a constant $c_0 > 0$ and for every $j = 0, 1, \cdots$ there is a constant K_j so that*

$$t^{j/2} |\mathcal{D}^j u(t)|_\infty \leq K_j |f|_\infty \quad \textit{for} \quad 0 < t \leq \frac{c_0}{|f|_\infty^2}. \tag{1.3}$$

The constants c_0 and K_j are independent of t and f.

Let us briefly discuss some key ideas of the Kreiss and Lorenz paper. Rewrite (1.1) as

$$u_t = \triangle u + Q$$

where

$$Q = -\nabla p - u \cdot \nabla u.$$

Applying D^j for $j \geq 0$ and using *Duhamel's principle*, one obtains

$$v(t) = D^j e^{\triangle t} f + \int_0^t e^{\triangle (t-s)} D^j Q(s) ds, \quad v := D^j u. \tag{1.4}$$

Roughly speaking, obtaining the desired result of the Kreiss and Lorenz paper is a twofold in view of equation (1.4): first, estimates on the solution of the heat equation. Second, estimates on the derivatives of Q. Also, notice in (1.4), one can move one derivative D to the heat semi-group and consequently requiring an estimate for $|D^{j-1} Q|_\infty$ to estimate $|v(t)|_\infty$. Clearly, it is necessary to determine the pressure term p of the Navier-Stokes equations so that (u, p) solves (1.1) and (1.2); the estimate of the derivatives of p is being used

to estimate $|D^{j-1}Q|_\infty$. To proceed towards obtaining the required estimates on the pressure, Kreiss and Lorenz determine the pressure from the Poisson equation

$$\triangle p = -\nabla \cdot (u \cdot \nabla)u \tag{1.5}$$

which is given by

$$p = \sum_{i,j} R_i R_j (u_i u_j), \tag{1.6}$$

where $R_i = (-\triangle)^{-1/2} D_i$ is the ith Riesz transform. Since the Riesz transforms are not bounded in $L^\infty(\mathbb{R}^n)$, the pressure term $p \in L^1_{loc}(0,T;BMO)$ where BMO is the space of functions of bounded mean oscillation. Because of the non-local nature of the pressure, the proof of Theorem 1.1 of the Kreiss and Lorenz paper is complicated, however. This is where the method proposed in this paper deviates significantly from the approach adopted by Kreiss and Lorenz in their paper [8].

For the purpose of proving Theorem 1.1 for $n \geq 3$, we start by transforming momentum equation of the Navier-Stokes equations into the abstract ordinary differential equation for u

$$u_t = \triangle u - \mathbb{P}(u \cdot \nabla)u \tag{1.7}$$

by eliminating the pressure, where $\mathbb{P}$ is the Leray projector defined by

$$\mathbb{P} = (\mathbb{P}_{ij})_{1 \leq i,j \leq n}, \quad \mathbb{P}_{ij} = \delta_{ij} + R_i R_j;$$

where R_i is as in (1.6) and δ_{ij} is the Kronecker delta function. Note that the equation (1.7) is obtained from (1.1) by applying the Leray projector with the properties $\mathbb{P}(\nabla p) = 0, \mathbb{P}(\triangle u) = \triangle u$, since $\nabla \cdot u = 0$. We use the solution operator $e^{\triangle t}$ of the heat equation to transform the abstract differential equation into an integral equation

$$u(t) = e^{\triangle t} f - \int_0^t e^{\triangle (t-s)} \mathbb{P}(u \cdot \nabla u)(s) ds \quad t > 0. \tag{1.8}$$

In a paper by Giga and others [5] for $n \geq 2$, they consider the initial data $f \in BUC(\mathbb{R}^n)$ which is the space of all bounded uniformly continuous functions or in $L^\infty(\mathbb{R}^n)$ which is the space of all essentially bounded functions, and construct a unique local in time solution of (1.8). Such solution of (1.8) is called mild solution of (1.1) and (1.2). They later proved in the same paper that

such mild solution is indeed a strong solution of the Navier-Stokes equations (1.1) and (1.2) in some maximum interval of time. In addition, for essentially bounded initial data, existence and uniqueness of a solution of (1.1) and (1.2) is also proved in [2]; however, Giga and others in [5] claim that their approach is simpler than the method proposed in [2]. In the same paper [5], while constructing such mild solution of (1.1) and (1.2), it requires to obtain the estimate $t^{1/2}|\nabla u|_\infty \leq C|f|_\infty$ for some constant $C > 0$ independent of t and f in some maximum interval of time. However, such maximum norm estimates for higher order derivatives of the velocity field had not been achieved until H-O Kreiss and J. Lorenz obtained in [8] for $f \in L^\infty(\mathbb{R}^3)$.

