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Stieltjes Transformation as the Iterated Laplace Transformation

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Abstract

Following the approach of Pilipovic, Stankovic and Takaci we express the Modified Stieltjes transformationT_{r+1} of a distribution from S'_+ as the iterated Laplace transformation.

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1 Introduction

Among various approaches to define the Stieltjes transformation of a distribution, wẽ following Pilipovic, Stankovic, Takaci and Lighthill [5, 7], which is related to a subspace of tempered distribution support $[0, \infty)$. The definition of the Stieltjes transformation given by Marichev [6] is modified in such a way that it is available for the whole space of tempered distributions

defined on R with supports inR'_+ . Using the notion of quasiasymptotic behaviour at infinity, theorems of Abelian type for the Laplace transformation is proved by an approach, which are quite different form the one given in [5,7,8].**Areas of application:** Ruin theory, Statistical physics, Stoichiometry and modelling Chemical reactions, Ecology, particularly population modeling, Evolutionary biology, Optimization in computer science, Telecommunications, Applied Probability and queueing theory etc. In the first part of the paper we give the definition of quasiasymptotic behaviour of distribution at infinity of an element form S'_+ . In this paper we give the definition of space$L'(r)$, Stieltjes transformation, modified Stieltjes transformationT_{r+1} and generalized modified Stieltjes transformation $\tilde{T}_{r+1}$. This enables us to obtain, in the second part of the paper, the quasiasymptotic behaviour of distributions at infinity of modified Stieltjes transformation of distribution form S'_+ as the iterated Laplace transformation.
Notation: As usually R,C,N are the spaces of real complex and natural numbers; $N_0 = N \cup \{0\}$. S'_+ denotes the space of tempered distributions with support in the $[0, \infty)$. The space of rapidly decreasing functions is denoted S , S' is the space of all distribution of slow growth . T_{r+1} denotes modified Stieltjes transformation with index r .

2 Preliminary Notes

Definition 2.1 Slowly Varying Function:*A positive continuous function L is defined on $(0, \infty)$ is called slowly varying function at∞ if for every $k > 0$* $\lim_{k\to\infty} \frac{L(kx)}{L(k)} = 1$. *We denote by$\sum_\infty$ the set of slowly varying functions at ∞ . For properties of slowing varying functions we refer the reader [10].*
If L is slowly varying functions at ∞ , then for every $\epsilon > 0$ there is $A_\epsilon > 0$ such $x^{-\epsilon} < L(x) < x^\epsilon$ if$x > A_\epsilon$

Definition 2.2 The quasiasymptotic behaviour of distribution (q.a.b.) at infinity; *If T be a distribution from S'_+ such that distributional limit* $\lim_{k\to\infty} \frac{T(kx)}{\rho(k)} = \gamma(x)$ *exists in $S'(\gamma(x) \neq 0)$, then T is called the quasiasymptotic behaviour at infinity related to regularly function $\rho(k) = k^\alpha L(k)$ with the limit γ; we can write this as $T \sim^q \gamma$ as $x \to \infty$. Here ρ is regularly varying at infinity and limit γ from S'_+, is of the form $\gamma(x) = Cf_{\alpha+1}(x)$*
We shall repeat in this section some well-known facts about quasiasymptotic behaviour from [8].
Let $f \in S'_+$. It is said that f has the q.a.b. at ∞ with the limit$g \neq 0$ with respect $k^\alpha L(k), L \in \sum_\infty \left((\frac{1}{k})^\alpha L(\frac{1}{k}), L \in \sum_0\right), \alpha \in \mathbf{R}$, if

$$\lim_{k\to\infty}\left\langle \frac{f(kt)}{k^\alpha L(k)},\phi(t)\right\rangle = \langle g(t),\phi(t)\rangle, \phi\in S.$$

Definition 2.3 *A function $\rho:(a,\infty)\to\mathbf{R}, a\in\mathbf{R}$ is called regularly varying function at infinity if it is positive , measurable and there exists a real number α such that for each $x>0$,*

$$\lim_{k\to\infty}\frac{\rho(kx)}{\rho(k)} = x^\alpha$$

where α is called the index of ρ .

