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A Class of Nonlinear Perturbed Difference Equations

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Abstract

In this paper we study a class of nonlinear singularly perturbed difference equations with boundary value conditions. Provide that the singular perturbation parameter is small enough, we give sufficient conditions to guarantee the existence and uniqueness of the solution. An iterative process is proposed to determine the asymptotic representation of the solution.

Mathematics Subject Classification: 39A10

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1 Introduction

Many systems are described through difference equations containing small parameters, representing discrete processes or resulting from the discretization of continuous models. The presence of a small parameter naturally leads to suggesting perturbation methods. The method of asymptotic expansions has been widely used when solving singularly perturbed differential equations, but has not yet been developed for singularly perturbed difference equations which remains an open field of research. In previous papers [5, 6, 7, 8, 9, 10], we studied linear models, we noticed that there is no need to add *correction terms* in the asymptotic expansion, as for the singularly perturbed differential equations. In this work, we extend results obtained in [5, 6], to a class of nonlinear difference equations containing a small parameter. We study for boundary value problems, the existence and uniqueness of the solution, and we present a recursive procedure to find asymptotic approximate solutions. The method relies on the use of Faa Di Bruno's formula [4] and the Implicit Function Theorem [2]. The paper is organized as follows. We study in Section 2, a BVP presenting a *left-end perturbation*, and in Section 3, the method is immediately extended to difference equations presenting a *right-end perturbation*.

2 Left End Perturbation

Let $\mathcal{U}_k : \overline{E}_0 \times \overline{E}_1 \times \cdots \times \overline{E}_r \longrightarrow \Re$, and $\mathcal{A}_k : \overline{E}_0 \times \overline{E}_1 \times \cdots \times \overline{E}_{r-1} \longrightarrow \Re$, be sequences of functions, n-differentiable in their arguments, where E_k are open intervals, $E_{k+1} \subset E_k$, and the bar denote the closure. We consider the BVP

$$\begin{gathered} \varepsilon \mathcal{U}_k \left(x_k, x_{k+1}, \cdots, x_{k+r}\right) + \mathcal{A}_k \left(x_k, x_{k+1}, \cdots, x_{k+r-1}\right) = 0, \\ 0 \leq k \leq N - r, \end{gathered} \tag{1}$$

$$x_0 = \alpha_0, \quad x_1 = \alpha_1, \quad \cdots, \quad x_{r-2} = \alpha_{r-2}, \quad x_N = \beta, \tag{2}$$

where N is a fixed integer; α_i, $i = 0, \cdots, r-2$, β, are fixed real numbers, and $|\varepsilon| \ll 1$. Some linear cases of this model are studied in [1, 3, 5, 6]. We seek for the solution sequence $(x_k(\varepsilon))_{k=0,\cdots,N}$, a uniform asymptotic development in ε,

$$x_k = x_k^{(0)} + \varepsilon x_k^{(1)} + \varepsilon^2 x_k^{(2)} + \cdots + \varepsilon^n x_k^{(n)} + \mathcal{O}(\varepsilon^{n+1}). \tag{3}$$

For convenience, $D_1^{k_1} D_2^{k_2} \cdots D_n^{k_n} f$ will denote the partial derivative $\frac{\partial^k f(x_1, \cdots, x_n)}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}$.

2.1 Reduced Problem

We obtain the *reduced problem*:

$$\begin{gathered} \mathcal{A}_k \left(x_k, x_{k+1}, \cdots, x_{k+r-1}\right) = 0, \quad 0 \leq k \leq N - r, \\ x_0 = \alpha_0, \quad x_1 = \alpha_1, \quad \cdots, \quad x_{r-2} = \alpha_{r-2}, \end{gathered} \tag{4}$$

$$x_N = \beta, \tag{5}$$

by canceling the small parameter ε in (1) − (2). The order of equation (1) reduces to $r - 1$, exhibiting a *boundary layer behavior* at the final value (5), i.e., the values $x_0, \ldots, x_{N-1}$, can be computed without knowing the final condition; it is stated as *singular perturbation* in accordance with the singular perturbation theory of ODES. Thus, we will only have to solve recursively the IVP (4) since the boundary conditions are uncoupled. To guarantee that BVP (4)−(5) has a unique solution, we set the following hypothesis.

