International Journal of Mathematical Analysis
Vol. 14, 2020, no. 4, 177 - 185
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ijma.2020.912101

Isometries of the Zygmund F-Algebra on the Upper Half Plane

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Abstract

In [4] the author introduced the Zygmund F-algebra $N\log^{\alpha} N(D)$ $(\alpha > 0)$ of holomorphic functions f on the upper half plane $D = \{z \in \mathbf{C} \,|\, \mathtt{Im}z > 0\}$ that satisfy

$$\sup_{y>0} \int_{\mathbf{R}} \varphi_{\alpha}\left(\log(1 + |f(x+iy)|)\right) dx < +\infty,$$

where $\varphi_{\alpha}(t) = t\{\log(c_{\alpha} + t)\}^{\alpha}$ for $t \geq 0$ and $c_{\alpha} = \max(e\,, e^{\alpha})$. In this paper we shall characterize linear isometries of $N\log^{\alpha} N(D)$ onto $N\log^{\alpha} N(D)$.

Mathematics Subject Classification: 30H15, 46E10

Keywords: Zygmund F-algebra, linear isometry, upper half plane

1 Introduction

Linear isometries of function spaces of holomorphic functions have been studied since the 1960s. Many mathematicians seek representations of linear isometries of function spaces not only on the unit disk but on the upper half plane. In this paper we consider linear isometries of the Zygmund F-algebra on the upper half plane.

2 Preliminaries

Throughout this paper, let U and T denote the unit disk and the unit circle in $\mathbf{C}$, respectively, and $d\sigma$ the normalized Lebesgue measure on T. The upper half plane in $\mathbf{C}$ is denoted by D. An *F-algebra* is a topological algebra in which the topology arises from a complete metric.

Now we recall definitions and some properties of the Privalov class, the Smirnov class, and the Zygmund F-algebra on U. For each $0 < q \leq \infty$, the Hardy space on U is denoted by $H^q(U)$ with the norm $||\cdot||_{H^q(U)}$.

2.1 Privalov class $N^p(U)$

For $p > 1$, we denote by $N^p(U)$ the class of functions f holomorphic on U and satisfying

$$\sup_{0\leq r<1} \int_T \Big(\log(1+|f(r\zeta)|)\Big)^p d\sigma(\zeta) < +\infty.$$

Privalov [8] introduced the class $N^p(U)$. Letting $p = 1$, we have the Nevanlinna class $N(U)$. We easily see that $N^p(U) \subset N(U)$ $(p > 1)$. It is well-known that each function f in $N(U)$ has the nontangential limit $f^*(\zeta) = \lim_{r\to 1^-} f(r\zeta)$ (a.e. $\zeta \in T$). Define a metric

$$d_{N^p(U)}(f,g) = \left\{ \int_T \Big(\log(1+|f^*(\zeta)-g^*(\zeta)|)\Big)^p d\sigma(\zeta) \right\}^{\frac{1}{p}}$$

for $f, g \in N^p(U)$. With this metric the Privalov class becomes an F-algebra [12].

Linear isometries of $N^p(U)$ were characterized by Iida and Mochizuki in [5].

2.2 Smirnov class $N_*(U)$

The Smirnov class $N_*(U)$ consists of all holomorphic functions f on U such that $\log(1+|f(z)|) \leq Q[\phi](z)$ $(z \in U)$ for some $\phi \in L^1(T)$, $\phi \geq 0$, where the right side denotes the Poisson integral of ϕ on U. It is well-known that, if $f \in N(U)$, f belongs to $N_*(U)$ if and only if

$$\lim_{r\to 1^-} \int_T \log\left(1+|f(r\zeta)|\right) d\sigma(\zeta) = \int_T \log\left(1+|f^*(\zeta)|\right) d\sigma(\zeta).$$

We have $H^q(U) \subset N^p(U) \subset N_*(U) \subset N(U)$ $(0 < q \leq \infty\,, p > 1)$. These relations are proper. Under the metric

$$d_{N_*(U)}(f,g) = \int_T \log\left(1+|f^*(\zeta)-g^*(\zeta)|\right) d\sigma(\zeta)$$

for $f, g \in N_*(U)$, the class becomes an F-algebra (see [11]).

