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Equivalence of K-Functionals and Modulus of Smoothness Generated by the q-Dunkl Operator

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Abstract

On this paper, we study the equivalence between K-functionals and modulus of smoothness tied to a q-Dunkl operator.

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1 Introduction

Given a positive real number r and a positive integer m, the classical modulus of smoothness is defined for a function $f \in L^2(\mathbb{R})$ by

$$w_m(f,r) = sup_{0<h\leq r} \parallel \triangle_h^m \parallel_2$$

where

$$\triangle_h^m = (\tau^h - I)^m f,$$

I being the unit operator and τ^h stands for the usual translation operator given by $\tau^h f(x) = f(x+h)$.While the classical K-functional, introduced in [4], is defined by

$$K_m(f,r) = inf\{\parallel f-g \parallel_2 + r \parallel D^m g \parallel_2; g \in W_2^m\},$$

where W_2^m be the Sobolev space constructed by the operator $D = \frac{d}{dx}$,

$$W_2^m = \{f \in L^2(\mathbb{R}) : D^j f \in L^2(\mathbb{R}), j = 1, ..., m\}.$$

An oustanding result of the theory of approximation of functions on $\mathbb{R}$, which establishes the equivalence between modulus of smoothness and $K-$functionals, can be formulated as follows:

Theorem 1.1. $(see[1])$ *There are two positive constants c_1 and c_2 such that for all $f \in L^2(\mathbb{R})$ and $r > 0$:*

$$c_1 w_m(f,r) \leq K_m(f,r^m) \leq c_2 w_m(f,r)$$

IN the classical theory of approximation of functions on $\mathbb{R}$, the modulus of smoothness are basically built by means of the translation operators $f \longrightarrow f(x+y)$. The translation operator is used for the the construction of modulus of continuity and smoothness which are the fundamental elements of direct and inverse theorems in the approximation theory.
Many generalized modulus of smoothness are often more convenient than the usual ones for the study of the connection between the smoothness properties of a function and the best approximations of this function in weight functional spaces $(see[5]-[8])$.
In addition to modulus of smoothness, the K-functionals introduced by J.Peetre [4] have turned out to be a simple efficient tool for the description of smoothness properties of functions. The study of the connection between these two quantities is one of the main problems in the theory of approximation of functions. In the classical setting, the equivalence of modulus of smoothness these problems are studied, for example , in [1].
The present paper is organized as follows: In Section 2, we present some preliminary results and notations that will be useful in the sequel and we establish some results associated with the q-Dunkl operator. In section 3, the main result is the proof of the theorem on the equivalence of a $K-$functional and the modulus of smoothness constructed by the q-Dunkl oprtator.

2 Preliminaries

Throughout this paper, we assume $0 < q < 1$.
In this section , we provide some facts about harmonic analysis related to the q-Dunkl Operator $\Lambda_{\alpha,q}$. We cite here, as briefly as possible, only those properties actually required for the discussion. For more details we refer to [2].
We put $\mathbb{R}_q = \{\pm q^n : n \in Z\}$, $\widehat{\mathbb{R}_q} = \mathbb{R}_q \cup \{0\}$.
For a $\in \mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_0 = 1; (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), n = 1,2,....; (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$$

We also denote

$$[x]_q = \frac{1-q^x}{1-q}, \quad x \in \mathbb{C}, \quad \text{and } [n]_{q!} = \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}$$

A q-analogue of the classical exponential function is given by $(see[6,\ 7])$

$$e(z;q^2) = cos(-iz;q^2) + isin(-iz;q^2)$$

where

$$cos(x;q^2) = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)} \frac{x^{2n}}{[2n]_{q!}}, \quad sin(x;q^2) = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)} \frac{x^{2n+1}}{[2n+1]_{q!}}$$

These three functions are entire on $\mathbb{C}$ and when q tends to 1, they tend to the corresponding classical ones pointwise and uniformly on compacts.
The q-Jackson integrals are defined by $(see[3])$.

