International Journal of Mathematical Analysis
Vol. 14, 2020, no. 3, 125 - 132
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ijma.2020.91287

Solution of Fractional k-Hypergeometric Differential Equation

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Abstract

The main objective of this paper to introduce the new technique of fractional power series to solve the k-fractional hypergeometric differential equation. Also introduce the forms of the conformable fractional derivative and the integral representation of the k-fractional Gaussian function.

Keywords: Fractional k-Gauss differential equation; Local fractional derivatives; k- Gauss hypergeometric function; Fractional derivative

1 Introduction

Fractional calculus investigates the arbitrary order derivatives and integrals. There is no need for us to review the impact that Fractional calculusl and special functions theory has applications in mathematics, science, engineering and computations. (see[1], [3], [4]) and for statistics application (see[2], [3], [6]). By the middle of the last century, handbooks had been compiled that could be found on nearly everyones bookshelf. In our time, handbooks join forces with mathematical software and new applications making the subject as relevant today as it was over a century ago. We believe that the modern day extensions of these scalar functions are the multivariate Fractional calculu. Various definitions of fractional derivatives are obtained and compared. Fractional derivatives in the sense of Riemann-Liouvellie and Caputo are the most popular. In 2014, Khalil et al [6] introduced a conformable fractional derivative such as, Let
$f : M \subset (0, \infty) \longrightarrow R$ and $x \in M$. The fractional derivative of order $\lambda \in (0, 1]$ for f at x is defined as

$$D^{\lambda} f(x) = \lim_{\varepsilon \longrightarrow 0} \frac{f(x + \varepsilon x^{1-\lambda})}{\epsilon}, \tag{1}$$

whenever the limit exists. The function f is called λ-conformable fractional differentiable at x. This definition carries with it very important and natural properties. Let D^{λ} denote the operator which is called the fractional derivatives of order λ and we recall from [2],[4] and [5] some of its general properties. Let f and g λ- differentiable. Then the following properties satisfy:
i. Linearity: $D^{\lambda}(\alpha\kappa + \beta\nu) = \alpha D^{\lambda}(\kappa) + \beta D^{\lambda}(\nu) \quad \forall \alpha, \beta \in R$

ii. Product: $D^{\lambda}(\kappa\nu) = \kappa D^{\lambda}(\nu) + \nu D^{\lambda}(\kappa)$

iii. Quotient: $D^{\lambda}(\frac{\kappa}{\nu}) = \frac{\nu D^{\lambda}(\kappa) - \kappa D^{\lambda}(\nu)}{\nu^2}$

iv. chain rule: $D^{\lambda}(\kappa \o \nu) = D^{\lambda}(\kappa(\nu)) D^{\lambda}(\nu) \nu^{\lambda-1}$
For $\lambda = 1$ in λ-conformable fractional derivative we get the basic limit definition of the derivative.

2 Preliminaries

In this section, we briefly review some basic definitions and facts concerning the k-hypergeometric series and the ordinary differential equation [12]. Solutions to the hypergeometric differential equation are built out of the hypergeometric series. The equation

$$kx(1 - kx)D^{''}y + (c - (a + b + k)kx)D^{'}y - aby = 0. \tag{2}$$

is called Gauss hypergeometric differential equation. The point $x = 0$ is a regular singular point for the equation. Power series technique is a method to solve such equation. Some surveys and literature for k-hypergeometric series and the k-hypergeometric differential equation can be found in Diaz and Pariguan [11] have introduced and proved some identities of k-gamma function, k-beta function and k-pochhammer symbol. They have deduced an integral representation of k-gamma and k-beta function respectively given by

$$\Gamma_k(\mu) = \int_o^\infty t^{\mu-1} e^{-\frac{t^k}{k}} dt, \qquad Re(\mu) > 0, \quad k > 0. \tag{3}$$

$$\beta_k(\mu, \nu) = \frac{1}{k} \int_o^1 t^{\frac{\mu}{k}-1} (1-t)^{\frac{\nu}{k}-1} dt, \qquad \mu > 0, \quad \nu > 0. \tag{4}$$