Linear isometries of $N_*(U)$ were described by Stephenson in [10].

2.3 Zygmund F-algebra $N\log^\alpha N(U)$

Let $\alpha > 0$. The Zygmund F-algebra $N\log^\alpha N(U)$ is the set of all holomorphic functions f on U such that

$$\sup_{0\le r<1}\int_T \varphi_\alpha\left(\log^+|f(r\zeta)|\right)\,d\sigma(\zeta) < +\infty, \tag{1}$$

where $\varphi_\alpha(t) = t\{\log(c_\alpha + t)\}^\alpha$ for $t \ge 0$ and $c_\alpha = \max(e\,,\, e^\alpha)$. It is known that (1) is equivalent to the condition

$$\sup_{0\le r<1}\int_T \varphi_\alpha\left(\log(1+|f(r\zeta)|)\right)\,d\sigma(\zeta) < +\infty. \tag{2}$$

This class was considered by Zygmund in his monograph [14]. After that the topological properties of the class were investigated in [1, 2, 13]. We have the following strict inclusions

$$\bigcup_{q>0} H^q(U) \subset N\log^\alpha N(U) \subset N_*(U) \subset N(U) \quad (\alpha > 0).$$

This implies that every $f \in N\log^\alpha N(U)$ has a finite nontangential limit almost everywhere on T. Therefore the characteristic defined by (2) satisfies the relation:

$$d_{N\log^\alpha N(U)}(f,0) := \int_T \varphi_\alpha\left(\log(1+|f^*(\zeta)|)\right)\,d\sigma(\zeta).$$

We can define the metric

$$d_{N\log^\alpha N(U)}(f,g) = \int_T \varphi_\alpha\left(\log(1+|f^*(\zeta)-g^*(\zeta)|)\right)\,d\sigma(\zeta)$$

for $f, g \in N\log^\alpha N(U)$. With this metric $N\log^\alpha N(U)$ is an F-algebra [1].

Linear isometries of $N\log^\alpha N(U)$ were characterized by Ueki in [13].

Next we shall show definitions and some properties of the Nevanlinna class, the Smirnov class, the Privalov class, and the Zygmund F-algebra on D. For $0 < q < \infty$, the Hardy space $H^q(D)$ is the set of all holomorphic functions f on D satisfying

$$\|f\|_{H^q(D)} := \sup_{y>0}\int_{\mathbf{R}} |f(x+iy)|^q\,dx < +\infty.$$

2.4 Smirnov class $N_*(D)$

Mochizuki [7] considered the Nevanlinna class $N_0(D)$ and the Smirnov class $N_*(D)$ on the upper half plane $D := \{z \in \mathbf{C} \,|\, \mathtt{Im} z > 0\}$: the class $N_0(D)$ consists of all holomorphic functions f on D for which

$$\sup_{y>0} \int_{\mathbf{R}} \log\left(1 + |f(x+iy)|\right) dx < +\infty$$

and $N_*(D)$ the set of all holomorphic functions f on D which satisfy $\log(1 + |f(z)|) \leq P[\phi](z) \quad (z \in D)$ for some $\phi \in L^1(\mathbf{R})$, $\phi \geq 0$, where the right side means the Poisson integral of ϕ on D. It is known that each function f in $N_0(D)$ has the nontangential limit $f^*(x) = \lim_{y \to 0^+} f(x+iy) \quad$ (a.e. $x \in \mathbf{R}$). Let $f \in N_0(D)$. Then f belongs to $N_*(D)$ if and only if

$$\lim_{y \to 0^+} \int_{\mathbf{R}} \log\left(1 + |f(x+iy)|\right) dx = \int_{\mathbf{R}} \log\left(1 + |f^*(x)|\right) dx$$

holds. Moreover, under the metric

$$d_{N_*(D)}(f,g) = \int_{\mathbf{R}} \log\left(1 + |f^*(x) - g^*(x)|\right) dx$$

for $f, g \in N_*(D)$, this class becomes an F-algebra [7].