H1 Suppose $(\alpha_0, \alpha_1, \cdots, \alpha_{r-2}) \in \overline{E}_0 \times \overline{E}_1 \times \cdots \times \overline{E}_{r-2}$, and each of the functions $x \mapsto \mathcal{A}_k(x_k^{(0)}, x_{k+1}^{(0)}, \cdots, x_{k+r-2}^{(0)}, x)$, where $x_k^{(0)} = \alpha_k, k = 0, \cdots, r-2$, has a range which contains the value 0, and

$$\forall x \in \overline{E}_{r-1}, \quad D_r\mathcal{A}_k(x_k^{(0)}, x_{k+1}^{(0)}, \cdots, x_{k+r-2}^{(0)}, x) \neq 0.$$

Proposition 2.1 *If H1 holds, then problem (4)−(5) has a unique solution.*

2.2 Preliminaries

Observe that the BVP (1)−(2) may be reviewed as a system of equations depending on a parameter. Let $X = (x_0, x_1, \dots, x_N)$, we define the function

$$\begin{aligned}
&\mathcal{F} : (-1, 1) \times \Re^{N+1} \longrightarrow \Re^{N+1}, \quad \mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_N),\\
&\mathcal{F}_k(\varepsilon, X) = x_k - \alpha_k, \quad 0 \leq k \leq r-2,\\
&\mathcal{F}_{k+r-1}(\varepsilon, X) = \varepsilon\mathcal{U}_k(x_k, \cdots, x_{k+r}) + \mathcal{A}_k(x_k, \cdots, x_{k+r-1}), \quad 0 \leq k \leq N-r,\\
&\mathcal{F}_N(\varepsilon, X) = x_N - \beta,
\end{aligned}$$

then $\mathcal{F}$ is of class C^n, just as are $\mathcal{A}_k$ and $\mathcal{U}_k$, and (1) − (2) is equivalent to $\mathcal{F}(\varepsilon, x_0, x_1, \cdots, x_N) = 0$. The Implicit Function Theorem ensures, under some assumptions, the existence of a function $g(\varepsilon) = (g_0(\varepsilon), g_1(\varepsilon), \dots, g_N(\varepsilon))$, having the same regularity as $\mathcal{F}$, such that $\mathcal{F}(\varepsilon, g(\varepsilon)) = 0$. Therefore,

$$\begin{gathered}
\varepsilon\mathcal{U}_k(g_k(\varepsilon), \cdots, g_{k+r}(\varepsilon)) + \mathcal{A}_k(g_k(\varepsilon), \cdots, g_{k+r-1}(\varepsilon)) = 0,\\
k = 0, \cdots, N-r,\\
g_0(\varepsilon) = \alpha_0, \quad g_1(\varepsilon) = \alpha_1, \quad \cdots, g_{r-2}(\varepsilon) = \alpha_{r-2}, \quad g_N(\varepsilon) = \beta.
\end{gathered} \tag{6}$$

For $\varepsilon \simeq 0$, the Maclaurin expansion of $g_k(\varepsilon)$, which is of class C^n, is

$$g_k(\varepsilon) = g_k(0) + \varepsilon \frac{dot g_k(0)}{1!} + \varepsilon^2 \frac{\ddot{g}_k(0)}{2!} + \cdots + \varepsilon^n \frac{g_k^{(n)}(0)}{n!} + \mathcal{O}(\varepsilon^{n+1}). \tag{7}$$

In order to determine the coefficients in (7), we use Faa di Bruno's formula [4] to give explicit sequential differentiation of equations (6). We drop the arguments for $\mathcal{U}_k$, $\mathcal{A}_k$ and g_k to write concise formulas.