The main work of this paper will focus on deriving estimate (1.3) of Theorem 1.1 by a "different approach" in a few ways than that of the Kreiss and Lorenz paper adopts. At the same time, this paper will also demonstrate the fact, the absence of the pressure term in the transformed abstract differential equation (1.7) eliminates significant amount of work of the paper by Kreiss and Lorenz while obtaining the uniform estimates of the pressure and its derivatives. However, there are some intriguing developments in the work of this paper due to the application of the Leray projector in our "different approach".

Major difficulty in proving Theorem 1.1 lies in the fact that the Leray projector $\mathbb{P}$ is not a bounded operator in $L^\infty(\mathbb{R}^n)$, since the Riesz transforms are not bounded in this space although they are bounded in $L^r(\mathbb{R}^n)$ for $1 < r < \infty$. To overcome the difficulty, we obtain an uniform bound on the composite operator $D^j e^{\triangle t}\mathbb{P}$ for $j = 0, 1 \cdots$ in section 2.

This paper is organized in the following ways: In section 2, we introduce a few estimates for the solution of the heat equations. In the same section, we also state and prove a few known results which will be used later in sections 3 and 4. In section 3, we introduce an analogous system of (1.1) for the illustrative purpose, introduce the analogous theorem 3.1 of Theorem 1.1 and provide proof of Theorem 3.1. In section 4, we prove Theorem 1.1 using the same techniques as in the proof of Theorem 3.1 of section 3. Finally, in section 5 we outline remarks on the use of the estimate (1.3) obtained in Theorem 1.1.

2 Some auxiliary results

Let us consider $f \in L^\infty(\mathbb{R}^n)$. The solution of

$$u_t = \triangle u, \quad u = f \quad \text{at} \quad t = 0,$$

is denoted by

$$u(t) := u(\cdot, t) = e^{\triangle t} f = \theta * f$$

where $\theta(t) := \theta(x,t) = 1/(4\pi t)^{n/2} e^{-|x|^2/4t}, t > 0$ is the n dimensional heat kernel in $\mathbb{R}^n$ and $*$ is the convolution operator. It is well known that

$$|e^{\triangle t} f|_\infty \leq |f|_\infty, \quad t \geq 0 \tag{2.1}$$

and

$$|\mathcal{D}^j e^{\triangle t} f|_\infty \leq C_j t^{-j/2} |f|_\infty, \quad t > 0, \quad j = 1, 2, \cdots \tag{2.2}$$

Here, and in the following C, C_j, c, etc are positive constants that are independent of t and the initial function f.

Lemma 2.1. *Let $\theta(t) = \theta(x,t)$ be the n-dimensional heat kernel in $\mathbb{R}^n$. Then, for every $j = 1, 2 \cdots$ and every $t > 0, D^j\theta(t)$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ and*

$$||D^j\theta(t)||_{\mathcal{H}^1(\mathbb{R}^n)} \leq C_j t^{-j/2}. \tag{2.3}$$

for some constant C_j.

Proof. First, let us recall the definition of the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$.

$$\mathcal{H}^1(\mathbb{R}^n) = \{u \in L^1(\mathbb{R}^n) \quad s.t \quad \sup_{s>0} |h_s * u| \in L^1(\mathbb{R}^n)\}$$

for some Schwartz class function h where $h_s(x) = s^{-n} h(\frac{x}{s}), s > 0$ such that $0 \leq h \leq 1$ and $\int h = 1$. We may endow $\mathcal{H}^1(\mathbb{R}^n)$ with the norm

$$||u||_{\mathcal{H}^1(\mathbb{R}^n)} = ||u||_{L^1(\mathbb{R}^n)} + ||\sup_{s>0} |h_s * u|||_{L^1(\mathbb{R}^n)}.$$