They have also provided some useful results

$$\beta_k(\mu, \nu) = \frac{\Gamma_k(\mu)\Gamma_k(\nu)}{\Gamma_k(\mu+\nu)}. \tag{5}$$

$$(\mu)_{n,k} = \frac{\Gamma_k(\mu+nk)}{\Gamma_k(\mu)}. \tag{6}$$

$$\sum_{n=0}^{\infty} (a)_{n,k} \frac{\mu^n}{n!} = (1-k\mu)^{\frac{a}{k}}. \tag{7}$$

Definition 1. The k-hypergeometric function with three parameters μ, ν, c, two parameters μ, ν, in the numerator and one parameter c in the denominator, are defined by

$${}_2F_{1,k}\left((\mu,k),(\nu,k);(c,k);\mu\right) = \sum_{n=0}^{\infty} \frac{(\mu)_{n,k}(b)_{n,k}\mu^n}{(c)_{n,k} n!}. \tag{8}$$

Definition 2. The Pochhammer k-symbol is defined as
$(\sigma)_{n,k} = \sigma(\sigma+k)(\sigma+2k)...(\sigma+(n-1)k), \qquad \sigma \neq 0,\ (\sigma)_{0,k} = 1$, where $k > 0$.

3 The Solutions of Fractional k-Hypergeometric Equations

In this section, by mean of fractional power series, we expound upon the series solution of fractional form of the Gauss k-hypergeometric differential equation. More precisely:

$$kx^\lambda(1-kx^\lambda)D^\lambda D^\lambda y + \lambda(c-(\mu+\nu+k)kx^\lambda)D^\lambda y - \lambda^2\mu\nu y = 0. \tag{9}$$

where $\lambda \in (0,1]$, $k > 0$ and μ, ν, c are reals.
Definition: The point $x = 0$ is called λ-regular singular point for the the equation

$$D^\lambda D^\lambda y + P(x)D^\lambda y + Q(x)y = 0.$$

if the $\lim\limits_{x\longrightarrow 0}\frac{P(x)}{x^{\lambda}}$ and $\lim\limits_{x\longrightarrow 0}\frac{P(x)}{x^{2\lambda}}$ both exist.
Clearly, $x=0$ is an λregular singular point for equation. In [1] fractional power series defined as, $\sum_{n=0}^{\infty} d_n x^{n\lambda}$. We will use $D^{n\lambda}$ to denote $D^{\lambda}...D^{\lambda}$ n-times. If $D^{n\lambda}f$ exist for all n in some interval $[0,\chi]$ then one can write f in the form of a fractional power series. Let $y=\sum_n d_n x^{n\lambda+\chi}$. Then from the basic properties of the conformable derivative we get:

$$D^{\lambda}y=\sum_{n=0}^{\infty} d_n(n\lambda+\chi)x^{n\lambda+\chi-\lambda},$$

and

$$D^{\lambda}D^{\lambda}y=\sum_{n=0}^{\infty} d_n(n\lambda+\chi)(n\lambda+\chi-\lambda)x^{n\lambda+\chi-2\lambda},$$

Substituting in Equation (9), we get:

$$\sum_{n=0}^{\infty}(k(n\lambda+\chi)(n\lambda+\chi-\lambda)+\lambda c(n\lambda+\chi))d_n x^{n\lambda+\chi-\lambda}$$

$$-\sum_{n=0}^{\infty}(k^2(n\lambda+\chi)(n\lambda+\chi-\lambda)+\lambda k(n\lambda+\chi)(\mu+b+k))d_n x^{n\lambda+\chi}=0,$$

However,

$$k^2(n\lambda+\chi)(n\lambda+\chi-\lambda)+\lambda k(n\lambda+\chi)(\mu+b+k)=$$

$$(k(n\lambda+\chi)+\lambda\mu)(k(n\lambda+\chi)+\lambda b).$$

Using above in (9), we get

$$\sum_{n=0}^{\infty}(k(n\lambda+\chi)(n\lambda+\chi-\lambda)+\lambda c(n\lambda+\chi))d_n x^{n\lambda+\chi-\lambda},$$

$$-\sum_{n=0}^{\infty}(k(n\lambda+\chi)+\lambda\mu)(k(n\lambda+\chi)+\lambda b)d_n x^{n\lambda+\chi}=0,$$