Mochizuki also characterized linear onto isometries of $N_*(D)$ in [7].

2.5 Privalov class $N^p(D)$

In [3] the author introduced the Privalov class on D. We denote by $N^p(D) \;\; (p > 1)$ the set of all holomorphic functions f on D such that

$$\sup_{y>0} \int_{\mathbf{R}} \Big(\log(1 + |f(x+iy)|)\Big)^p dx < +\infty.$$

Each $f \in N^p(D)$ has the nontangential limit $f^*(x)$ for a.e. $x \in \mathbf{R}$, and under the metric

$$d_{N^p(D)}(f,g) = \left\{ \int_{\mathbf{R}} \Big(\log(1 + |f^*(x) - g^*(x)|)\Big)^p dx \right\}^{\frac{1}{p}}$$

for $f, g \in N^p(D)$, the class $N^p(D)$ becomes an F-algebra [3].

Linear onto isometries of $N^p(D)$ were investigated by Iida and Takahashi in [6].

2.6 Zygmund F-algebra $N\log^\alpha N(D)$

Let $\alpha > 0$. $N\log^\alpha N(D)$ is the set of all holomorphic functions f on D satisfying

$$\sup_{y>0}\int_{\mathbf{R}} \varphi_\alpha\left(\log(1+|f(x+iy)|)\right)\,dx < +\infty,$$

where $\varphi_\alpha(t) = t\{\log(c_\alpha + t)\}^\alpha$ for $t \geq 0$ and $c_\alpha = \max(e\,,\, e^\alpha)$. This class was introduced by the author in [4]. It is known that, if $f \in N\log^\alpha N(D)$, the nontangential limit $f^*(x)$ exists a.e. for $x \in \mathbf{R}$. Moreover we can define, for $f, g \in N\log^\alpha N(D)$, a metric

$$d_{N\log^\alpha N(D)}(f,g) = \int_{\mathbf{R}} \varphi_\alpha\left(\log(1+|f^*(x) - g^*(x)|)\right)\,dx.$$

With this metric $N\log^\alpha N(D)$ is also an F-algebra [4].

3 Main result

In this section we shall characterize linear onto isometries of the Zygmund F-algebra on the upper half plane D.

3.1 Linear isometrie of $N\log^\alpha N(D)$

Theorem 3.1. *Let $\alpha > 0$ and let A be a linear isometry of $N\log^\alpha N(D)$ onto $N\log^\alpha N(D)$. Then there exist $c \in \mathbf{C}\,,\ |c| = 1,$ and $\alpha \in \mathbf{R}$ such that*

$$(Af)(z) = cf(z+\alpha) \quad (z \in D\,,\ f \in N\log^\alpha N(D)).$$

Conversely, given such c and α, the transformation A defines a linear isometry of $N\log^\alpha N(D)$ onto $N\log^\alpha N(D)$.

3.2 Proof of main theorem

We need some lemmas.

Lemma 3.2. ([7]) *Let $q > 0\,,\ q \neq 2$ and let A be a linear isometry of $H^q(D)$ onto $H^q(D)$. Then A is written in the form*

$$(Af)(z) = \tag{3}$$

$$c(\psi'(\Psi(z)))^{\frac{1}{q}}\left(\frac{1}{z+i}\right)^{\frac{2}{q}}\left(\frac{2i}{1-(\psi\circ\Psi)(z)}\right)^{\frac{2}{q}} f((\Psi^{-1}\circ\psi\circ\Psi)(z)) \quad (z \in D)$$