For any $j = 0, 1, \cdots$, we want to prove $D^j\theta(t) \in \mathcal{H}^1(\mathbb{R}^n)$, that means, it suffices to show that $\sup_{s>0} |h_s * D^j\theta(t)| \in L^1(\mathbb{R}^n)$. For that, we take $h(x) = \theta(x, 1)$ and notice

$$|h_s * D^j\theta(t)(x)| = |D^j\theta(t+s, x)|$$

so that

$$\sup_{s>0} |h_s * D^j\theta(t)| = |D^j\theta(t)| \in L^1(\mathbb{R}^n).$$

Finally, we arrive at

$$||D^j\theta(t)||_{\mathcal{H}^1(\mathbb{R}^n)} \leq C_j t^{-j/2}, \quad t > 0.$$

□

Lemma 2.2. *For any $f \in L^\infty(\mathbb{R}^n)$. Let $j \geq 1$, there is a constant C_j such that*

$$|\mathcal{D}^j e^{\triangle t}\mathbb{P}f|_\infty \leq C_j t^{-j/2}|f|_\infty \quad \textit{for} \quad 0 < t \leq T. \tag{2.4}$$

Proof. For $1 \leq i \leq n$ and $t > 0$, by the definition of the Leray projector, we write

$$\begin{aligned}(D^j e^{\triangle t}\mathbb{P}f)_i &= D^j e^{\triangle t} f_i + \sum_{l=1}^n D^j e^{\triangle t} R_i R_l f_l \\ &= D^j \theta(t) * f_i + \sum_{l=1}^n D^j R_i R_l \theta(t) * f_l \\ &= \sum_{l=1}^n (\delta_{il} + R_i R_l)((D^j\theta(t)) * f_l) \\ &= \sum_{l=1}^n k_{il}(t) * f_l\end{aligned}$$

where the kernel $k_{il}(t) = (\delta_{il} + R_i R_l)(D^j\theta(t))$. Since the Riesz transforms are bounded in $\mathcal{H}^1(\mathbb{R}^n)$ and $||\cdot||_{L^1(\mathbb{R}^n)} \leq ||\cdot||_{\mathcal{H}^1(\mathbb{R}^n)}$, we have

$$\begin{aligned}||k_{il}(t)||_{L^1(\mathbb{R}^n)} &\leq ||k_{il}(t)||_{\mathcal{H}^1(\mathbb{R}^n)} \\ &\leq ||D^j\theta(t)||_{\mathcal{H}^1(\mathbb{R}^n)}.\end{aligned}$$

Thus, from previous Lemma 2.1 we obtain

$$||k_{il}(t)||_{L^1(\mathbb{R}^n)} \leq C_j t^{-j/2} \quad \text{for} \quad t > 0.$$

Finally, by the Young's inequality of convolution we estimate as

$$\begin{aligned}|(D^j e^{\triangle t}\mathbb{P}f)_i|_\infty &\leq \sum_{l=1}^n |k_{il} * f_l|_\infty \\ &\leq C||k_{il}(t)||_{L^1(\mathbb{R}^n)}|f_l|_\infty \\ &\leq C_j t^{-j/2}|f_l|_\infty.\end{aligned}$$

Hence Lemma 2.2 is proved. □

Corollary 2.3. *Let $F \in L^\infty(\mathbb{R}^n \times [0,T])$ then the solution of*

$$u_t = \triangle u + D_i \mathbb{P}F, \quad u = 0 \quad \textit{at} \quad t = 0 \tag{2.5}$$

satisfies

$$|u(t)|_\infty \leq Ct^{1/2} \max_{0 \leq s \leq t} |F(s)|_\infty, \quad 0 < t \leq T \tag{2.6}$$

for some $T > 0$.