$$\int_a^b f(x)d_qx = (1-q)b\sum_{n=0}^{\infty} q^n f(bq^n) - (1-q)a\sum_{n=0}^{\infty} q^n f(aq^n)$$

and

$$\int_{-\infty}^{+\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n) + (1-q)\sum_{n=-\infty}^{\infty} q^n f(-q^n).$$

Notation 2.1. *We denote by*
$\bullet S_q(\mathbb{R}_q)$ *the space of functions f defined on $\mathbb{R}_q$ satisfying, for all m, n non negative integers,*

$$P_{m,n,q}(f) = \sup_{x\in\mathbb{R}_q} \mid x^m \delta_q^n f(x) \mid < \infty$$

and

$$\lim_{x\longrightarrow 0} \delta_q^n f(x) \ \ (in\ \mathbb{R}_q) \ \ exists$$

$\bullet S_q^{'}(\mathbb{R}_q)$ *the space of tempered distributions on $\mathbb{R}_q$. It is the topological dual of $S_q(\mathbb{R}_q)$.*
$\bullet L_q^p(\mathbb{R}_q) = \{f : \|f\|_{p,q} = (\int_{-\infty}^{+\infty} |f(x)|^p d_qx)^{\frac{1}{p}} < \infty\}.$
$\bullet L_q^{\infty}(\mathbb{R}_q) = \{f : \|f\|_{\infty,q} = \sup_{x\in\mathbb{R}_q} \mid f(x) \mid < \infty\}.$

The q-Dunkl operator is defined by

$$\Lambda_{\alpha,q} = \delta_q[f_e + q^{2\alpha+1} f_o](x) + [2\alpha+1]_q \frac{f(x) - f(-x)}{2x}, \tag{1}$$

with f_e, f_o are respectively the even and odd parts of f and δ_q is the q^2-analogue differential operator constructed by R. L. Rubin. It is shown in [1] that for all $\lambda \in \mathbb{C}$, the differential-difference equation

$$\Lambda_{\alpha,q}(f) = i\lambda f, \qquad f(0) = 1, \tag{2}$$

admits a unique C^∞ solution on $\mathbb{R}$, denoted by $\Psi_\lambda^{\alpha,q}$ given by

$$\Psi_\lambda^{\alpha,q}(x) = j_\alpha(\lambda x; q^2) + \frac{i\lambda x}{[2\alpha+2]_q} j_{\alpha+1}(\lambda x; q^2) \tag{3}$$

with $j_\alpha(\lambda x; q^2)$ is the normalized third Jackson's $q-$Bessel function given by:

$$j_\alpha(\lambda x; q^2) = \sum_{n=0}^{+\infty} (-1)^n \frac{\Gamma_{q^2}(\alpha+1) q^{n(n+1)}}{\Gamma_{q^2}(\alpha+n+1)\Gamma_{q^2}(\alpha+1)} (\frac{x}{1+q})^{2n}.$$

The $q-$ Dunkl kernel $\Psi_\lambda^{\alpha,q}$ admits the following properties

Proposition 2.1. *For all $x, a, \lambda \in \mathbb{R}$ and $a \in \mathbb{C}$, we have*

i) $\Psi_\lambda^{\alpha,q}(x) = \Psi_x^{\alpha,q}(\lambda), \qquad \Psi_{a\lambda}^{\alpha,q}(x) = \Psi_\lambda^{\alpha,q}(ax).$

ii) If $\alpha = -\frac{1}{2}$, then $\Psi_\lambda^{\alpha,q}(x) = e(i\lambda x; q^2)$.
For $\alpha > -\frac{1}{2}$, $\Psi_\lambda^{\alpha,q}$ has the following $q-$integral representation of Mehler type

$$\Psi_\lambda^{\alpha,q}(x) = \frac{(1+q)\Gamma_{q^2}(\alpha+1)}{2\Gamma_{q^2}(\frac{1}{2})\Gamma_{q^2}(\alpha+\frac{1}{2})} \int_{-1}^{1} \frac{(t^2q^2; q^2)_\infty}{(t^2q^{2\alpha+1}; q^2)_\infty} (1+t) e(i\lambda xt; q^2) d_q t. \tag{4}$$

iii) For all $\lambda \in \mathbb{R}_q$, $\Psi_\lambda^{\alpha,q}$ is bounded on $\widehat{\mathbb{R}_q}$ and we have

$$\mid \Psi_\lambda^{\alpha,q}(x) \mid \leq \frac{4}{(q;q)_\infty} \ \forall x \in \widehat{\mathbb{R}_q}$$