for $f \in H^q(D)$, where $c \in \mathbf{C}\,,\ |c| = 1\,,\ \Psi(z) = (z-i)(z+i)^{-1}\,(z \in D)\,,\ \psi$ is a conformal map of U onto U. If we put $\phi = \Psi^{-1}\circ\psi\circ\Psi$, we have

$$(Af)(z) = c(\phi'(z))^{\frac{1}{q}} f(\phi(z)) \quad (z \in D). \tag{4}$$

Lemma 3.3. ([13]) *There exist a bounded continuous function θ_α on $[0,\infty)$ and a positive constant K_α such that*

$$\varphi_\alpha(\log(1+x)) = (\log c_\alpha)^\alpha x - K_\alpha x^2 + x^3\theta_\alpha(x) \quad \text{for} \quad x \in [0,\infty).$$

Lemma 3.4. *Let $f \in N\log^\alpha N(D)$ $(\alpha > 0)$ and we define $D_\delta = \{z \in \mathbf{C} \mid \mathtt{Im} z > \delta\}$ for $\delta > 0$. Then we have the following:*

$$\sup\{\, \varphi_\alpha(\log(1+|f(z)|)) \mid z \in \overline{D}_\delta\} \le \frac{C_\alpha}{\delta}\, d_{N\log^\alpha N(D)}(f,0),$$

where C_α is a constant independent of f and δ.

Proof. φ_α is increasing and strictly convex on $[0,\infty)$, so the subharmonic function $\varphi_\alpha(\log(1+|f|))$ has the above property by [9, Chapter II, Theorem 4.6]. □

Lemma 3.5. *Let $\alpha > 0$ and V be the family of holormorphic functions f on D such that $|f(z)||z+i|^2$ are bounded. Then V is a linear subspace of $N\log^\alpha N(D)$ and dense in $N\log^\alpha N(D)$.*

Proof. We follow [7, Lemma 5.4]. Let $f \in N\log^\alpha N(D)$ $(\alpha > 0)$. For $s > 0$, we put f_s by $f_s(z) = f(z+is)$ $(z \in D)$. It is clear that $f_s \in N\log^\alpha N(D)$. Now there exists a sequence $\{g_i\}$ of continuous functions on $\overline{D}$ which are holomorphic on D and such that $|g_i(z)| \le 1$ $(z \in \overline{D})$, $|g_i(z)||z+i|^2 \to 0$ as $|z| \to +\infty$ in $\overline{D}$, and $g_i(z) \to 1$ as $i \to \infty$ $(z \in \overline{D})$. Lemma 3.4 implies $|f_s(z)| \le M$ $(z \in D)$. If we let $f_i = f_s g_i$, it follows that $f_i \in V$ and $d_{N\log^\alpha N(D)}(f_i\,,\, f_s) \to 0$ as $i \to \infty$. □

Lemma 3.6. *Let $\alpha > 0$ and suppose A be a linear isometry of $N\log^\alpha N(D)$ onto $N\log^\alpha N(D)$. Then A transforms V onto V as an $H^1(D)$-isometry.*

Proof. We note that A transforms $H^1(D)$ onto $H^1(D)$ as an $H^1(D)$-isometry, as in [13, Lemma 1]. Hence Af is written in the form (3), with $q = 1$, for $f \in H^1(D)$. Let $f \in V$, $|f(z)||z+i|^2 \le M$ $(z \in D)$. Since $2i(1-(\psi\circ\Psi)(z))^{-1} = (\Psi^{-1}\circ\psi\circ\Psi)(z) + i$, an easy calculation shows

$$\left|\frac{2i}{1-(\psi\circ\Psi)(z)}\right|^2 |f((\Psi^{-1}\circ\psi\circ\Psi)(z))| \le M.$$

Here ψ is of the form : $\psi(w) = b(a-w)(1-\bar{a}w)^{-1}$ $(w \in U)$ with $b \in T$ and $a \in U$, so we obtain $|\psi'(w)| \le 2(1-|a|)^{-1}$. We see that

$$|(Af)(z)| \le \frac{2}{1-|a|}\left|\frac{1}{z+i}\right|^2 M.$$

Hence it follows that $Af \in V$. The same argument for A^{-1} shows that A transforms V onto V. □