Proof. The solution of (2.5) is given by

$$u(t) = \int_0^t e^{\triangle(t-s)} D_i \mathbb{P} F(s) ds, \quad 0 < t \leq T$$

and

$$|u(t)|_\infty \leq \int_0^t |e^{\triangle(t-s)} D_i \mathbb{P} F(s)|_\infty ds.$$

Since D_i commutes with the heat semi-group, applying Lemma 2.2 for $j = 1$ yields

$$|u(t)|_\infty \leq \max_{0 \leq s \leq t} |F(s)|_\infty \int_0^t (t-s)^{-1/2} ds.$$

Hence we obtain

$$|u(t)|_\infty \leq C t^{1/2} \max_{0 \leq s \leq t} |F(s)|_\infty.$$

□

3 Estimates for the system $u_t = \triangle u + D_i \mathbb{P} g$

In this section, we state and prove an analogous theorem of Theorem 1.1 for the solution of an analogous system of the Navier-Stokes equations (1.1) and (1.2). For that purpose, let us recall $\mathbb{P}(u \cdot \nabla) u = \sum_i D_i \mathbb{P}(u_i u)$ for $1 \leq i \leq n$. Therefore, it is appropriate to consider the illustrative system to be

$$u_t = \triangle u + D_i \mathbb{P} g(u(x,t)), \quad x \in \mathbb{R}^n, \quad t \geq 0 \tag{3.1}$$

with initial function

$$u(x,0) = f(x) \quad \text{where} \quad f \in L^\infty(\mathbb{R}^n). \tag{3.2}$$

Here $g : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be quadratic in u. We will prove the maximum norm estimates of the derivatives of the solution of (3.1) and (3.2) by the maximum norm estimate of the initial function f. It is well-known that the solution is C^∞ in a maximal interval $0 < t < T_f$ where $0 < T_f \leq \infty$.

Theorem 3.1. *Under the assumptions on f and g mentioned above, the solution of (3.1) and (3.2) satisfies the following:*

(a) There is a constant $c_0 > 0$ with

$$T_f > \frac{c_0}{|f|_\infty^2}$$

and

$$|u(t)|_\infty \le 2|f|_\infty \quad \textit{for} \quad 0 \le t \le \frac{c_0}{|f|_\infty^2}. \tag{3.3}$$

(b) For every $j = 1, 2, \cdots$, there is a constant $C_j > 0$ with

$$t^{j/2}|\mathcal{D}^j u(t)|_\infty \le C_j |f|_\infty \quad \textit{for} \quad 0 < t \le \frac{c_0}{|f|_\infty^2}. \tag{3.4}$$

The constants c_0 and C_j are independent of t and f.

Proof of part (a) will be given in the following lemma. Subsequently, we will also derive the estimate (3.4) of part (b). Consider u as the solution of the inhomogeneous heat equation $u_t = \triangle u + D_i \mathbb{P} F$ where

$$F(x,t) := g(u(x,t)) \quad \text{for} \quad x \in \mathbb{R}^n, \quad 0 \le t < T_f.$$

Since g is quadratic in u, there is a constant C_g such that we have the following:

$$|g(u)| \le C_g |u|^2, \quad |g_u(u)| \le C_g |u| \quad \text{for all} \quad u \in \mathbb{R}^n. \tag{3.5}$$

Next lemma estimates the maximum norm of u.

Lemma 3.2. *Let C_g denote the constant in (3.5) and let C denote the constant in (2.6); set $c_0 = \frac{1}{16C^2C_g^2}$. Then we have $T_f > c_0/|f|_\infty^2$ and*

$$|u(t)|_\infty < 2|f|_\infty \quad \textit{for} \quad 0 \le t < \frac{c_0}{|f|_\infty^2}. \tag{3.6}$$

Proof. Suppose (3.6) does not hold, then we can find the smallest time t_0 such that $|u(t_0)|_\infty = 2|f|_\infty$. Since t_0 is the smallest time so we have $t_0 < c_0/|f|_\infty^2$. Now by (2.1) and (2.6) we have

$$\begin{aligned}
2|f|_\infty &= |u(t_0)|_\infty \\
&\le |f|_\infty + C t_0^{1/2} \max_{0 \le s \le t_0} |g(s)|_\infty \\
&\le |f|_\infty + C C_g t_0^{1/2} \max_{0 \le s \le t_0} |u(s)|_\infty^2 \\
&\le |f|_\infty + C C_g t_0^{1/2} 4 |f|_\infty^2.
\end{aligned}$$

This gives

$$1 \leq 4CC_g t_0^{1/2}|f|_\infty,$$

therefore $t_0 \geq 1/(16C^2C_g^2|f|_\infty^2) = c_0/|f|_\infty^2$ which is a contradiction. Therefore (3.6) must hold. The estimate $T_f > c_0/|f|_\infty^2$ is valid since $\limsup_{t\to T_f}|u(t)|_\infty = \infty$ if T_f is finite. $\square$