Lemma 2.1. *for all $x \in \mathbb{R}_q$*

1. $\mid j_\alpha(x, q^2) \mid \leq \frac{2}{(q;q)_\infty}$.
2. $\mid 1 - \Psi_\lambda^{\alpha,q} \mid \leq \mid \lambda x \mid$.
3. *There is $c_1 > 0$ such that $|1 - \Psi_\lambda^{\alpha,q}| \geq c_1$ with $|x| \geq 1$.*

Proof. Analog of lemma 2.9 in [8] □

For $\alpha \geq -\frac{1}{2}$, the $q-$Dunkl transform is defined on $L^1_{\alpha,q}(\mathbb{R}_q)$ by

$$\mathcal{F}_D^{\alpha,q}(f)(\lambda) = c_{\alpha,q} \int_{-\infty}^{+\infty} f(x)\Psi_{-\lambda}^{\alpha,q}(x) \mid x^{2\alpha+1} \mid d_q x, \;\; \lambda \in \widehat{\mathbb{R}_q}, \tag{5}$$

where $c_{\alpha,q} = \frac{(1+q)^{-\alpha}}{2\Gamma_{q^2}(\alpha+1)}$

Proposition 2.2. *i) If $f \in L^1_{\alpha,q}(\mathbb{R}_q)$ then $\mathcal{F}_D^{\alpha,q}(f) \in L^1_{\alpha,q}(\mathbb{R}_q)$,*

$$\| \mathcal{F}_D^{\alpha,q}(f) \|_{\infty,q} \leq \frac{4c_{\alpha,q}}{(q;q)_\infty} \| f \|_{1,\alpha,q} \tag{6}$$

ii) Let $f \in L^1_{\alpha,q}(\mathbb{R}_q)$ and $\mu \in S'_q(\mathbb{R}_q)$, then for $n = 1, 2, \dots$ we have

$$\mathcal{F}_D^{\alpha,q}(\Lambda^n_{\alpha,q} f)(\lambda) = (i\lambda)^n \mathcal{F}_D^{\alpha,q}(f)(\lambda) \tag{7}$$

$$\mathcal{F}_D^{\alpha,q}(\Lambda^n_{\alpha,q} \mu) = (-i\lambda)^n \mathcal{F}_D^{\alpha,q}(\mu) \tag{8}$$

iii) For all $f \in L^1_{\alpha,q}(\mathbb{R}_q)$, we have

$$\forall x \in \mathbb{R}_q, \;\; f(x) = c_{\alpha,q} \int_{\mathbb{R}} \mathcal{F}_D^{\alpha,q}(f)(\lambda)\Psi_\lambda^{\alpha,q}(x) \mid \lambda \mid^{2\alpha+1} d_q\lambda. \tag{9}$$

Theorem 2.1. *i) For $\alpha \geq -\frac{1}{2}$, the $q-$Dunkl transform $\mathcal{F}_D^{\alpha,q}$ is an isomorphism from $S_q(\mathbb{R}_q)$ onto itself. Moreover, for all $f \in S_q(\mathbb{R}_q)$, we have*

$$\| \mathcal{F}_D^{\alpha,q}(f) \|_{2,\alpha,q} = \| f \|_{2,\alpha,q} \,. \tag{10}$$

ii) The $q-$Dunkl transform can be uniquely extended to an isometric isomorphism on $L^2_{\alpha,q}(\mathbb{R}_q)$. Its inverse transform $(\mathcal{F}_D^{\alpha,q})^{-1}$ is given by:

$$(\mathcal{F}_D^{\alpha,q})^{-1}(f)(x) = c_{\alpha,q} \int_{-\infty}^{+\infty} f(x)\Psi_\lambda^{\alpha,q}(x) \mid \lambda^{2\alpha+1} \mid d_q\lambda = (\mathcal{F}_D^{\alpha,q})(f)(-x)$$

