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A Note on gw-Continuity Induced by Generalized w-Open Sets in Associated w-Spaces

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Abstract

The purpose of this paper is to introduce the notions of gw-continuous and gw^*-continuous functions induced by gw-open sets in associated w-spaces, and to study some properties and the relationships among such notions and other continuity.

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1 Introduction

Siwiec [20] introduced the notions of weak neighborhoods and weak base in a topological space. We introduced the weak neighborhood systems defined by using the notion of weak neighborhoods in [11]. The weak neighborhood system induces a weak neighborhood space which is independent of neighborhood spaces [4] and general topological spaces [2]. The notions of weak structure,

w-space, W-continuity and W^*-continuity were investigated in [12]. In [13], The notions of associated w-spaces, WO-continuity and WK-continuity were investigated. Levine [5] introduced the notion of g-closed subsets in topological spaces. In fact, the set of all g-closed subsets is a kind of weak structure. In the same way, we introduced the notions of gw-closed sets [15] and gw_τ-closed sets [16] in weak spaces, and investigated some basic properties of such notions. The notions of gW-continuous, gW^*-continuous, gW-irresolute, and gW^*-irresolute functions induced by gw-open sets introduced in [18], and also the notions of $gw_\tau W$-continuous, $gw_\tau W^*$-continuous, $gw_\tau W$-irresolute, and $gw_\tau W^*$-irresolute functions investigated in [18]. The purpose of this note is to introduce the notions of gw-continuity and gw^*-continuity and to study the relationships among such notions and the other continuity in associated w-spaces.

2 Preliminaries

Let X be a nonempty set. A subfamily w_X of the power set $P(X)$ is called a *weak structure* [12] on X if it satisfies the following:

(1) $\emptyset \in w_X$ and $X \in w_X$.

(2) For $U_1, U_2 \in w_X$, $U_1 \cap U_2 \in w_X$.

Then the pair (X, w_X) is called a *w-space* on X. Then $V \in w_X$ is called a *w-open* set and the complement of a w-open set is a *w-closed* set. The collection of all w-open sets (resp., w-closed sets) in a w-space X will be denoted by $W(X)$ (resp., $WC(X)$). We set $W(x) = \{U \in W(X) : x \in U\}$.

Let S be a subset of a topological space X. The closure (resp., interior) of S will be denoted by clS (resp., $intS$). A subset S of X is called a *preopen* set [9] (resp., α-open set [19], *semi-open* [6]) if $S \subset int(cl(S))$ (resp., $S \subset int(cl(int(S))), S \subset cl(int(S))$). The complement of a preopen set (resp., α-open set, *semi-open*) is called a *preclosed* set (resp., α-closed set, *semi-closed*). The family of all preopen sets (resp., α-open sets, semi-open sets) in X will be denoted by $PO(X)$ (resp., $\alpha(X), SO(X)$). We know the family $\alpha(X)$ is a topology finer than the given topology on X. And a subset A of X is said to be g-closed [5] (resp., gp-closed [7], gs-closed [1, 3]) if $cl(A)$ (resp., $pCl(A)$, $sCl(A)) \subset U$ whenever $A \subset U$ and U is open in X.

Then the family τ, $GO(X)$, $g\alpha O(X)$, and $g\alpha^* O(X)$, are all weak structures on X. But $PO(X)$, $GPO(X)$ and $SO(X)$ are not weak structures on X. A subfamily m_X of the power set $P(X)$ of a nonempty set X is called a *minimal structure* on X [8] if $\emptyset \in w_X$ and $X \in w_X$. Thus clearly every weak structure is a minimal structure.

For a subset A of X, the *w-closure* of A and the *w-interior* of A are defined as follows in [12]:

(1) $wC(A) = \cap\{F : A \subseteq F, X - F \in w_X\}$.

(2) $wI(A) = \cup\{U : U \subseteq A, U \in w_X\}$.

Theorem 2.1 ([12])**.** Let (X, w_X) be a w-space and $A \subseteq X$.

(1) $x \in wI(A)$ if and only if there exists an element $U \in W(x)$ such that $U \subseteq A$.

(2) $x \in wC(A)$ if and only if $A \cap V \neq \emptyset$ for all $V \in W(x)$.

(3) If $A \subseteq B$, then $wI(A) \subseteq wI(B)$; $wC(A) \subset wC(B)$.