Lemma 2.2 *Assume that g_k, $\mathcal{U}_k$ and $\mathcal{A}_k$ satisfy (6), and that all the necessary derivatives are defined. Then we have for $n \geq 2$,*

$$\begin{aligned}
\textstyle\sum_{l=0}^{r-1} D_{1+l}\mathcal{A}_k g_{k+l}^{(n)} = &-\underline{\textstyle\sum_0 \cdots \sum_n} \frac{n! D_1^{p_1^{(0)}} \cdots D_r^{p_r^{(0)}} \mathcal{A}_k \prod_{i=1}^n (g_k^{(i)})^{q_{i1}^{(0)}} \cdots (g_{k+r-1}^{(i)})^{q_{ir}^{(0)}}}{\prod_{i=1}^n (i!)^{k_i^{(0)}} \prod_{i=1}^n \prod_{j=1}^r q_{ij}^{(0)}!}\\
&- \textstyle\sum_0 \cdots \sum_{n-1} \frac{n! D_1^{p_1^{(1)}} \cdots D_{r+1}^{p_{r+1}^{(1)}} \mathcal{U}_k \prod_{i=1}^{n-1} (g_k^{(i)})^{q_{i1}^{(1)}} \cdots (g_{k+r}^{(i)})^{q_{ir+1}^{(1)}}}{\prod_{i=1}^{n-1} (i!)^{k_i^{(1)}} \prod_{i=1}^{n-1} \prod_{j=1}^{r+1} q_{ij}^{(1)}!},
\end{aligned} \tag{8}$$

where the coefficients $k_i^{(l)}$, $q_{ij}^{(l)}$ *and* $p_j^{(l)}$, $l = 0, 1$, *are all nonnegative integer solutions of the Diophantine equations*

$$\begin{array}{c} \sum_0 \to k_1^{(l)} + 2k_2^{(l)} + \cdots + (n-l)k_{n-l}^{(l)} = n - l, \quad l = 0,1, \\ \sum_i \to q_{i1}^{(l)} + q_{i2}^{(l)} + \cdots + q_{ir+l}^{(l)} = k_i^{(l)}, \quad i = 1, \cdots, n-l, \quad l = 0,1, \\ p_j^{(l)} = q_{1j}^{(l)} + q_{2j}^{(l)} + \cdots + q_{nj}^{(l)}, \quad j = 1, \cdots, r+l, \quad l = 0,1, \\ k^{(l)} = p_1^{(l)} + p_2^{(l)} + \cdots + p_{r+l}^{(l)} = k_1^{(l)} + k_2^{(l)} + \cdots + k_{n-l}^{(l)}, \quad l = 0,1, \end{array} \tag{9}$$

in $\underline{\sum_0 \cdots \sum_n}$ *we fix* $k_n^{(0)} = 0$*; the case* $k_n^{(0)} = 1$ *is omitted and corresponds to the left side of equation (8).*

Proof 2.3 *By induction, we prove that for* $n \geq 1$*, we have*

$$\begin{array}{c} \varepsilon \frac{d^n}{d\varepsilon^n}[\mathcal{U}_k(g_k(\varepsilon), \cdots, g_{k+m}(\varepsilon))] + n\frac{d^{n-1}}{d\varepsilon^{n-1}}[\mathcal{U}_k(g_k(\varepsilon), \cdots, g_{k+m}(\varepsilon))] \\ + \frac{d^n}{d\varepsilon^n}[\mathcal{A}_k(g_k(\varepsilon), \cdots, g_{k+m-1}(\varepsilon)] = 0. \end{array} \tag{10}$$

Cancelling the parameter ε *in (10), we find*

$$\frac{d^n}{d\varepsilon^n}[\mathcal{A}_k(g_k(0), \cdots, g_{k+m-1}(0))] + n\frac{d^{n-1}}{d\varepsilon^{n-1}}[\mathcal{U}_k(g_k(0), \cdots, g_{k+m}(0))] = 0. \tag{11}$$

Expanding Faa Di Bruno's Formula into the total derivatives $\frac{d^n \mathcal{A}_k}{d\varepsilon^n}$ *and* $\frac{d^{n-1}\mathcal{U}_k}{d\varepsilon^{n-1}}$ *in (11), we obtain (8) by arranging the equation (9) so that on the left hand side, we have the terms corresponding to* $k_n^{(0)} = 1$.