The generalized $q-$Dunkl translation operator is defined for $f \in L^2_{\alpha,q}(\mathbb{R}_q)$ and $x, y \in \mathbb{R}_q$ by

$$T_y^{\alpha;q}(f)(x) = c_{\alpha,q} \int_{-\infty}^{+\infty} \mathcal{F}_D^{\alpha,q}(f)(\lambda)\psi_\lambda^{\alpha,q}(x)\psi_\lambda^{\alpha,q}(y) \mid \lambda \mid^{2\alpha+1} d_q\lambda,$$

$$T_0^{\alpha;q}(f) = f.$$

Proposition 2.3. *i) For $f \in L^2_{\alpha,q}(\mathbb{R}_q), x, y \in \mathbb{R}_q$, we have*

$$\mathcal{F}_D^{\alpha,q}(T_y^{\alpha;q} f)(\lambda) = \psi_\lambda^{\alpha,q}(y)\mathcal{F}_D^{\alpha,q}(f)(\lambda). \tag{11}$$

ii) *For $f \in S_q(\mathbb{R}_q)$ and $y \in \mathbb{R}_q$, we have*

$$\Lambda_{\alpha,q} T_y^{\alpha;q} f = T_y^{\alpha;q} \Lambda_{\alpha,q} f. \tag{12}$$

iii) *If $f \in L^2_{\alpha,q}(\mathbb{R}_q)$ (resp. $S_q(\mathbb{R}_q)$) then $T_y^{\alpha;q}(f) \in L^2_{\alpha,q}(\mathbb{R}_q)$ (resp. $S_q(\mathbb{R}_q)$) and we have*

$$\| T_y^{\alpha;q}(f) \|_{2,\alpha,q} \leq \frac{4}{(q;q)_\infty} \| f \|_{2,\alpha,q} \tag{13}$$

Notation 2.2. *From now on assume $m = 1, 2, ...$ Let $W^m_{2,\alpha,q}$ be the Sobolev type space constructed by the $q-$Dunkl operator $\Lambda_{\alpha,q}$, i.e.,*

$$W^m_{2,\alpha,q} = \{f \in L^2_{\alpha,q}(\mathbb{R}_q) \,:\, \Lambda^j_{\alpha,q} f \, \in L^2_{\alpha,q}(\mathbb{R}_q), \ j = 1, 2..., m\}$$

More explicitly, $f \in W^m_{2,\alpha,q}$ if and only if for each $j = 1, 2..., m$, there is a function in $L^2_{\alpha,q}(\mathbb{R}_q)$ abusively denoted by $\Lambda^j_{\alpha,q} f$, such that $\Lambda^j_{\alpha,q} T_f^{\alpha;q} = T^{\alpha;q}_{\Lambda^j_{\alpha,q} f}$

Proposition 2.4. *For $f \in W^m_{2,\alpha,q}$ we have*

$$\mathcal{F}_D^{\alpha,q}(\Lambda^m_{\alpha,q} f)(\lambda) = (-i\lambda)^m \mathcal{F}_D^{\alpha,q}(f)(\lambda). \tag{14}$$

Proof. using proposition 5.7 in [2] □

3 Equivalence of K-Functionnals and Modulus of Smoothness

Definition 3.1. *Let $f \in L^2_q(\mathbb{R}_q)$ and $r > 0$. Then*
• *The generalized modulus of smoothness is defined by*

$$w_m(f, r)_{2,\alpha,q} = sup_{0<h\leq r} \| \triangle^m_h f \|_{2,\alpha,q}$$

where

$$\triangle^m_h f = (T_h^{\alpha;q} - I)^m f,$$

I being the unit operator.
• *The generalized K-functional is defined by*

$$K_m(f, r)_{2,\alpha,q} = inf\{\| f - g \|_{2,\alpha,q} + r \| \Lambda^m_{\alpha,q} g \|_{2,\alpha,q}; g \in W^m_{2,\alpha,q}\}.$$

The following theorem establishes the equivalence of the modulus of Smoothness and the K-functional.