Now, we prove estimate (3.4) of Theorem 3.1 by induction on j. Let $j \geq 1$ and assume

$$t^{k/2}|\mathcal{D}^k u(t)|_\infty \leq K_k|f|_\infty, \quad \text{for} \quad 0 \leq t \leq \frac{c_0}{|f|_\infty^2} \quad \text{and} \quad 0 \leq k \leq j-1. \tag{3.7}$$

where c_0 is the same constant as in the previous lemma. Next, we begin by applying D^j to the equation $u_t = \triangle u + D_i\mathbb{P}g(u)$ to obtain

$$v_t = \triangle v + D^{j+1}\mathbb{P}g(u), \quad v := D^j u,$$
$$v(t) = D^j e^{\triangle t} f + \int_0^t e^{\triangle(t-s)} D^{j+1}(\mathbb{P}g(u))(s)ds.$$

Using (2.2) we get

$$t^{j/2}|v(t)|_\infty \leq C|f|_\infty + t^{j/2}\left|\int_0^t e^{\triangle(t-s)} D^{j+1}(\mathbb{P}g(u))(s)ds\right|_\infty. \tag{3.8}$$

We split the integral into

$$\int_0^{t/2} + \int_{t/2}^t =: I_1 + I_2$$

and obtain

$$\begin{aligned}|I_1(t)| &= \left|\int_0^{t/2} D^{j+1}e^{\triangle(t-s)}(\mathbb{P}g(u))(s)ds\right|_\infty \\ &\leq \int_0^{t/2} |D^{j+1}e^{\triangle(t-s)}(\mathbb{P}g(u))(s)ds|_\infty ds.\end{aligned}$$

Using the inequality in Lemma 2.2, we get

$$\begin{aligned}|I_1(t)|_\infty &\leq C\int_0^{t/2}(t-s)^{-(j+1)/2}|g(u(s))|_\infty ds \\ &\leq C|f|_\infty^2 t^{(1-j)/2}.\end{aligned}$$

The integrand in I_2 has singularity at $s = t$. Therefore, we can move only one derivative from $D^{j+1}\mathbb{P}g(u)$ to the heat semigroup.(If we move two or more derivatives then the singularity becomes non-integrable.) Thus, we have

$$|I_2(t)|_\infty = \left| -\int_{t/2}^{t} De^{\triangle(t-s)}(D^j\mathbb{P}g(u))(s)ds \right|_\infty .$$

Since the Leray projector commutes with any derivatives, therefore

$$|I_2(t)|_\infty = \left| -\int_{t/2}^{t} De^{\triangle(t-s)}(\mathbb{P}D^j g(u))(s)ds \right|_\infty .$$

If we use Lemma 2.2 for $j = 1$, we obtain

$$|I_2(t)|_\infty \leq C \int_{t/2}^{t} (t-s)^{-1/2} |D^j g(u(s))|_\infty ds. \tag{3.9}$$

Since $g(u)$ is quadratic in u, therefore

$$|D^j g(u)|_\infty \leq C|u|_\infty |\mathcal{D}^j u|_\infty + \sum_{k=1}^{j-1} |\mathcal{D}^k u|_\infty |\mathcal{D}^{j-k} u|_\infty .$$

By induction hypothesis (3.7) we obtain

$$\sum_{k=1}^{j-1} |\mathcal{D}^k u(s)|_\infty |\mathcal{D}^{j-k} u(s)|_\infty \leq C s^{-j/2} |f|_\infty^2 . \tag{3.10}$$

Integral (3.9) can be estimated as below:

$$\begin{aligned} |I_2(t)|_\infty &\leq C \int_{t/2}^{t} (t-s)^{-1/2} \left(C|u(s)|_\infty |\mathcal{D}^j u(s)|_\infty + \sum_{k=1}^{j-1} |\mathcal{D}^k u(s)|_\infty |\mathcal{D}^{j-k} u(s)|_\infty \right) ds \\ &= J_1 + J_2 . \end{aligned}$$

Using (3.10), and since $\int_{t/2}^{t} (t-s)^{-1/2} s^{-j/2} ds = C t^{(1-j)/2}$, where C is independent of t, we obtain $|J_2(t)|_\infty \leq C|f|_\infty^2 t^{(1-j)/2}$.