2.4 SPACE $L'(r)$ We extend the definition of the space $I'(r)$ given in [8] and using the same idea we give the definition of **space** $L'(r)$.
$L'(r), r\in\mathbf{R}\backslash(-N)$ denotes the space of all distributions $f\in S'_+(\mathbf{R})$ such that there exists $k\in N_0$ and locally integrable function F, supp $F\subset[0,\infty)$, so that f is of the form
$f=t^{-r}D^kF$ (2.1)
and there exists $C=C(F)$ and $\epsilon=\epsilon(F)>0$ such that
$|F(x)|\le C(1+x)^{r+k-\epsilon}, x\ge 0$ (2.2)
The **Stieltjes transformation** $S_r(f)(s), r\in\mathbf{R}\backslash(-N)$ is complex valued function, defined by
$S_r(f(t))(s)=\int_0^\infty\frac{f(t)}{(s+t)^{r+1}}dts\in C\backslash(-\infty,0], 0<t<\infty, r\in\mathbf{R}\backslash(-N)$ (2.3)
Modified Stieltjes Transformation is introduced by Marichev is defined as $T_\alpha(f(x))=\frac{1}{\Gamma(\alpha)}\int_0^\infty(1+\frac{x}{y})^{-\alpha}.\frac{1}{y}f(y)dy, x\in C\backslash(-\infty,0], 0<y<\infty, r\in\mathbf{R}\backslash(-N)$ (2.4)
It can be written as,
$T_\alpha(f)(x)=\frac{1}{\Gamma(\alpha)}\int_0^\infty\frac{(y^{\alpha-1})f(y)}{(x+y)^\alpha}dy, x\in C\backslash(-\infty,0].$ (2.5)
setting, $r=\alpha-1, f(t)=y^{\alpha-1}f(y)$ in (2.3) we get
$S_{\alpha-1}(y^{\alpha-1}f(y))(x)=\int_0^\infty\frac{y^{\alpha-1}f(y)}{(x+y)^\alpha}dy, x\in C\backslash(-\infty,0),$ (2.6)
$\Gamma(r+1)T_{r+1}(f)(s)=S_r(y^rf)(z), z\in C\backslash(-\infty,0], r\in R\backslash(-N)$ (2.7)
Modified Stielejes Transformation T_{r+1} is defined $T_{r+1}(f), r\in R\backslash(-N)$ is complex valued function defined by
$\Gamma(r+1)T_{r+1}(f)(s)=(r+1)_k\int_0^\infty\frac{F(t)}{(s+t)^{(r+1+k)}}dt, r\in R\backslash(-N), s\in C\backslash 0<t<\infty.$
Ganeralized modified Stieltjes transformation $\tilde{T}_{r+1}$ The $\tilde{T}_{r+1}$ - transformation of distribution $f\in S'_+(R)$ is complex valued function $\tilde{T}_{r+1}$ defined by $\Gamma(r+1)T_{r+1}(f)(s)=\lim w\to\infty < f(t),\eta(t)(s+t)^{-r-1}\exp(-wt) >, w\in R, s\in\wedge\subset(C\backslash(-\infty,0]), \eta\in A(s)$. Here $\wedge$ is the set of complex numbers for which this limit exists and A(s) is the family of all smooth functions defined on R.

3 Main Results

Mainly, the results of this section are from [5]

The Modified Stieltjes transformation as the iterated Laplace transformation:

The Stieltjes transformation $T_{r+1}$of a distributionS'_+ can be expressed by the iterated Laplace transform [5]. We can obtain the well-known classical theorem of Abelian type for the Laplace transformation to obtain results for the distributional modified Stieltjes transformation.In this section we shall explain the all results in one-dimensions for modified Stieltjes transformations.