2.3 Main results

In this section, we establish the expansion (3), and we describe how to compute its coefficients. The coefficients of 0^{th}-order approximation define the solution sequence of the *reduced* problem (4)−(5). Substituting

$$x_k^{(n)} = \frac{g_k^{(n)}(0)}{n!}, \quad k = 0, \cdots, N, \tag{12}$$

into (8), we find the following. For 1^{st}-order approximation, we have

$$\begin{array}{c} \sum_{l=0}^{r-1} D_{1+l}\mathcal{A}_k(x_k^{(0)}, \cdots, x_{k+r-1}^{(0)})x_{k+l}^{(1)} = -\mathcal{U}_k(x_k^{(0)}, \cdots, x_{k+r}^{(0)}), \\ 0 \leq k \leq N - r, \\ x_0^{(1)} = 0, \quad x_1^{(1)} = 0, \quad \cdots \quad , \quad x_{r-2}^{(1)} = 0, \quad x_N^{(1)} = 0. \end{array} \tag{13}$$

The coefficients $x_0^{(1)}, x_1^{(1)}, \cdots, x_{N-1}^{(1)}$, can be computed recursively with the initial values regardless of the final value $x_N^{(1)}$, but we need the 0^{th}-order solution which must be calculated from (4)−(5). To give shorter formulas, we remove the arguments for $\mathcal{A}_k$ and $\mathcal{U}_k$, which are always $(x_k^{(0)}, \cdots, x_{k+r-1}^{(0)})$ and $(x_k^{(0)}, \cdots, x_{k+r}^{(0)})$, respectively. For the 2^{th}-order developement, we have

$$\begin{array}{c} \sum_{l=0}^{r-1} D_{1+l}\mathcal{A}_k x_{k+l}^{(2)} = -\frac{1}{2!}\sum_{l=0}^{r-1} D_{1+l}^2\mathcal{A}_k \left(x_{k+l}^{(1)}\right)^2 - \sum_{l=0}^{r} D_{1+l}\mathcal{U}_k x_{k+l}^{(1)} \\ -\sum_{0\le l<m\le r-1} D_{1+l}D_{1+m}\mathcal{A}_k x_{k+l}^{(1)} x_{k+m}^{(1)}, \quad 0 \le k \le N-r, \\ x_0^{(2)} = 0, \quad x_1^{(2)} = 0, \quad \cdots \quad , \quad x_{r-2}^{(2)} = 0, \quad x_N^{(2)} = 0. \end{array} \tag{14}$$

For the n^{th}-order development, we have

$$\begin{array}{c} \sum_{l=0}^{r-1} D_{1+l}\mathcal{A}_k x_{k+l}^{(n)} = -\underline{\sum_0 \cdots \sum_n} \frac{D_1^{p_1^{(0)}} \cdots D_r^{p_r^{(0)}} \mathcal{A}_k \prod_{i=1}^{n-1} (x_k^{(i)})^{q_{i1}^{(0)}} \cdots (x_{k+r-1}^{(i)})^{q_{ir}^{(0)}}}{\prod_{i=1}^{n-1}\prod_{j=1}^{r} q_{ij}^{(0)}!} \\ -\sum_0 \cdots \sum_{n-1} \frac{D_1^{p_1^{(1)}} \cdots D_{r+1}^{p_{r+1}^{(1)}} \mathcal{U}_k \prod_{i=1}^{n-1} (x_k^{(i)})^{q_{i1}^{(1)}} \cdots (x_{k+r}^{(i)})^{q_{ir+1}^{(1)}}}{\prod_{i=1}^{n-1}\prod_{j=1}^{r+1} q_{ij}^{(1)}!}, \end{array} \tag{15}$$

