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$n^{th}SM$ Sum Graphs and Some Parameters

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Abstract

We introduce and analyse a new class of graphs called SM-sum graphs. This has been done by using the concept that any positive integer can be expressed as a sum of distinct powers of 2. This relationship between powers of 2 and other positive integers are very much used in Computer Science especially in the binary number system, bit string operations, in signal processing etc. We are making use of this relationship to introduce a new class of graphs called SM sum graphs. Thus we are trying to provide a graph theoretical approach to study the various aspects of this relationship. Since the parameters of graphs are more significant, we are examining some parameters of the two newly defined graphs $SM(\sum_n)$ and $SMD(\sum_n)$. The domination number , chromatic number, vertex independence number, the distance based (topological) indices like Wiener indices of these graphs are obtained and analysed.

Mathematics Subject Classification: 05C99

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Introduction

Graph parameters have been playing a vital role in the development of information technology and thus the development of technology in other parts of science and engineering. In this paper we are introducing a special type of graphs having certain properties related with the various graph parameters. For a fixed positive integer n, consider the set $P = \{2^m,\ 0 \leq m \leq n-1\}$. Any positive integer less than 2^n and not in P can be expressed as the sum of two or more distinct elements of P. If $p \notin P$ and $p = \sum x_i$, with distinct $x_i \in P$ then each x_i is called an additive component of p. We define a new simple graph, $SM(\sum_n)$ with vertex set $\{v_1, v_2, \ldots, v_{2^n-1}\}$ and adjacency of vertices defined by two distinct vertices v_i and v_j are adjacent if either i is an additive component of j or j is an additive component of i. Some parameters like domination number, independence number, Wiener index and Hyper Wiener index of these graphs were computed.

1 $n^{th}SM$ Sum Graphs

Definition 1.1. *If $p < 2^n$, is a positive integer which is not a power of 2, then $p = \sum_1^n x_i$, with $x_i = 0$ or 2^m, for some $0 \leq m \leq n-1$ and x_is are distinct. Here we call each $x_i \neq 0$ as an additive component of p.*

Definition 1.2. *For a fixed integer $n \geq 2$, define a simple graph $SM(\sum_n)$, called n^{th} SM sum graph, with vertex set $\{v_1, v_2, \ldots, v_{2^n-1}\}$ and adjacency of vertices defined by, v_i and v_j are adjacent if either i is an additive component of j or j is an additive component of i.*

Definition 1.3. *If $G(V, E)$ is a graph, then Hit_{high} number (H_H) of G is defined as the number of vertices of maximum degree and Hit_{low} number (H_L) of G is defined as the number of vertices of minimum degree.*

Note:For a fixed integer $n \geq 2$, let $P = \{2^m : 0 \leq m \leq n-1\}$, $N = \{1, 2, 3, \ldots, 2^n - 1\}$. Then consider $P^c = N - P$ throughout this paper unless otherwise specified.

Observation 1.4.

1. *Suppose $G(V, E) = SM(\sum_n)$. Then for $n \geq 3$, $\min\limits_{v \in V} deg\ v = 2$ and $\max\limits_{v \in V} deg\ v = 2^{n-1} - 1$.*
2. *For $n \geq 4$, $H_H = n$ and $H_L = \binom{n}{2}$.*

Proposition 1.5. *In $SM(\sum_n)$, degree of the vertex v_{2^n-1} is n and $\sum\limits_{v \in V} deg\ v = 2n(2^{n-1} - 1)$.*

Theorem 1.6. *The graph $SM(\sum_n)$ is connected for all $n \geq 2$.*

Theorem 1.7. *The graph $SM(\sum_n)$ is bipartite for all n.*

Proof. We can partition the vertex set of $SM(\sum_n)$ as $X \cup Y$ where $X = \{v_i : \ where \ i = 2^r \ for \ 0 \leq r \leq n-1\}$ and $Y = V - X$. □

Corollary 1.8. *The chromatic number of $SM(\sum_n)$ is 2 for all $n \geq 2$.*

Theorem 1.9. *For any $n \geq 2$, $SM(\sum_n)$ is a subgraph of $SM(\sum_{n+1})$.*

2 Some Parameters of $SM(\sum_n)$

In most of the cases graph parameters are useful to study the nature of graph up to isomorphism. So we are trying to find out some graph parameters like independence number, domination number, Wiener index, Hyper Wiener index etc.

Theorem 2.1. *If $G = SM(\sum_n)$, then the vertex independence number, $\alpha(G) = 2^n - n - 1$, for $n \geq 2$.*

Proof. Let $P = \{2^m : 0 \leq m \leq n-1\}$. $\{v_i : i \in P^c\}$, is a maximal independent set. □

Corollary 2.2. *For any $SM(\sum_n)$ graph, the vertex covering number, $\beta(G) = n$.*

Lemma 2.3. *If $G = SM(\sum_n)$, $P = \{2^m : 0 \leq m \leq n-1\}$. , then*

$$d(v_i, v_j) = \begin{cases} 1 & \textit{, if } i \textit{ is an additive component of } j \textit{ or } j \textit{ is an additive component of } i \\ 2 & \textit{, if } i, j \in P \textit{ or } i, j \notin P, \ i \textit{ and } j \textit{ have atleast one common additive component} \\ 3 & \textit{, neither } i \textit{ nor } j \textit{ is an additive component but exactly one of them belongs to } P \\ 4 & \textit{, } i, j \notin P, i \textit{ and } j \textit{ have no common additive component.} \end{cases}$$