Theorem 3.1. *There are two positive constants c_1 and c_2 such that*

$$c_1 w_m(f, r)_{2,\alpha,q} \leq K_m(f, r^m)_{2,\alpha,q} \leq c_2 w_m(f, r)_{2,\alpha,q},$$

for all $f \in L^2_q(\mathbb{R}_q)$ and $r > 0$.

In order to prove Theorem 3.1, we shall need some preliminary results.

Lemma 3.1. *Let* $f \in L^2_q(\mathbb{R}_q)$ *and* $h > 0$. *Then*

$$\| \triangle_h^m f\|_{2,\alpha,q} \leq \frac{4^m}{(q;q)^m_\infty}\|f\|_{2,\alpha,q} \tag{15}$$

$$\mathcal{F}_D^{\alpha,q}(\triangle_h^m f)(\lambda) = (\psi_\lambda^{\alpha,q}(h) - 1)^m \mathcal{F}_D^{\alpha,q}(f)(\lambda). \tag{16}$$

Proof. The result follows easily by using $(11), (13)$, and an induction on m. □

Lemma 3.2. *Let* $f \in W^m_{2,q}$, $r > 0$. *The following inequality is true:*

$$w_m(f,r)_{2,\alpha,q} \leq c_2 r^m \|\Lambda^m_{\alpha,q} f\|_{2,\alpha,q}$$

Proof. Assume that $h \in]0, r]$. By $(14), (16)$ and theorem 2.1 we have

$$\begin{aligned}
\| \triangle_h^m f\|^2_{2,\alpha,q} &= ||\mathcal{F}_D^{\alpha,q}(\triangle_h^m f)(\lambda)||^2_{2,\alpha,q} \\
&= ||(1-\psi_\lambda^{\alpha,q}(h))^m \mathcal{F}_D^{\alpha,q}(f)(\lambda)||^2_{2,\alpha,q} \\
&= h^{2m}||\tfrac{(1-\psi_\lambda^{\alpha,q}(h))^m}{(hi\lambda)^m}(i\lambda)^m \mathcal{F}_D^{\alpha,q}(f)(\lambda)||^2_{2,\alpha,q}
\end{aligned}$$

According to Lemma 2.1, with all $l \in \widehat{\mathbb{R}_q}$ we have the inequality $|\frac{(1-\psi_\lambda^{\alpha,q}(l))^m}{l^m}| \leq (\beta\lambda)^m$, where $\beta \in \mathbb{R}$. Then we deduce $\|\triangle_h^m f\|^2_{2,\alpha,q} \leq h^{2m}\beta^{2m}||(i\lambda)^m \mathcal{F}_D^{\alpha,q}(f)(\lambda)||^2_{2,\alpha,q} = h^{2m}\beta^{2m}||\mathcal{F}_D^{\alpha,q}(\Lambda^m_{\alpha,q} f)(\lambda)||^2_{2,\alpha,q} = h^{2m}\beta^{2m}||\Lambda^m_{\alpha,q} f||^2_{2,\alpha,q}$.
Calculating the supremum with respect to all $h \in]0, r]$, we obtain $w_m(f,r)_{2,\alpha,q} \leq c_2 r^m \|\Lambda^m_{\alpha,q} f\|_{2,\alpha,q}$, where $c_2 = \beta^m$. □