(4) $wC(X - A) = X - wI(A)$; $wI(X - A) = X - wC(A)$.

(5) If A is w-closed (resp., w-open), then $wC(A) = A$ (resp., $wI(A) = A$).

Let (X, w_X) be a w-space and $A \subseteq X$. Then A is called *a generalized w-closed set* (simply, a gw-closed set) [15] if $wC(A) \subseteq U$, whenever $A \subseteq U$ and U is w-open. Then the union of two gw-closed sets is a gw-closed set, but the intersection of two gw-closed sets is not always gw-closed. The family of all w-closed sets (resp., gw-closed sets, gw-open sets) in X will be denoted by $WC(X)$ (resp., $GWC(X), GW(X)$). We set $gW(x) = \{U \in GW(X) : x \in U\}$. And A is called *a generalized w-open set* (simply, a gw-open set) if $X - A$ is gw-closed. Then A is gw-open if and only if $F \subseteq wI(A)$ whenever $F \subseteq A$ and F is w-closed. For a subset A of X, *gw-closure* of A and *gw-interior* [15] of A are defined as the following:

(1) $gwC(A) = \cap\{F : A \subseteq F, F \text{ is } gw\text{-closed}\}$.

(2) $gwI(A) = \cup\{U : U \subseteq A, U \text{ is } gw\text{-open}\}$.

Theorem 2.2 ([15])**.** Let (X, w_X) be a w-space and $A \subseteq X$.

(1) If A is gw-open (gw-closed), then $gwI(A) = A$ ($gwC(A) = A$).

(2) If $A \subseteq B$, then $gwI(A) \subseteq gwI(B)$; $gwC(A) \subseteq gwC(B)$.

(3) $gwC(X - A) = X - gwI(A)$; $gwI(X - A) = X - gwC(A)$.

(4) $x \in gwI(A)$ iff there exists a gw-open set U containing x such that $U \subseteq A$.

(5) $x \in gwC(A)$ iff $A \cap V \neq \emptyset$ for all gw-open set V containing x.

3 Main Results

First, we recall that: Let X be a nonempty set and let (X, τ) be a topological space. A subfamily w of the power set $P(X)$ is called *an associated weak structure* (simply, w_τ) [13] on X if $\tau \subseteq w$ and w is a weak structure. Then the pair (X, w_τ) is called *an associated w-space* with τ.

Definition 3.1. Let $f : X \to Y$ be a function in two associated w-spaces. Then f is said to be

(1) *gw-continuous* if for $x \in X$ and for each open set V containing $f(x)$, there is a gw-open set U containing x such that $f(U) \subseteq V$:

(2) *gw^*-continuous* if for every open set V in Y, $f^{-1}(V)$ is a gw-open set in X.

Obviously we obtain the following theorem:

Theorem 3.2. Every gw^*-continuous function is gw-continuous.

The following example supports that the converse of the above theorem is not true in general.

Example 3.3. Let $X = \{a, b, c, d\}$, a topology $\tau = \{\emptyset, \{a, c\}, X\}$ and an associated w-structure $w = \{\emptyset, \{a, c\}, \{a\}, \{b\}, \{c\}, \{a, d\}, X\}$ in X. Then for the power set $P(X)$ of X, $GW(X) = P(X) - \{\{b, c, d\}, \{b, d\}\}$ is the set of all gw-open sets. Consider a function $f : (X, w) \to (X, w)$ defined by $f(a) = b; f(b) = a; f(c) = d; f(d) = c$. Then f is gw-continuous. For an open set $\{a, c\}$, $f^{-1}(\{a, c\}) = \{b, d\}$ is not gw-open, and so f is not gw^*-continuous.

We recall that: Let (X, w_τ) be an associated w-space with a topology τ and $A \subseteq X$. Then A is called *a generalized w_τ-closed set* (simply, gw_τ-closed set) [16] if $cl(A) \subseteq U$, whenever $A \subseteq U$ and U is w-open.

Let $f : X \to Y$ be a function in two associated w-spaces w-spaces. Then f is said to be

(1) *gw_τ-continuous* [17] if for $x \in X$ and for each open set V containing $f(x)$, there is a gw_τ-open set U containing x such that $f(U) \subseteq V$:

(2) *gw^*_τ-continuous* [17] if for every open set V in Y, $f^{-1}(V)$ is a gw_τ-open set in X.