where in $\underline{\sum_0 \cdots \sum_n}$ we have $k_n^{(0)} = 0$, the initial values are

$$x_0^{(n)} = 0, \quad x_1^{(n)} = 0, \quad \cdots \quad , \quad x_{r-2}^{(n)} = 0, \tag{16}$$

and the final value is

$$x_N^{(n)} = 0. \tag{17}$$

Above are the consecutive steps to compute approximate solutions. After completing the calculations, we substitute in (3), i.e., we obtain the desired n^{th}-order approximate solution to the BVP (1)−(2). This calculation process is validated in the following theorem.

Theorem 2.4 *If H1 holds, there exists $\epsilon > 0$, such that for all $|\varepsilon| < \epsilon$, the boundary value problem (1)−(2) has a unique solution which satisfy (3); where $x_k^{(0)}$, $x_k^{(1)}$, $x_k^{(2)}$, $x_k^{(n)}$, are the solutions of (4)−(5), (13), (14), (15)−(16)−(17), respectively.*

Proof 2.5 *To give a detailed proof, we adapt the Implicit Function Theorem proof technique to our case. We consider the function $\widehat{\mathcal{F}}\left(\tilde{X}\right) = \left(\varepsilon, \mathcal{F}\left(\tilde{X}\right)\right)$, where $\varepsilon \in (-1, 1)$, $\tilde{X} = (\varepsilon, X)$, and let $\mathcal{D}\widehat{\mathcal{F}}$ denotes its jacobian matrix. Obviously from H1, $\det \mathcal{D}\widehat{\mathcal{F}}\left(\tilde{X}^{(0)}\right) = \prod_{i=0}^{N-r} \mathcal{D}_2\mathcal{A}_i\left(x_i^{(0)}, x_{i+1}^{(0)}, \cdots, x_{i+r-1}^{(0)}\right) \neq 0$, i.e., $L = \left[\mathcal{D}\widehat{\mathcal{F}}\left(\tilde{X}^{(0)}\right)\right]^{-1}$ is well defined, and we can prove that $\widehat{\mathcal{F}}$ is locally invertible. Because $\mathcal{D}\widehat{\mathcal{F}}$ is continuous, we can choose $\rho > 0$ such that,*

$$\forall \tilde{X} \in B\left(\tilde{X}^{(0)}, \rho\right) : \|\mathcal{D}\widehat{\mathcal{F}}\left(\tilde{X}\right) - \mathcal{D}\widehat{\mathcal{F}}\left(\tilde{X}^{(0)}\right)\| < \frac{1}{2\|L\|}. \tag{18}$$

