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Spectral Synthesis in the Heisenberg Group

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Abstract

We introduce some sets of spectral synthesis in some commutative Banach algebras of integrable functions on the Heisenberg group.

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1. Introduction and Preliminaries

Consider the Heisenberg group H as the set {(z, t): z $\in \mathbb{C}$, t $\in \mathbb{R}$} with the group operation (z, t) • (w, s) = (z + w, t + s + 2Im z w). Let us denote by L_0^1(H) the integrable functions on H which are radial on $\mathbb{C}$: L_0^1(H) = {f $\in L^1$ (H): f(z, t) = f($|z|$,t) }. It is well-known that L_0^1(H) is a commutative Banach algebra under the convolution f*g(z, t) = $\iint f\,(z\text{-}w,\ t - s - 2\ \mathrm{Im}\ z\overline{w})g(s,w)dsdw$ where dw is Lebesgue measure on $\mathbb{C}$ ([2]). The regular maximal ideals of L_0^1(H) are the annihilators of the spherical functions ψ_ν^λ, $\lambda \in \mathbb{R}$, $\lambda \neq 0$, $\nu = \{0, 1, 2, . . .\}$ (the Laguerre part) and B_ρ, $\rho \geq 0$ (the Bessel part) where ψ_ν^λ(z,t) = $e^{i\lambda t} \cdot e^{-|\lambda||z|^2} L_\nu(4\pi\,|\lambda|\,|z|^2)$ where $L_\nu = \frac{e^x}{\nu!}\frac{d\nu}{dx^\nu}(e^{-x}x^\nu)$ (the Laguerre polynomials) and B_ρ(z) = $J_0(\rho|z|)$ where J_0 is the Bessel function of the first kind of order 0. Hulanicki and Ricci showed that Wiener Tauberian theorem holds for $L_0^1(H)$ ([4]). That it, every proper closed ideal is contained in some regular maximal ideal.

The dual space of $L_0^1(H)$ is $L_0^\infty(H) = \{f \in L^\infty(H): f(z, t) = f(|z|, t)\}$. By duality, Weiner theorem implies that for every $f \in L_0^\infty(H)$, $f \neq 0$, the subspace $V(f)$ = *w**- closure of $\{f * h: h \in L_0^1(H)\}$ contains a spherical function. The spectrum of f is defined as Spe (f) = $\{\rho\} \cup \{(\lambda, \nu)\}$ when B_ρ and ψ_ν^λ belong to V(f). The basic *"radial translate"* of a function f in $L_0^\infty(H)$ is $T_{r,s}(f)(z, t) = \frac{1}{2\pi}\int_0^{2\pi} f(z - re^{i\theta}, t - s - 2\mathrm{Im}(z\, re^{-i\theta}))d\theta$ and then V(f) is the *w**- closed subspace spanned by all *"radial translates"* of f .

For $f \in L^\infty(\mathbb{R})$ its spectrum is defined as the set of $\lambda \in \mathbb{R}$ such that $e^{i\lambda t}$ belongs to the translation invariant *w**- closed subspace generated by f ([5]).

The purpose of this note is to study some sets of spectral synthesis in $L_0^\infty(H)$ and to give a simple proof to Wiener theorem for $L_{00}^1(H)$ introduced in [1].

2. Main results

Theorem 1. Each point is a set of spectral synthesis. That is, if I is a proper closed ideal in $L_0^1(H)$ which is contained in exactly one regular maximal ideal M then I = M. By duality, for $f \in L_0^\infty(H)$ if Spe (f) = $\{\rho\}$ then $f(z, t) = C \cdot J_0(\rho|z|)$ and if Spe (f) = $\{(\lambda, \nu)\}$ then $f(z, t) = C \cdot \psi_\nu^\lambda(z, t)$.