For J_1, we have

$$\begin{aligned} |J_1(t)|_\infty &= C \int_{t/2}^{t} (t-s)^{-1/2} |u(s)|_\infty |\mathcal{D}^j u(s)|_\infty ds \\ &\leq C|f|_\infty \int_{t/2}^{t} (t-s)^{-1/2} s^{-j/2} s^{j/2} |\mathcal{D}^j u(s)|_\infty ds \\ &\leq C|f|_\infty t^{(1-j)/2} \max_{0 \leq s \leq t} \{ s^{j/2} \mathcal{D}^j u(s)|_\infty \}. \end{aligned}$$

We use these bounds to bound the integral in (3.8). We have $v = D^j u$. Then maximizing the resulting estimate for $t^{j/2}|D^j u(t)|_\infty$ over all derivatives D^j of order j and setting

$$\phi(t) := t^{j/2}|\mathcal{D}^j u(t)|_\infty$$

and from (3.8), we obtain the following estimate

$$\phi(t) \le C|f|_\infty + Ct^{1/2}|f|_\infty^2 + C|f|_\infty t^{1/2} \max_{0\le s\le t} \phi(s) \quad \text{for} \quad 0 \le t \le \frac{c_0}{|f|_\infty^2}.$$

Since $t^{1/2}|f|_\infty \le \sqrt{c_0}$ then $Ct^{1/2}|f|_\infty^2 \le C\sqrt{c_0}|f|_\infty$. Therefore

$$\phi(t) \le C_j|f|_\infty + C_j|f|_\infty t^{1/2} \max_{0\le s\le t} \phi(s) \quad \text{for} \quad 0 \le t \le c_0/|f|_\infty^2. \tag{3.11}$$

Let us fix C_j so that the above estimate holds and set

$$c_j = \min\left\{c_0, \frac{1}{4C_j^2}\right\}.$$

First, let us prove the following

$$\phi(t) < 2C_j|f|_\infty \quad \text{for} \quad 0 \le t < \frac{c_j}{|f|_\infty^2}.$$

Suppose there is a smallest time t_0 such that $0 < t_0 < c_j/|f|_\infty^2$ with $\phi(t_0) = 2C_j|f|_\infty$. Then using (3.11) we obtain

$$2C_j|f|_\infty = \phi(t_0) \le C_j|f|_\infty + 2C_j^2|f|_\infty^2 t_0^{1/2},$$

thus

$$1 \le 2C_j|f|_\infty t_0^{1/2} \quad \text{gives} \quad t_0 \ge c_j/|f|_\infty^2$$

which contradicts the assertion. Therefore, we proved the estimate

$$t^{j/2}|\mathcal{D}^j u(t)|_\infty \le 2C_j|f|_\infty \quad \text{for} \quad 0 \le t \le c_j/|f|_\infty^2. \tag{3.12}$$

If

$$T_j := \frac{c_j}{|f|_\infty^2} < t \le \frac{c_0}{|f|_\infty^2} =: T_0 \tag{3.13}$$

then we start the corresponding estimate at $t - T_j$. Using Lemma 3.2, we have $|u(t - T_j)|_\infty \le 2|f|_\infty$ and obtain

$$T_j^{j/2}|\mathcal{D}^j u(t)|_\infty \le 4C_j|f|_\infty. \tag{3.14}$$

Finally, for any t satisfying (3.13)

$$t^{j/2} \le T_0^{j/2} = \left(\frac{c_0}{c_j}\right)^{j/2} T_j^{j/2}$$

and (3.14) yield

$$t^{j/2}|\mathcal{D}^j u(t)|_\infty \le 4C_j \left(\frac{c_0}{c_j}\right)^{j/2} |f|_\infty.$$

This completes the proof of Theorem 3.1.