We denote $\epsilon = \frac{\rho}{2\|L\|}$*, the mapping* $G_Y(\tilde{X}) = \tilde{X} - L\left(\widehat{\mathcal{F}}(\tilde{X}) - Y\right)$ *satisfies* $\|\mathcal{D}G_Y\| < \frac{1}{2}$ *if* $|\varepsilon| < \epsilon$ *and* Y *in* $B\left(\widehat{\mathcal{F}}\left(\tilde{X}^{(0)}\right), \epsilon\right) = B(0,\epsilon)$*, i.e.,* G_Y *is a contraction that maps* $B\left(\tilde{X}^{(0)}, \rho\right)$ *to itself, then* G_Y *has a unique fixed point. There exists a unique* $\tilde{X}$ *in* $B\left(\tilde{X}^{(0)}, \rho\right)$*, such that* $Y = \widehat{\mathcal{F}}(\tilde{X})$ *for* Y *fixed in* $B(0,\epsilon)$*. Consider* $g(\varepsilon) = (g_0(\varepsilon), \cdots, g_N(\varepsilon))$*, if* $|\varepsilon| < \epsilon$*, there exists a unique* $(\varepsilon, g(\varepsilon))$ *in* $B\left(\tilde{X}^{(0)}, \rho\right)$*, such that* $(\varepsilon, 0, \cdots, 0) = \widehat{\mathcal{F}}(\varepsilon, g(\varepsilon))$*. Then for* ε *fixed,* $|\varepsilon| < \epsilon$*, there exists a unique* $g(\varepsilon)$ *such that* $\mathcal{F}(\varepsilon, g(\varepsilon)) = 0$*, i.e., the BVP (1)–(2) has a unique solution. Also, as* $\widehat{\mathcal{F}}$ *and* $\widehat{\mathcal{F}}^{-1}$*, the function* g *is* $C^n(-\epsilon, \epsilon)$*, we have* $\dot{g}(\varepsilon) = -\frac{\partial \mathcal{F}(\varepsilon, g(\varepsilon))}{\partial \varepsilon}\left(\frac{\partial \mathcal{F}(\varepsilon, g(\varepsilon))}{\partial X}\right)^{-1}$ *and we can compute the derivatives of* $\dot{g}$ *by Lagrange's Inversion Formula or any other inversion formula. Because the asymptotic development is unique, it is not necessary to do further computations, Lemma 2.2 gives these derivatives.*

The iterative problems given above are defined for any order provide $\mathcal{U}_k$ and $\mathcal{A}_k$ are *smooth* functions. In this case, the development (3) converges to the solution of (1)–(2). We deduce from Theorem 2.4, the following result.

Theorem 2.6 *If H1 holds,* $\mathcal{U}_k$ *and* $\mathcal{A}_k$ *are smooth, then there exists* $\epsilon > 0$*, for all* $|\varepsilon| < \epsilon$*, the boundary value problem (1)–(2) has a unique solution* $x_k(\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n x_k^{(n)}$*, where* $x_k^{(0)}$*,* $x_k^{(1)}$*,* $x_k^{(2)}$*,* $x_k^{(n)}$*, are solutions of (4)–(5), (13), (14), (15)–(16)–(17), respectively.*

3 Right End Perturbation

We consider the perturbed difference equation

$$\mathcal{U}_k(x_{k+1}, \cdots, x_{k+r}) + \varepsilon \mathcal{A}_k(x_k, \cdots, x_{k+r}) = 0, \quad 0 \le k \le N - r, \tag{19}$$

called *right-end perturbation*, and we associate the boundary conditions

$$x_0 = \alpha, \quad x_{N-r+2} = \beta_{r-2}, \quad \cdots, \quad x_{N-1} = \beta_1, \quad x_N = \beta_0. \tag{20}$$

We follow the same procedure described in Section 2. For the *reduced problem*

$$x_0 = \alpha, \tag{21}$$

$$\begin{array}{l} \mathcal{U}_k(x_{k+1}, x_{k+2}, \cdots, x_{k+r}) = 0, \quad 0 \le k \le N - r, \\ x_{N-r+2} = \beta_{r-2}, \quad \cdots, \quad x_{N-1} = \beta_1, \quad x_N = \beta_0, \end{array} \tag{22}$$

we can compute backward the values $x_1, \ldots, x_{N-r+1}$, without using (21); the *boundary layer behavior* is located at the initial value.

H2 Suppose that $(\beta_{r-2}, \cdots, \beta_0) \in \overline{E}_{N-r+2} \times \overline{E}_{N-1} \times \cdots \times \overline{E}_{r-2}$, and each of the functions that each of the functions $x \mapsto \mathcal{U}_k \left(x, x_{k+2}^{(0)}, \cdots, x_{k+r}^{(0)}\right)$, has range containing zero, and $D_1\mathcal{U}_k(x, x_{k+2}^{(0)}, \cdots, x_{k+r}^{(0)}) \neq 0$ for all x in $\overline{E}_{k+1}$.