Proof:

Suppose first that Spe (f)= $\{\rho\}$ for some $\rho \geq 0$ and assume that f is continuous. We claim that f (z, t) = f (z) for each $t \in \mathbb{R}$ and $z \in \mathbb{C}$. Suppose that k(t) = $f(z_0, t) \neq C$ for some $z_0 \in \mathbb{C}$. Since the singleton {0} is a set of spectral synthesis in $L^\infty(\mathbb{R})$ ([5]) there exists $\lambda_0 \neq 0$ in the spectrum of k. Let $\varphi \in L^1(\mathbb{R})$ so that $\hat{\varphi}$ is supported in $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ and $k * \varphi \neq 0$.

Then $F(z, t) = \int f(z, t-s)\, \varphi(s)\, ds \neq 0$, $F \in V(f)$ and for each z the spectrum of $F(z, \cdot)$ lies in $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$.

For each fixed θ let P (z, t) = F(z – r$e^{i\theta}$, t – s – 2Im(z·r $e^{-i\theta}$)). The function P(z,·) is a translate in t of F(z - r$e^{i\theta}$, ·) implying that the spectrum of P(z ,·) lies in (λ_0- ε, λ_0+ ε) for each z. Taking average over θ preserve this property implying that each *" radial translate"* $T_{r,s}F$ and hence each function in V(F) shares this property. So V(F) cannot contain a spherical Bessel function and by Wiener theorem there exists (λ, ν) in Spe (F) $\subseteq$ Spe (f) contradicting that Spe (f) = { ρ } proving that f (z, t) = f (z) $\forall$ t $\in \mathbb{R}$ and z $\in \mathbb{C}$.

In this case V (f) is the *w**- closure of {f * h: h radial in $L^1(\mathbb{R}^2)$}. Since, as proved by Hertz ([He]), the circle C_ρ= (z $\in \mathbb{C}$: $|z|$ =ρ } is a set of spectral synthesis, each radial bounded function on $\mathbb{R}^2$ with spectrum C_ρ is of the form C $J_0(\rho|z|)$ which proves the first part of the theorem.

Suppose now that Spe (f) = {(λ_0, ν_0)}. We claim that for each z the spectrum of f(z, ·) is {λ_0}. Suppose that for some z_0 the spectrum of f (z_0,·) contains μ $\neq \lambda_0$. Let φ $\in$ $L^1(\mathbb{R})$ such that $\hat{\varphi}$ is supported in U = [μ-ε, μ+ε] so that $\lambda_0 \notin$ U and g (z, t) = $\int$ f(z, t -s) φ(s)ds $\neq$ 0. The function g $\in$ V(f) and for each z the spectrum of g(z, ·) is contained in U. This property is shared by all functions in V(g) $\subseteq$ V(f). By Wiener theorem Spe (g) contains some (λ, ν) with $\lambda \neq \lambda_0$, contradicting our assumption.

Since the point {λ_0} is of spectral synthesis we obtain f(z, t) = $e^{i\lambda_0 t}\chi(z)$, for some $\chi \neq 0$.

It remains to show that $\chi(z) = W_{\nu_0}^{\lambda_0}(z) = e^{-|\lambda_0||z|^2} L_{\nu_0}(4\pi\, |\lambda_0|\, |z|^2)$.

We consider $W_\nu^{\lambda_0}$ as a finite singular measure on H supported on $\mathbb{C}$. Hence E(z, t) $\in$ V(f) where

E(z, t) = $(e^{i\lambda_0 t}\chi(z)) * W_\nu^{\lambda_0}$

$= \int e^{i\lambda_0(t-2\mathrm{Im}\, z\overline{w})}\chi(z-w)\, W_\nu^{\lambda_0}(w)dw$

$= e^{i\lambda_0 t}\, (\chi \overset{\lambda_0}{*} W_\nu^{\lambda_0})(z) = e^{i\lambda_0 t}(W_\nu^{\lambda_0} \overset{\lambda_0}{*} \chi)(z)$

$= (e^{i\lambda_0 t}\, W_\nu^{\lambda_0}) * \chi$

$= e^{i\lambda_0 t}\, W_\nu^{\lambda_0} < \chi, W_\nu^{\lambda_0} >$ since $e^{i\lambda_0 t}\, W_\nu^{\lambda_0}$ is a character.