Hence, from the first term, we have

$$k(\chi)(\chi-\lambda)+\lambda c(\chi)d_0=0,$$

which is indicial equation. Since $d_0\neq 0$, we have

$$k(\chi)(\chi-\lambda)+\lambda c(\chi)=0,$$

Hence, the solutions of the above indicial equation are given below:

$$\chi=0, \qquad and \qquad \chi=\lambda-\frac{\lambda c}{k},$$

Also, from the rest of the terms, put $\chi = 0$, we have

$$d_n = \frac{(k(n-1)+a)(k(n-1)+\nu)}{(k(n-1)+c)} d_{n-1}.$$

for $n \geq 1$. Let us now simplify this relation by giving d_n in terms of d_0 instead of d_{n-1}. For $n = 1$

$$d_1 = \frac{(a)(\nu)}{1(c)} d_0.$$

For $n = 2$

$$d_2 = \frac{(a)(\nu)}{1(c)} \frac{(a+k)(\nu+k)}{2(c+k)} d_0.$$

From the recurrence relation, we have the following:

$$d_n = \sum_{n=0}^{\infty} \frac{(a)_{n,k}(\nu)_{n,k}}{(c)_{n,k} n!} d_0.$$

Hence, our assumed solution takes the form

$$Y_1 = d_0 \sum_{n=0}^{\infty} \frac{(a)_{n,k}(\nu)_{n,k} x^{n\lambda}}{(c)_{n,k} n!}. \tag{10}$$

Also, from (9),

$$d_{n+1} = \frac{(k(n\lambda+\chi)+\lambda a)(k(n\lambda+\chi)+\lambda\nu)}{(k(n\lambda+\lambda+\chi)(k(n\lambda+\chi)+\lambda c)} d_n,$$

Put $\chi = \lambda - \frac{\lambda c}{k}$

$$d_{n+1} = \frac{(k(n+1)+a-c)(k(n+1)+\nu-c)}{(k(n+2)-c)(n+1)} d_n,$$

Replace n into $n-1$

$$d_n = \frac{(kn+\mu-c)(kn+\nu-c)}{(k(n+1)-c)(n)} d_{n-1},$$

From the recurrence relation, we have the following:

$$d_n = \sum_{n=0}^{\infty} \frac{(\mu-c+k)_{n,k}(\nu-c+k)_{n,k}}{(2k-c)_{n,k} n!} d_0$$

another independent solution is,

$$Y_2 = \sum_{n=0}^{\infty} \frac{(\mu-c+k)_{n,k}(\nu-c+k)_{n,k}}{(2k-c)_{n,k} n!} x^{n\lambda}. \tag{11}$$

and general solution is:

$$y = A\sum_{n=0}^{\infty}\frac{(\mu)_{n,k}(\nu)_{n,k}x^{n\lambda}}{(c)_{n,k}n!} + Bx^{\lambda-\frac{\lambda c}{k}}\sum_{n=0}^{\infty}\frac{(\mu-c+k)_{n,k}(\nu-c+k)_{n,k}}{(2k-c)_{n,k}n!}x^{n\lambda}.$$

and also

$$y = A{}_2F_{1,k}\left((\mu,k),(\nu,k);(c,k);x^{\lambda}\right) + x^{\lambda-\frac{\lambda c}{k}}B{}_2F_{1,k}\left((\mu-c+k,k),(\nu-c+k,k);(2k-c,k);x^{\lambda}\right). \quad (12)$$

Here A and B are arbitrary constants.