Proposition 3.1 *If H2 holds, then (21)−(22) has a unique solution.*

In the following theorem, we prove that, under some conditions, there exists an open neighborhood $V(0)$, and $g(\varepsilon) = (g_0(\varepsilon), \dots, g_N(\varepsilon)) \in C^n(V)$ such that

$$\begin{array}{c} \varepsilon\mathcal{U}_k(g_{k+1}(\varepsilon), \cdots, g_{k+r}(\varepsilon)) + \mathcal{A}_k(g_k(\varepsilon), \cdots, g_{k+r}(\varepsilon)) = 0, \quad k = 0, \dots, N-r \\ g_0(\varepsilon) = \alpha, \quad g_{N-r+2}(\varepsilon) = \beta_{r-2}, \quad \cdots \quad , \quad g_{N-1}(\varepsilon) = \beta_1, \quad g_N(\varepsilon) = \beta_0. \end{array} \tag{23}$$

The coefficients in the Maclaurin expansion (7) satisfy the formula given in the following Lemma (the arguments are droped and can be deduced easily).

Lemma 3.2 *Assume that the functions g_k, $\mathcal{U}_k$ and $\mathcal{A}_k$ satisfy (23), and that all the necessary derivatives are defined. Then we have for $n \geq 2$,*

$$\begin{array}{l} \sum_{l=0}^{r-1} D_{1+l}\mathcal{U}_k g_{k+l}^{(n)} = -\underline{\sum_0 \cdots \sum_n} \frac{n! D_1^{p_1^{(0)}} \cdots D_r^{p_r^{(0)}} \mathcal{U}_k \prod_{i=1}^{n} (g_{k+1}^{(i)})^{q_{i1}^{(0)}} \cdots (g_{k+r}^{(i)})^{q_{ir}^{(0)}}}{\prod_{i=1}^{n} (i!)^{k_i^{(0)}} \prod_{i=1}^{n} \prod_{j=1}^{r} q_{ij}^{(0)}!} \\ \quad - \sum_0 \cdots \sum_{n-1} \frac{(n)! D_1^{p_1^{(1)}} \cdots D_{r+1}^{p_{r+1}^{(1)}} \mathcal{A}_k \prod_{i=1}^{n-1} (g_k^{(i)})^{q_{i1}^{(1)}} (g_{k+1}^{(i)})^{q_{i2}^{(1)}} \cdots (g_{k+r}^{(i)})^{q_{ir+1}^{(1)}}}{\prod_{i=1}^{n-1} (i!)^{k_i^{(1)}} \prod_{i=1}^{n-1} \prod_{j=1}^{r+1} q_{ij}^{(1)}!}. \end{array} \tag{24}$$

where the coefficients in (24) are the solutions of the Diophantine (9); in $\underline{\sum_0 \cdots \sum_n}$ we fix $k_n^{(0)} = 0$, the case $k_n^{(0)} = 1$ corresponds to the left side of equation (24).

We can already indicate the main result of this section. The proofs are similar to that in Section 2. From (3), (12) and (24), we deduce the 1^{st}-order development

$$\begin{array}{c} \sum_{l=0}^{r-1} D_{1+l}\mathcal{U}_k(x_{k+1}^{(0)}, \cdots, x_{k+r}^{(0)}) x_{k+l+1}^{(1)} = -\mathcal{A}_k(x_k^{(0)}, \cdots, x_{k+r}^{(0)}), \\ k = 0, \cdots, N-r, \\ x_0^{(1)} = 0, \quad x_{N-r+2}^{(1)} = 0, \quad \cdots \quad , \quad x_{N-1}^{(1)} = 0, \quad x_N^{(1)} = 0, \end{array} \tag{25}$$

we compute the values $x_0^{(1)}$, $x_1^{(1)}$, $\cdots$, $x_{N-1}^{(1)}$ backward from the final values, and we need the $0th$-order solution. For $2nd$-order development, we have