Obviously, the following things are obtained:

Theorem 3.4. (1) Every gw_τ-continuous function is gw-continuous.

(2) Every gw^*_τ-continuous function is gw^*-continuous.

Proof. Since every gw_τ-open set is gw-open, the things are obvious. □

The following example supports that the converses of the above theorem are not true in general.

Example 3.5. Let $X = \{a, b, c, d\}$, a topology $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $w_X = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, d\}, X\}$ be a w-structure in X. Note that:

$WC(X) = \{\emptyset, \{b,c,d\}, \{c,d\}, \{b,d\}, \{b,c\}, \{c\}, X\}$;
$GW_\tau C(X) = \{\emptyset, \{b,c\}, \{c,d\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}, X\}$;
$GW_\tau(X) = \{\emptyset, \{a\}, \{b\}, \{d\}, \{a,b\}, \{a,d\}, X\}$;
$GWC(X) = \{\emptyset, \{b\}, \{c\}, \{d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}, X\}$;
$GW(X) = \{\emptyset, \{a\}, \{b\}, \{d\}, \{a,d\}, \{a,c\}, \{a,b\}, \{a,b,c\}, \{a,c,d\}, \{a,b,d\}, X\}$.

Consider a function $f : X \to X$ defined as: $f(a) = f(c) = a; f(b) = b; f(d) = d$. Then f is gw-continuous and gw^*-continuous. But for an open set $\{a\}$, $f^{-1}(\{a\}) = \{a,c\}$ is not gw_τ-open. So f is not neither gw_τ-continuous nor gw_τ^*-continuous.

Let $f : X \to Y$ be a function in two associated w-spaces w-spaces. Then f is said to be

(1) *WO-continuous* [13] if for $x \in X$ and for each open set V containing $f(x)$, there is a w-open set U containing x such that $f(U) \subseteq V$;

(2) *WK-continuous* [13] if for every open set V in Y, $f^{-1}(V)$ is a w-open set in X.

Obviously, the following things are obtained:

Theorem 3.6. (1) Every WO-continuous function is gw-continuous.

(2) Every WK-continuous function is gw-continuous.

Proof. Since every w-open set is gw-open, they are obtained. □

The following example supports that the converses of the above theorem are not true in general.

Example 3.7. Consider the function f defined in Example 3.5. Then f is gw-continuous and gw^*-continuous but neither WO-continuous nor WK-continuous.

Let $f : X \to Y$ be a function on w-spaces. Then f is said to be

(1) gW*-continuous* [18] if for $x \in X$ and for each w-open set V containing $f(x)$, there is a gw-open set U containing x such that $f(U) \subseteq V$;

(2) gW^**-continuous* [18] if for every w-open set V in Y, $f^{-1}(V)$ is a gw-open set in X.

Obviously, the following things are obtained:

Theorem 3.8. (1) Every gW-continuous function is gw-continuous.

(2) Every gW^*-continuous function is gw^*-continuous.

Proof. Since every open set is w-open, the things are obtained. □

The following example supports that the converses of the above theorem are not true in general.

Example 3.9. (1) The function f defined in Example 3.5 is obviously gw^*-continuous but not gW^*-continuous.

(2) In Example 3.5, consider a function $g : X \to X$ defined by $g(a) = b; g(b) = a; g(c) = c; g(d) = d$. Then g is gw-continuous. For a w-open set $V = \{a, c\}$ and for $g(c) = c \in U$, there is no any gw-open set U containing c such that $g(U) \subseteq V$. So, g is not gW^*-continuous.

Let $f : (X, w_\tau) \to (Y, w_\mu)$ be a function on two associated w-spaces with τ and μ. Then f is said to be

(1) *gW-irresolute* [18] if for $x \in X$ and for each gw-open set V containing $f(x)$, there is gw-open set U containing x such that $f(U) \subseteq V$;

(2) *gW^*-irresolute* [18] if for every gw-open set V in Y, $f^{-1}(V)$ is gw-open in X.

In [18], we showed that every gW-irresolute is gW-continuous and very gW^*-irresolute function is gW^*-continuous. From Theorem 3.4, the following theorem is directly obtained:

Theorem 3.10. (1) Every gW-irresolute is gw-continuous.

(2) Every gW^*-irresolute function is gw^*-continuous.

Remark 3.11. For a function from an associated w-space to an associated w-space, we have the following diagram:

Continuity $\to$ WK-conti. $\to$ WO-conti.