Definition 3.2. *For any function* $f \in L^2_q(\mathbb{R}_q)$ *and any number* $\nu > 0$ *let us define the function*

$$P_\nu(f)(x) := c_{\alpha,q}\int_{-\nu}^{\nu} \mathcal{F}_D^{\alpha,q}(f)(\lambda)\Psi_\lambda^{\alpha,q}(x) d_q\lambda = (\mathcal{F}_D^{\alpha,q})^{-1}(\mathcal{F}_D^{\alpha,q}(f)\chi_\nu(x)),$$

where $\chi_\nu(\lambda)$ *is the characteristic function of the segment* $[-\nu,\nu], (\mathcal{F}_D^{\alpha,q})^{-1}$ *is the inverse Fourier transform. One can easily prove that the function* $P_\nu(f)$ *is infinitely differentiable and belongs to all classes* $W^m_{2,\alpha,q}, m \in \mathbb{N}$.

Lemma 3.3. *There is a positive constant* c_3 *such that*

$$\|f - P_\nu(f)\|_{2,\alpha,q} \leq c_3 \| \triangle^m_{1/\nu} f\|_{2,\alpha,q}$$

for any $f \in L^2_q(\mathbb{R}_q)$ *and* $\nu > 0$.

Proof. Using the Plancherel formula, we obtain

$$\begin{aligned}\|f-P_\nu(f)\|^2_{2,\alpha,q} &= \|\mathcal{F}^{\alpha,q}_D(f-P_\nu f)\|^2_{2,\alpha,q}\\ &= \|(1-\chi_\nu(\lambda))\mathcal{F}^{\alpha,q}_D(f)(\lambda)\|^2_{2,\alpha,q}\\ &= \int_{\mathbb{R}} \mid 1-\chi_\nu(\lambda)\mid^2\mid \mathcal{F}^{\alpha,q}_D(f)(\lambda)\mid^2 d_q\lambda\end{aligned}$$

By lemma 2.1 there is a constant $c_1>0$ such that

$$\mid 1-\Psi^{\alpha,q}_{\lambda/\nu}(x)\mid \geq c_1,$$

for all $\lambda\in\mathbb{R}$ with $\mid\lambda\mid\geq\nu$. From this, (16) we get

$$\begin{aligned}\|f-P_\nu(f)\|^2_{2,\alpha,q} &\leq c_1^{-2m}\int_{|\lambda|\geq\nu}\mid 1-\Psi^{\alpha,q}_\lambda(x/\nu)\mid^{2m}\mid \mathcal{F}^{\alpha,q}_D(f)(\lambda)\mid^2 d_q\lambda\\ &= c_1^{-2m}\int_{\mathbb{R}}\mid \mathcal{F}^{\alpha,q}_D(\triangle^m_{1/\nu}f)(\lambda)\mid^{2m} d_q\lambda\\ &= c_1^{-2m}\|\triangle^m_{1/\nu}f\|^2_{2,\alpha,q}.\end{aligned}$$

We get the inequality $\|f-P_\nu(f)\|_{2,\alpha,q}\leq c_3\|\triangle^m_{1/\nu}f\|_{2,\alpha,q}$, where $c_3=\frac{1}{(c_1)^m}$. □

Corollary 3.1. *There is a positive constant c_3 such that*

$$\|f-P_\nu(f)\|_{2,\alpha,q}\leq c_3 w_m(f,1/\nu)_{2,q}$$

for any $f\in L^2_q(\mathbb{R}_q)$ and $\nu>0$.

Lemma 3.4. *There is a positive constant c_4 such that*

$$\|\Lambda^m_{\alpha,q}(P_\nu(f))\|_{2,\alpha,q}\leq c_4\nu^m\|\triangle^m_{1/\nu}f\|_{2,\alpha,q}$$

for any $f\in L^2_q(\mathbb{R}_q)$ and $\nu>0$.