$$\begin{array}{c} \sum_{l=0}^{r-1} D_{1+l}\mathcal{U}_k x_{k+l+1}^{(2)} = -\frac{1}{2!} \sum_{l=0}^{r-1} D_{1+l}^2 \mathcal{U}_k \left(x_{k+l+1}^{(1)}\right)^2 - \sum_{l=0}^{r} D_{1+l}\mathcal{A}_k x_{k+l}^{(1)} \\ - \sum_{0 \leq l < m \leq r-1} D_{1+l} D_{1+m} \mathcal{U}_k x_{k+l+1}^{(1)} x_{k+m+1}^{(1)}, \quad k = 0, \dots, N-r, \\ x_0^{(2)} = 0, \quad x_{N-r+2}^{(2)} = 0, \quad \cdots \quad , \quad x_{N-1}^{(2)} = 0, \quad x_N^{(2)} = 0, \end{array} \tag{26}$$

For *n-th* order development, $n \geq 2$, we have

$$\begin{aligned} \textstyle\sum_{l=0}^{r-1} D_{1+l}\mathcal{U}_k x_{k+l}^{(n)} = &-\underline{\textstyle\sum_0 \cdots \sum_n} \frac{D_1^{p_1^{(0)}} \cdots D_r^{p_r^{(0)}} \mathcal{U}_k \prod_{i=1}^{n} (x_{k+1}^{(i)})^{q_{i1}^{(0)}} \cdots (x_{k+r}^{(i)})^{q_{ir}^{(0)}}}{\prod_{i=1}^{n} \prod_{j=1}^{r} q_{ij}^{(0)}!} \\ &- \textstyle\sum_0 \cdots \sum_{n-1} \frac{D_1^{p_1^{(1)}} \cdots D_{r+1}^{p_{r+1}^{(1)}} \mathcal{A}_k \prod_{i=1}^{n-1} (x_k^{(i)})^{q_{i1}^{(1)}} \cdots (x_{k+r}^{(i)})^{q_{ir+1}^{(1)}}}{\prod_{i=1}^{n-1} \prod_{j=1}^{r+1} q_{ij}^{(1)}!}, \end{aligned} \tag{27}$$

in $\underline{\sum_0 \cdots \sum_n}$ we fix $k_n^{(0)} = 0$. The iteration is done backward from the final values

$$x_{N-r+2}^{(n)} = 0, \quad \cdots \quad , \quad x_{N-1}^{(n)} = 0, \quad x_N^{(n)} = 0, \tag{28}$$

while the initial value remains fixed,

$$x_0^{(n)} = 0. \tag{29}$$

Theorem 3.3 *If H2 holds, there exists* $\epsilon > 0$, *such that for all* $|\varepsilon| < \epsilon$, *the boundary value problem (19)–(20) has a unique solution which satisfy 3; the coefficients* $x_k^{(0)}$, $x_k^{(1)}$, $x_k^{(2)}$, $x_k^{(n)}$, *are the solutions of (21)–(22), (25), (26), (27)–(28)–(29), respectively.*

Theorem 3.4 *If H2 holds, and* $\mathcal{U}_k$, $\mathcal{A}_k$ *are smooth functions, then there exists* $\epsilon > 0$ *such that for all* $|\varepsilon| < \epsilon$,*the boundary value problem (19)–(20) has a unique solution which satisfy* $x_k(\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n x_k^{(n)}$ *where the coefficients* $x_k^{(0)}$, $x_k^{(1)}$, $x_k^{(2)}$, $x_k^{(n)}$ *are the solution of the problems (22)–(21), (25), (26), (27)–(28)–(29), respectively.*

3.1 Conclusion

Contrary to singularly perturbed differential equations, we can find uniform development for singularly perturbed difference equations, however both are similar in giving mode decoupling. The difficulty of solving a boundary problem was overcome by a straightforward iterative method giving asymptotic approximate solutions. Exactly, the same procedure can be applied to initial value problems with *left-end perturbation*, or to final value problems with *right-end perturbation*. For both kinds of perturbation, the method is easy to use and produces reliable results with few iterations.

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