$\searrow$ $\swarrow$

gw^*-conti. $\to$ gw-conti.

$\nwarrow$ $\nearrow$

gw^*_τ-conti.$\to$ gw_τ-conti.

$\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$

$gw_\tau W^*$-conti.$\to$ $gw_\tau W$-conti.

$\swarrow$ $\searrow$

gW^*-conti. $\to$ gW-conti.

$\nearrow$ $\nwarrow$

W^*-conti. $\to$ W-conti.

Continuity $\longrightarrow$ gw^*_τ-continuity$\longrightarrow$ gw_τ-continuity

Theorem 3.12. Let $f : X \to Y$ be a function on w-spaces. Then f is gw^*-continuous if and only if for every closed set F in Y, $f^{-1}(F)$ is gw-closed in X.

Proof. It is obvious. □

Theorem 3.13. Let $f : X \to Y$ be a function on w-spaces. Then the following statements are equivalent:

(1) f is gw-continuous.
(2) $f(gwC(A)) \subseteq cl(f(A))$ for $A \subseteq X$.
(3) $gwC(f^{-1}(V)) \subseteq f^{-1}(cl(V))$ for $V \subseteq Y$.
(4) $f^{-1}(int(V)) \subseteq gwI(f^{-1}(V))$ for $V \subseteq Y$.

Proof. Obvious. □

Corollary 3.14. Let $f : X \to Y$ be a function on w-spaces. Then the following statements are equivalent:

(1) f is gw-continuous.
(2) $f^{-1}(V) = gwI(f^{-1}(V))$ for every open set $V \in Y$.
(3) $f^{-1}(B) = gwC(f^{-1}(B))$ for every closed set $B \subseteq Y$.

Proof. From Theorem 2.2 and Theorem 3.13, it is obvious. □

Let (X, w) be a w-space. Let $gW(x)$ (resp., $W(x)$) denote the set of all gw-open (resp., w-open) set containing x in X. A collection $\mathcal{H}$ of subsets of X is called an *m-family* [10] on X if $\cap\mathcal{H} \neq \emptyset$. Let $\mathcal{H}$ be an m-family on X. Then we say that an m-family $\mathcal{H}$ *gw-converges* (resp., *converges*) to $x \in X$ if $\mathcal{H}$ is finer than $gW(x)$ (resp., $O(x)$) i.e., $gW(x) \subseteq \mathcal{H}$ (resp., $O(x) \subseteq \mathcal{H}$). Let $f : X \to Y$ be a function; then it is obvious $f(\mathcal{H}) = \{f(F) : F \in \mathcal{H}\}$ is an m-family on Y.

Theorem 3.15. Let $f : X \to Y$ be a function on w-spaces. If f is gw-continuous, then for an m-family $\mathcal{H}$ gw-converging to $x \in X$, an m-family $< f(\mathcal{H}) >= \{F : H \subseteq F \text{ for some } H \in f(\mathcal{H})\}$ converges to $f(x)$.

Proof. Let f be gw-continuous and let $\mathcal{H}$ be an m-family gw-converging to $x \in X$. By gw-continuity, for an open set V containing $f(x)$, there exists a gw-open set U containing x such that $f(U) \subseteq V$. Since $f(gW(x)) \subseteq f(\mathcal{H})$, $V \in< f(\mathcal{H}) >$, and so $O(f(x)) \subseteq< f(\mathcal{H}) >$. Hence the m-family $< f(\mathcal{H}) >$ converges to $f(x)$. □

Theorem 3.16. Let $f : X \to Y$ be a bijective function on w-spaces. Then f is gw^*-continuous iff for an m-family $\mathcal{H}$ gw-converging to $x \in X$, the m-family $f(\mathcal{H})$ converges to $f(x)$.

Proof. Suppose f is gw^*-continuous and $\mathcal{H}$ is an m-family gw-converging to $x \in X$. By hypothesis and surjectivity, $O(f(x)) \subseteq f(gW(x)) \subseteq f(\mathcal{H})$, and so the m-family $f(\mathcal{H})$ converges to $f(x)$.

For the converse, let $U \in O(f(x))$ for $U \subseteq Y$. Since the family $gW(x)$ clearly gw-converges to x, by hypothesis, we get $O(f(x)) \subseteq f(gW(x))$ for $x \in X$. Since f is injectivity, $f^{-1}(U) \in gW(x)$. □

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