Lemma 3.7. *Let $\alpha > 0$ and A be a linear isometry of $N\log^\alpha N(D)$ onto $N\log^\alpha N(D)$. Then A transforms V onto V as an $H^3(D)$-isometry.*

Proof. By Lemma 3.6, if $f \in V$ then $g := Af \in V$. Putting $|tf^*(x)|$ and $|tg^*(x)|$ $(t > 0\,,\ x \in \mathbf{R})$, respectively, in place of x in Lemma 3.3, we have

$$-K_\alpha \int_{\mathbf{R}} |f^*(x)|^2\,dx + t\int_{\mathbf{R}} |f^*(x)|^3\,\theta_\alpha(|tf^*(x)|)\,dx$$
$$= -K_\alpha \int_{\mathbf{R}} |g^*(x)|^2\,dx + t\int_{\mathbf{R}} |g^*(x)|^3\,\theta_\alpha(|tg^*(x)|)\,dx$$

for $t > 0$. Letting $t \to 0^+$, we gain

$$\int_{\mathbf{R}} |f^*(x)|^2\,dx = \int_{\mathbf{R}} |g^*(x)|^2\,dx.$$

Therefore we see that A transforms V onto V as an $H^2(D)$-isometry.

In the same manner we prove that A transforms V onto V as an $H^3(D)$-isometry, using the equality $\varphi_\alpha(\log(1+x)) = (\log c_\alpha)^\alpha x - K_\alpha x^2 + L_\alpha x^3 + x^4\theta_\alpha(x)$ (L_α is a constant), which we easily obtain from Lemma 3.3. □

Proof of Theorem 3.1. We follow [7, Theorem 5.2]. Since V is dense in $H^3(D)$, there is a linear isometry $\tilde{A}$ of $H^3(D)$ onto $H^3(D)$ such that $\tilde{A} = A$ on V. Hence $\tilde{A}f$ is of the form (4), with $q = 3$. If we let $f \in V$, then (4) holds for both $q = 1$ and $q = 3$. Thus we have

$$(Af)(z) = c_1(\phi_1'(z))f(\phi_1(z)) = c_3(\phi_3'(z))^{\frac{1}{3}}f(\phi_3(z)) \quad (z \in D).$$

Here ϕ_j $(j = 1, 3)$ are conformal maps of D onto D and $c_j \in T$. In this case ϕ_j is of the form:

$$\phi_j(z) = \frac{\alpha_j z + \beta_j}{\gamma_j z + \delta_j} \quad (z \in D),$$

where $\alpha_j, \beta_j, \gamma_j, \delta_j \in \mathbf{R}$ and $D_j := \alpha_j\delta_j - \beta_j\gamma_j > 0$. It follows that

$$\frac{D_3}{D_1^3}\left|\frac{f(\phi_3(z))}{f(\phi_1(z))}\right|^3 \frac{|\gamma_1 z + \delta_1|^6}{|\gamma_3 z + \delta_3|^2} = 1 \quad (z \in D).$$

Suppose $\gamma_1 \neq 0$ and put $f(z) = (z + i)^{-3}$. If we were to let $|z| \to +\infty$, we would have a contradiction. Therefore Af must be of the form

$$(Af)(z) = cf(\beta z + \alpha) \quad (c \in \mathbf{C},\ \beta > 0,\ \alpha \in \mathbf{R})$$

for $f \in V$. Moreover the fact that $||Af||_{H^q(D)} = ||f||_{H^q(D)}$ $(q = 1, 3)$ shows $|c| = \beta = 1$. Since V is dense in $N\log^\alpha N(D)$, the conclusion holds. The proof of the converse is clear. □

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Received: May 1, 2020; Published: May 14, 2020