Here $f \overset{\lambda}{*} h$ denotes the *twisted* convolution of functions on $\mathbb{C}$. But Spe (f) = $\{(\lambda_0, \nu_0)\}$ implying that $< \chi, W_\nu^{\lambda_0} > = 0$ for all $\nu \neq \nu_0$ and it follows that $\chi = C\, W_{\nu_0}^{\lambda_0}$.

Corollary. The set B = $\{\rho \geq 0\}$ is a set of spectral synthesis. That is, if Spe (f) = B then f is contained in the *w** - closure of the subspace spanned by $\{J_0(\rho|z|) : \rho \geq 0\}$.

Proof:
As in the first part of the proof of **Theorem 1** it follows that f (z, t) = f ($|z|$) implying that f is contained in the *w** - closure of the subspace spanned by $\{J_0(\rho|z|) : \rho \geq 0\}$.

Let us denote by L_{00}^1 (H) the closed sub-algebra of L_0^1(H) defined by L_{00}^1 (H) = $\{f \in L_0^1(H) : \int f(z, t)dt = 0, \ \forall z \in \mathbb{C}\}$. In the following we give a simple proof to Theorem 4.7 in [1].

Theorem 2. Wiener Tauberian theorem holds for $L_{00}^1(H)$. That is, each proper closed ideal is contained in a maximal regular ideal which is in the Laguerre part of the maximal ideal space of L_0^1(H).

Proof:
Let f (z, t) be a continuous function in L_0^∞(H) which is not a function of z only. Let Q(f) = the *w** - closure of $\{f * h : h \in L_{00}^1(H)\}$. By duality, we have to show that Q(f) contains a function $e^{i\lambda t} W_\nu^\lambda$ for some $\lambda \in \mathbb{R}\backslash\{0\}$ and integer ν. Suppose that $f(z_0, \cdot) \neq C$.

Let λ_0 be in the spectrum of $f(z_0, \cdot)$ and $\varphi \in L^1(\mathbb{R})$ with $\hat{\varphi}(\lambda_0) \neq 0$, $\hat{\varphi}(\lambda) = 0$ in U = [- ϵ, ϵ], $\lambda_0 \notin U$.

Let $F(z, t) = \int f(z, t-s)\varphi(s)\, ds \neq 0$ and F ϵ V(f). Then for each z the spectrum of $F(z, \cdot) \subseteq \mathbb{R} \setminus U$ and all functions in V(F) share this property as shown in the proof of Theorem 1. By Wiener Tauberian theorem for L_0^1(H) the subspace V(F) and hence V(f) contains a spherical function which cannot be in the Bessel part implying that $e^{i\lambda t}\, W_\nu^\lambda \in V(f)$ for some $\lambda \in \mathbb{R} \setminus \{0\}$ and integer ν. It remains to show that $e^{i\lambda t}\, W_\nu^\lambda \in Q(f)$.

We know that there exists a net $g_\tau \in L_0^1(H)$ such that $f^* g_\tau \to e^{i\lambda t}\, W_\nu^\lambda$ in *w**. Let $\chi \in L^1(\mathbb{R})$, $\hat{\chi}(0) = 0$ and $\hat{\chi}(\lambda) = 1$. Then $\int (f^* g_\tau)(z, t\text{-}s)\chi(s)ds \to e^{i\lambda t}\, W_\nu^\lambda$ in *w**.

By Fubini, it follows that $f^* E_\tau \to e^{i\lambda t}\, W_\nu^\lambda$ in *w** where $E_\tau(w, \alpha) = \int g_\tau (w, s)\varphi(\alpha + s)ds$ satisfy $\int E_\tau(w, \alpha)d\alpha = 0$. Hence $E_\tau \in L_{00}^1(H)$, and $e^{i\lambda t}\, W_\nu^\lambda \in Q(f)$ which completes the proof of the theorem.

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