Theorem: The conformable k-fractional derivative of the λ-Gaussian function has the form:

$$\frac{d^{\lambda}}{dx^{\lambda}}{}_2F_{1,k}\left((\mu,k),(\nu,k);(c,k);x^{\lambda}\right) = \frac{\lambda\mu\nu}{c}{}_2F_{1,k}\left((\mu+k,k),(\nu+k,k);(c+k,k);x^{\lambda}\right). \quad (13)$$

Proof:

$$\frac{d^{\lambda}}{dx^{\lambda}}{}_2F_{1,k}\left((\mu,k),(\nu,k);(c,k);x^{\lambda}\right) = \frac{d^{\lambda}}{dx^{\lambda}}\sum_{n=0}^{\infty}\frac{(\mu)_{n,k}(\nu)_{n,k}x^{n\lambda}}{(c)_{n,k}n!},$$

$$= \sum_{n=1}^{\infty}\frac{(\mu)_{n,k}(\nu)_{n,k}\lambda x^{n\lambda-\lambda}}{(c)_{n,k}(n-1)!},$$

$$= \sum_{n=0}^{\infty}\frac{(\mu)_{n+1,k}(\nu)_{n+1,k}\lambda x^{n\lambda}}{(c)_{n+1,k}(n)!},$$

Since $\quad (\mu)_{n+1,k} = \mu(\mu+k)_{n,k},$

$$\frac{d^{\lambda}}{dx^{\lambda}}{}_2F_{1,k}\left((\mu,k),(\nu,k);(c,k);x^{\lambda}\right) = \frac{\lambda\mu\nu}{c}\sum_{n=1}^{\infty}\frac{(\mu+k)_{n,k}(\nu+k)_{n,k}x^{n\lambda}}{(c+k)_{n,k}(n)!}.$$

Theorem: The λ-Gaussian function has an integral representation of the form:

$${}_2F_{1,k}\left((\mu,k),(\nu,k);(c,k);x^{\lambda}\right) = \frac{\Gamma_k(c)}{k\Gamma_k(c-\mu)\Gamma_k(\mu)}\int_0^1 t^{\frac{\mu}{k}-1}(1-t)^{\frac{c-\mu}{k}-1}(1-ktx^{\lambda})^{-\frac{\mu}{k}}dt. \quad (14)$$

Proof: We know from (2.4)

$$\beta_k(\mu+nk, c-\mu) = \frac{\Gamma_k(\mu+nk)\Gamma_k(c-\mu)}{\Gamma_k(nk+c)}.$$

$$\frac{(\mu)_{n,k}\Gamma_k(\mu)\Gamma_k(c-\mu)}{(c)_{n,k}\Gamma_k(c)} = \frac{1}{k}\int_0^1 t^{\frac{\mu}{k}+n-1}(1-t)^{\frac{c-\mu}{k}-1}dt,$$

$$\sum_{n=0}^{\infty}\frac{(\mu)_{n,k}(\nu)_{n,k}x^{n\lambda}}{(c)_{n,k}n!} = \frac{\Gamma_k(\mu)\Gamma_k(c-\mu)}{k\Gamma_k(c)}\int_0^1 t^{\frac{\mu}{k}-1}(1-t)^{\frac{c-\mu}{k}-1}\sum_{n=0}^{\infty}\frac{(\nu)_{n,k}(tx^{\lambda})^n}{n!}dt,$$

$$= \frac{\Gamma_k(c)}{k\Gamma_k(c-\mu)\Gamma_k(\mu)}\int_0^1 t^{\frac{\mu}{k}-1}(1-t)^{\frac{c-\mu}{k}-1}(1-ktx^{\lambda})^{-\frac{\nu}{k}}dt.$$

4 Conclusions

In this paper, we present a formula of the general solution of fractional k-hypergeometric ordinary differential Equation (9), provided that $\mu, \nu, c \in R$ with both c and 2k - c neither zero, nor negative integers. The solutions of this type of equations are denoted by in the form of k-hypergeometric series. Integral representation and conformable k-fractional derivative of the λ-Gaussian function. It would be interesting to have more research about this case.

Acknowledgements. The authors express their sincere gratitude to Dr. Ghulam Farid whose constant encouragement and useful discussions has enabled us to do this work.

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Received: March 24, 2020; Published: April 16, 2020