4 Estimates For the Navier-Stokes Equations

Recall the transformed abstract ordinary differential equation (1.7)

$$u_t = \triangle u - \mathbb{P}(u \cdot \nabla u), \qquad \nabla \cdot u = 0 \tag{4.1}$$

with

$$u(x, 0) = f(x). \tag{4.2}$$

Solution of (4.1) and (4.2) is given by

$$u(t) = e^{\triangle t} f - \int_0^t e^{\triangle (t-s)} \mathbb{P}(u \cdot \nabla u)(s) ds. \tag{4.3}$$

Using the solution (4.3) with previous estimates (2.1),(2.2) and (2.4), we prove the following lemma.

Lemma 4.1. *Set*

$$V(t) = |u(t)|_\infty + t^{1/2}|\mathcal{D}u(t)|_\infty, \qquad 0 < t < T(f). \tag{4.4}$$

There is a constant $C > 0$, independent of t and f, so that

$$V(t) \le C|f|_\infty + Ct^{1/2} \max_{0 \le s \le t} V^2(t), \qquad 0 < t < T(f). \tag{4.5}$$

Proof. Using estimate (2.1) of the heat equation in (4.3), we obtain

$$|u(t)|_\infty \le |f|_\infty + \left| \int_0^t e^{\triangle (t-s)} \mathbb{P}(u \cdot \nabla u)(s) ds \right|_\infty.$$

Apply identity $\mathbb{P}(u \cdot \nabla u) = \sum_i D_i \mathbb{P}(u_i u)$ with the fact, heat semi-group commutes with D_i, then use of the inequality (2.4) in Lemma 2.2 for $j = 1$ to proceed

$$\begin{aligned}
|u(t)|_\infty &\leq |f|_\infty + C \int_0^t (t-s)^{-1/2} |u(s)|_\infty^2 ds \\
&= |f|_\infty + C \int_0^t (t-s)^{-1/2} s^{-1/2} s^{1/2} |u(s)|_\infty^2 ds \\
&\leq |f|_\infty + C \max_{0 \leq s \leq t} \{ s^{1/2} |u(s)|_\infty^2 \} \int_0^t (t-s)^{-1/2} s^{-1/2} ds.
\end{aligned}$$

Since $\int_0^t (t-s)^{-1/2} s^{-1/2} ds = C > 0$ which is independent of t, we have the following estimate

$$\begin{aligned}
|u(t)|_\infty &\leq |f|_\infty + C \max_{0 \leq s \leq t} \{ s^{1/2} |u(s)|_\infty^2 \} \\
|u(t)|_\infty &\leq |f|_\infty + C t^{1/2} \max_{0 \leq s \leq t} V^2(s). \qquad (4.6)
\end{aligned}$$

Apply D_i to (4.1), then by the *Duhamel's principle* gives

$$v(t) = D_i e^{\triangle t} f - \int_0^t e^{\triangle (t-s)} D_i \mathbb{P}(u \cdot \nabla) u(s) ds. \qquad (4.7)$$

We can estimate the integral in (4.7) using Lemma 2.2 for $j = 1$ in the following way:

$$\begin{aligned}
\left| - \int_0^t D_i e^{\triangle (t-s)} \mathbb{P}(u \cdot \nabla u)(s) ds \right| &\leq \int_0^t |D_i e^{\triangle (t-s)} \mathbb{P}(u \cdot \nabla u)(s)| ds \\
&\leq C \int_0^t (t-s)^{-1/2} |u(s)|_\infty |\mathcal{D} u(s)|_\infty ds \\
&= C \int_0^t (t-s)^{-1/2} s^{-1/2} s^{1/2} |u(s)|_\infty |\mathcal{D} u(s)|_\infty ds \\
&\leq C \max_{0 \leq s \leq t} \{ s^{1/2} |u(s)|_\infty |\mathcal{D} u(s)|_\infty \} \int_0^t (t-s)^{-1/2} s^{-1/2} ds \\
&\leq C \max_{0 \leq s \leq t} \{ |u(s)|_\infty^2 + s |\mathcal{D} u(s)|_\infty^2 \}.
\end{aligned}$$

Therefore, we arrive at

$$\begin{aligned}
|v(t)|_\infty &\leq C t^{-1/2} |f|_\infty + C \max_{0 \leq s \leq t} \{ |u(s)|_\infty^2 + s |\mathcal{D} u(s)|_\infty^2 \} \\
t^{1/2} |\mathcal{D} u(t)|_\infty &\leq C |f|_\infty + C t^{1/2} \max_{0 \leq s \leq t} V^2(t). \qquad (4.8)
\end{aligned}$$