Note that the definition of Laplace transformation in [5] and [7] are different. Let $T = D^k F$, where $k \in N, F$ is locally integrable function with supp $F \subset [0, \infty)$ and let $F(x) = o(x^{r+k+\alpha}, x \to \infty, \alpha > 0, r \in R\backslash(-N)$. Clearly in our Notation $f \in L'(r)$ with this assumption on T.

Theorem 3.1 *[5] Let $\tilde{f}(p) = (Lf)(p) = \langle f(x, e^{-px}) \rangle$. Then*
$(\Gamma(r+1)T_{r+1}f)(s) = \frac{1}{\Gamma(r+1)} L(\theta(x)x^r \tilde{f}(x))(s) = \frac{1}{\Gamma(r+1)} \int_0^\infty x^r \tilde{f}(x) e^{-sx} dx,$
$Res > 0, (\theta$ is Heaviside's function).
The modified Stieltjes inverse transformation $\Gamma(r+\bar{1})T_{r+1}, r \in R\backslash(-N)$ defined in [5] as an operation, which maps elements of a suitable space of holomorphic functions (on $C\backslash(-\infty, 0]$).
Into $I'(r)$ such that $\Gamma(r+1)T_{r+1}(\Gamma(r+\bar{1})T_{r+1}G(s) = G(s), s \in C\backslash(-\infty, 0]$.

Theorem 3.2 *[5] Assume that G has the following properties (i) G is holomorphic in $c\backslash(-\infty, 0]$; (ii) there exists $\beta > 0$ such that$s^\beta |G(s)|$ is bounded when $|s| \to \infty$. Then $\Gamma(r+\bar{1})T_{r+1}G$ exists , it is unique and belongs to $L'(r)$. If $j(x) = x^{-r}(L^{-1}G)(x), x \in R$ then j can be analytically continued onto the half-plane $Rez > 0$. For $f(x) = (L^{-1}j)(x), Rez > 0$ we have $(\Gamma(r+\bar{1})T_{r+1}G)(x) = \Gamma(r+1)\bar{f}(x), x \in R$. ($\bar{f}$ is a conjugate function for f).*

The following proposition can obtain as a consequence of the structure theorem for tempered distributions.

Proposition 3.3 A necessary and sufficient condition for $f \in S'$ to hold is that there exist a continuous function F of bounded growth $(|F(x)| \le C(1+|x|^2)^{\frac{m}{2}}$ for a fixed $m \in N_0$) with support in R_+ such that $f = f_{-a+1} * F$

Lemma 3.4 If $r > -1, \delta > 0, \omega > 0,$ and $0 < \epsilon < Res, (r, \delta, \omega, \epsilon \in R)$ then
$e^{-(\omega,t)}\eta(t) \int_0^\infty e^{-((s+t)} uu^r du$ tend to zero in S as $\delta \to 0^+$. (3.1)

Here$\{t \ge \delta\} = \{t \in R, t > \delta\}$

Lemma 3.5 If $r > -1, \delta > 0, \omega > 0, and 0 < \epsilon < Res, (r, \delta, \omega, \epsilon \in R)$ then

$$\left\langle f(t),\eta(t)e^{-(\omega,t)}\int_\delta^\infty e^{-(\omega+t)u}.u^r du\right\rangle = \int_\delta^\infty \langle f(t),\eta(t)e^{-(s+u)t}\rangle e^{-(\delta u)}u^r du \quad (3.2)$$

Proposition 3.6 Suppose that $r > -1$ and $f \in S'$ there exists an $(\Gamma(r+1)T_{r+1}f)(s)$.
Then for $Res > 0.(\Gamma(r+1)T_{r+1}f)(s) = \frac{1}{\Gamma(r+1)}\lim_{\omega\to 0^+}\int_0^\infty \langle f(t),\eta(t)e^{-(s+u)t}\rangle e^{-(\delta,u)}u^r du$