Proof.

$$\begin{aligned}\|\Lambda^m_{\alpha,q}(P_\nu(f))\|_{2,\alpha,q} &= \|\mathcal{F}^{\alpha,q}_D(\Lambda^m_{\alpha,q}(P_\nu(f)))\|_{2,\alpha,q}\\ &= \|(i\lambda)^m\chi_\nu(\lambda)\mathcal{F}^{\alpha,q}_D(f)(\lambda)\|_{2,\alpha,q}\\ &= \|\lambda^m\chi_\nu(\lambda)\mathcal{F}^{\alpha,q}_D(f)(\lambda)\|_{2,\alpha,q}\\ &= \|\tfrac{\lambda^m\chi_\nu(\lambda)}{(1-\Psi^{\alpha,q}_{\lambda/\nu})^m}(1-\Psi^{\alpha,q}_{\lambda/\nu})^m\mathcal{F}^{\alpha,q}_D(f)(\lambda)\|_{2,\alpha,q}\end{aligned}$$

Note that $\sup_{\lambda\in\mathbb{R}}\frac{\lambda^m\chi_\nu(\lambda)}{|1-\Psi^{\alpha,q}_{\lambda/\nu}|^m}=\nu^m\sup_{|\lambda|\leq\nu}\frac{(\lambda/\nu)^m}{|1-\Psi^{\alpha,q}_{\lambda/\nu}|^m}=\nu^m\sup_{|t|\leq1}\frac{t^m}{|1-\Psi^{\alpha,q}_t|^m}$
Let $c_4=\sup_{|t|\leq1}\frac{t^m}{|1-\Psi^{\alpha,q}_t|^m}$ □

Corollary 3.2. *There is a positive constant c_4 such that*

$$\|\Lambda_{\alpha,q}^m(P_\nu(f))\|_{2,\alpha,q} \leq c_4\nu^m w_m(f,1/\nu)_{2,\alpha,q}$$

for any $f \in L_q^2(\mathbb{R}_q)$ and $\nu > 0$.

Proof. of Theorem 3.1

1. Let $h \in]0,r]$, $g \in W_{2,\alpha,q}^m$. Using lemma 3.2 and lemma 3.1, we have

$$\begin{aligned}\|\triangle_h^m f\|_{2,\alpha,q} &= \|\triangle_h^m(f-g+g)\|_{2,\alpha,q} \\ &\leq \|\triangle_h^m(f-g)\|_{2,\alpha,q} + \|\triangle_h^m(g)\|_{2,\alpha,q} \\ &\leq 2^m\|f-g\|_{2,\alpha,q} + c_2h^m\|\Lambda_{\alpha,q}^m(g)\|_{2,\alpha,q} \\ &\leq c_5(\|f-g\|_{2,\alpha,q} + r^m\|\Lambda_{\alpha,q}^m(g)\|_{2,\alpha,q}),\end{aligned}$$

   where $c_5 = max\{2^m, c_2\}$. Calculating the supremum with respect to $h \in]0,r]$ and the infimum with respect to all possible functions $g \in W_{2,\alpha,q}^m$, we obtain

$$w_m(f,r)_{2,\alpha,q} \leq c_5K_m(f,r^m)_{2,\alpha,q}.$$

2. Since $P_\nu(f) \in W_{2,\alpha,q}^m$, by the definition of a $K-$ functional we have

$$K_m(f,r^m)_{2,\alpha,q} \leq \|f-P_\nu(f)\|_{2,\alpha,q} + r^m\|\Lambda_{\alpha,q}^m(P_\nu(f))\|_{2,\alpha,q}$$

   Using Corollaries 3.1 and 3.2, we get

$$K_m(f,r^m)_{2,\alpha,q} \leq c_3w_m(f,1/\nu)_{2,\alpha,q} + c_4(r\nu)^m w_m(f,1/\nu)_{2,\alpha,q}.$$

   Since ν is an arbitrary positive value, choosing $r = \frac{1}{\nu}$, we obtain

$$K_m(f,r^m)_{2,\alpha,q} \leq c_6w_m(f,1/\nu)_{2,\alpha,q},$$

   where $c_6 = c_3 + c_4$. This concludes the proof.

□

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