Using (4.6) and (4.8), the lemma is proved. □

Lemma 4.2. *Let $C > 0$ denote the constant in estimate (4.5) and set*

$$c_0 = \frac{1}{16C^4}.$$

Then $T_f > c_0/|f|_\infty^2$ and

$$|u(t)|_\infty + t^{1/2}|\mathcal{D}u(t)|_\infty < 2C|f|_\infty \quad \textit{for} \quad 0 \le t < \frac{c_0}{|f|_\infty^2}. \tag{4.9}$$

Proof. We prove this lemma by contradiction after recalling the definition of $V(t)$ in (4.4). Suppose that (4.9) does not hold, then denote by t_0 the smallest time with $V(t_0) = 2C|f|_\infty$. Use (4.5) to obtain

$$\begin{aligned} 2C|f|_\infty &= V(t_0) \\ &\le C|f|_\infty + Ct_0^{1/2}4C^2|f|_\infty^2, \end{aligned}$$

thus

$$1 \le 4C^2 t_0^{1/2}|f|_\infty^2,$$

therefore $t_0 \ge c_0/|f|_\infty^2$. This contradiction proves (4.9) and $T_f > c_0/|f|_\infty^2$. □

Lemma 4.2 proves Theorem 1.1 for $j = 0$ and $j = 1$. By an induction argument as in the proof of Theorem 3.1 one proves Theorem 1.1 for any $j = 0, 1, \cdots$

5 Remarks

We can apply estimate (1.3) of Theorem 1.1 for

$$\frac{c_0}{2|f|_\infty^2} \le t \le \frac{c_0}{|f|_\infty^2} \tag{5.1}$$

and obtain

$$|\mathcal{D}^j u(t)|_\infty \le C_j |f|_\infty^{j+1} \tag{5.2}$$

in interval (5.1). Starting the estimate at $t_0 \in [0, T_f)$ we have

$$|\mathcal{D}^j u(t_0 + t)|_\infty \le C_j |u(t_0)|_\infty^{j+1} \tag{5.3}$$

for

$$\frac{c_0}{2|u(t_0)|_\infty^2} \le t \le \frac{c_0}{|u(t_0)|_\infty^2}. \tag{5.4}$$

Then, if t_1 is fixed with

$$\frac{c_0}{2|f|_\infty^2} \leq t_1 < T_f, \tag{5.5}$$

we can maximize both sides of (5.3) over $0 \leq t_0 \leq t_1$ and obtain

$$\max\left\{|\mathcal{D}^j u(t)|_\infty : \frac{c_0}{2|f|_\infty^2} \leq t \leq t_1 + \tau\right\} \leq C_j \max\{|u(t)|_\infty^{j+1} : 0 \leq t \leq t_1\} \tag{5.6}$$

with

$$\tau = \frac{c_0}{|u(t_1)|_\infty^2}$$

Estimate (5.6) says, essentially, that the maximum of the j-th derivatives of u measured by $|\mathcal{D}^j u|_\infty$, can be bounded in terms of $|u|_\infty^{j+1}$. Clearly, a time interval near $t = 0$ has to be excluded on the left-hand side of (5.6) for smoothing to become effective. The positive value of τ on the left-hand side of (5.6) shows that $|u|_\infty^{j+1}$ controls $|\mathcal{D}^j u|_\infty$ for some time into the future.

As is well known, if (u, p) solves the Navier-Stokes equations and $\lambda > 0$ is any scaling parameter, then the functions u_λ, p_λ defined by

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$$

also solve the Navier-Stokes equations. Clearly,

$$|u_\lambda(t)|_\infty = \lambda |u(\lambda^2 t)|_\infty, \quad |\mathcal{D}^j u_\lambda(t)|_\infty = \lambda^{j+1} |\mathcal{D}^j u(\lambda^2 t)|_\infty.$$

Therefore, $|\mathcal{D}^j u|_\infty$ and $|u|_\infty^{j+1}$ both scale like λ^{j+1}, which is, of course, consistent with the estimate (5.6). We do not know under what assumptions $|u|_\infty^{j+1}$ can conversely be estimated in terms of $|\mathcal{D}^j u|_\infty$.

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