Proof: For $Res > 0$ and $r > -1$ we have
$$\frac{\Gamma(r+1)}{(s+t)^{r+1}} = \int_0^\infty e^{-(s+t)u}u^r du \quad (3.3)$$
From Lemmas 3.4 and 3.5, it follows for every $\omega > 0$

$$\left\langle f(t),\eta(t)\exp^{-\omega t}\frac{1}{(s+t)^{r+1}}\right\rangle = \left\langle f(t),\eta(t)e^{-\omega t}\frac{1}{\Gamma(r+1)}\int_0^\infty e^{-(s+t)u}u^r du\right\rangle$$

$$= \frac{1}{\Gamma(r+1)}\left\langle f(t),\eta(t)e^{-\omega t}\int_\delta^\infty e^{-(s+t)u}u^r du\right\rangle + \frac{1}{\Gamma(r+1)}\langle f(t),\eta(t)e^{-\omega t}\int_0^\delta e^{-(s+t)u}u^r du\rangle$$

$$= \frac{1}{\Gamma(r+1)}\int_0^\infty \left\langle f(t),\eta(t)\exp^{-(u+v)t}\right\rangle e^{-su}u^r du.$$

Theorem 3.7 If $f \in S'$, then there exists a continuous function F of polynomial growth with support in $\bar{R}_+$ and $a \in R_+$ such that $f = f_{-a+1} * F$. If $(\Gamma(r+1)T_{r+1})$exists for a fixed $r > -1$, then$Res > 0$.
$$(\Gamma(r+1)T_{r+1})(s) = \frac{1}{\Gamma(r+1)}\lim_{\omega\to 0^+}\int_0^\infty\left(\int_0^\infty F(t)e^{-(\omega+u)t}dt\right)e^{-su},(\omega+u)^a u^r du \quad (3.4)$$

Proof: From proposition 1.7.3 and 1.7.6, it follows that
$$(\Gamma(r+1)T_{r+1}) = \frac{1}{\Gamma(r+1)}\lim_{\omega\to 0^+}\int_0^\infty \langle (f_{-a+1}F),\eta(t)e^{-(\omega+u)t}\rangle.e^{-su}.u^r du. \quad (3.5)$$

Using the properties of the Laplace transformation and the fact that $(Lf_{a+1})(z) = \langle f_{a+1}(t),e^{izt}\rangle = \frac{1}{(-iz)^a}, Imz > 0$, we see that (3.4) follows from (3.5)

Theorem 3.8 Suppose that $f \in S'_+$ and that it have the quasi-asymptotic at infinity related to $k^a L(k)$. Then there is $p \in N_0, p+a > 0$, such that $r > max(a,-1)$ and $Res > 0$.
$f(s) = \frac{1}{\Gamma(r+1)}\int_0^\infty(\int_0^\infty F(t)e^{-ut}dt)e^{-su}u^{r+p}du$, $F = f_{p+1}F$ and $F \sim cf_{a+p+1}$ at infinity related to $k^{a+p}L(k), (c \neq 0)$

Proof: For $r > max(a,-1)$, we have $(\Gamma(r+1)T_{r+1}f) = (r+1)_p\int_0^\infty \frac{F(t)}{(s+t)^{r+1+p}}dt$, $s \in (c/\backslash\bar{R}_-)$ where $F = f_{p+1}f, a+p > 0$ using equation (3.3) we have, for $Res > 0$ and $r.max(a,-1)$,
$$(\Gamma(r+1)T_{r+1}f) = \frac{1}{\Gamma(r+1)}\int_0^\infty\left(\int_0^\infty e^{-(s+t)u}u^{r+p}du\right)F(t)dt$$
$$= \frac{1}{\Gamma(r+1)}\int_0^\infty(\int_0^\infty F(t)e^{-ut}dt)e^{-su}u^{r+p}du.$$

Conclusion: Using the notion of quasiasymptotic behaviour at infinity, theorems of Abelian type for the Laplace transformation is proved by an approach, which are quite different form the one given in [5,7,8].

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