Proof. Let $G = SM(\sum_n)$, $P = \{2^m : 0 \leq m \leq n-1\}$, $V = \{v_1, v_2, \ldots, v_{2^n-1}\}$.
case 1: When i is an additive component of j or j is an additive component of i, then by the definition v_i and v_j are adjacent. Therefore, $d(v_i, v_j) = 1$.
case 2(i): When $i, j \in P$. In this case v_i and v_j are non adjacent then v_i and v_j have a common neighbour v_{i+j} and it follows $d(v_i, v_j) = 2$.
(ii) When $i, j \notin P$, i and j have atleast one common additive component, say k, then v_k is a common neighbour of v_i and v_j and hence $d(v_i, v_j) = 2$.
case 3: Neither i nor j is an additive component but exactly one of them belongs to P.
Suppose $i \in P$ and $j \notin P$. Let k be an additive component of j. Then $r = i + k \neq j$, by the choice of i. Therefore, $v_i v_r v_k v_j$ is a path of length 3 and since a path of length 2 is not possible, $d(v_i, v_j) = 3$.
case 4: When $i, j \notin P$,i and j have no common additive component.

Let r and k be additive components of i and j respectively. Suppose $t = r + k$. Then v_t is a common neighbour of v_r and v_k and hence $v_i v_r v_t v_k v_j$ is a path of length 4. Since there is no path length 3 or less, $d(v_i, v_j) = 4$. □

Theorem 2.4. *Suppose $G = SM(\sum_n)$ graph,*

$$diam(G) = \begin{cases} 2 & \text{if } n = 2 \\ 3 & \text{if } n = 3 \\ 4 & \text{if } n \geq 4 \end{cases}$$

Proof. From lemma 2.3 , the theorem follows. □

Theorem 2.5. *The domination number, $\gamma(SM(\sum_n)) = n - 1$, for all $n \geq 2$.*

Proof. Let $G = SM(\sum_n)$ with vertex set V. When $n = 2$, the theorem is obvious. Suppose $n \geq 3$. Consider the sets $P = \{2^m : 0 \leq m \leq n - 1\}$, $N = \{1, 2, 3, \ldots, 2^n - 1\}$, and $V = \{v_i, i \in N\}$. Let $Q = (P - \{2, 4\}) \cup \{6\}$ and $D = \{v_i : i \in Q\}$.
We have, $|Q| = n-1$. Now the vertices in D are not adjacent to each other,each $v_i \in D$ is adjacent to some $v_i \in V - D$. Therefore, D is a dominating set.
Claim: D is a minimal of such dominating sets.
Suppose $D_1 = D - \{v_1\}$. Now, the vertex $v_5 \in V - D$ is not adjacent to any vertex in D_1. So by Ore's lemma, D_1 is not a dominating set, $|D| = |Q| = n-1$ and hence $\gamma(G) = n - 1$. □

Definition 2.6. *[3] If $G(V, E)$ is a graph, then Wiener index, $w(G)$ is defined as the sum of distances between all unordered pairs of vertices of G. ie., $w(G) = \sum_{\{u,v\} \subseteq V} d(u, v)$, where $d(u, v)$ is the distance between u and v.*

Definition 2.7. *[3] If $G(V, E)$ is a graph, then hyper Wiener index, $ww(G)$ is defined as*
$ww(G) = \dfrac{1}{2} \sum_{\{u,v\} \subseteq V} [d(u, v)^2 + d(u, v)]$.

Proposition 2.8. *Let $G = SM(\sum_n)$ be an n^{th} SM sum graph. Let $d_r(v_i, v_j)$ denotes the number of unordered pairs of vertices for which $d(v_i, v_j) = r$.Then:*

$$d_r(v_i, v_j) = \begin{cases} n.(2^{n-1} - 1) & , \text{if } r = 1 \\ \dfrac{n(n-1)}{2} + [\dfrac{(2^n - n - 2)(2^n - n - 1)}{2} - \delta] & , \text{if } r = 2 \\ (n+1).2^n - (n+2)2^{n-1} - n^2 & , \text{if } r = 3 \\ \delta & , \text{if } r = 4. \end{cases}$$

where $\delta = \dfrac{1}{2} \sum_{r=2}^{n-2} \left[\binom{n}{r} \sum_{k=2}^{n-2} \binom{n-r}{k} \right]$.

Proof. The proof is obvious when $r = 1$.
When $r = 4$, it is the total number of unordered pairs of disjoint subsets containing atleast 2 elements of a set consisting of $n \geq 4$ elements. Hence the result follows.
Suppose, $r = 2$. Let $P = \{2^m : 0 \leq m \leq n-1\}$ implies $|P^c| = 2^n - n - 1$. So the total number of unordered pairs of elements of P^c is $\frac{(2^n-n-2)(2^n-n-1)}{2}$, which follows the result.
Suppose $r = 3$. We have, $|V| = 2^n - 1$.
Therefore the number of unordered pairs is $\frac{(2^n-1)(2^n-2)}{2} - (d_1 + d_2 + d_4)$ $=$
$(n+1).2^n - (n+2)2^{n-1} - n^2$. □

Remark 2.9. $\delta = 0$ *for* $n = 2$ *or*3

Theorem 2.10. *Let* $G = SM(\sum_n), n \geq 2$ *. We have,* $w(G) = 2^{2n} - 3.2^n - n^2 + n + 2 + 2\delta$.

Proof. By the definition of $w(G)$,

$$\begin{aligned} w(G) &= \sum_{\{u,v\}\subseteq V} d(u,v) \\ &= 1.n.(2^{n-1}-1) + 2\Big[\frac{n(n-1)}{2} + \frac{(2^n-n-2)(2^n-n-1)}{2} - \delta\Big] + \\ &\quad 3\big[n.2^n - 2.2^{n-1} - n^2 - n2^{n-1} + 2^n)\big] + 4\delta \\ &= 2^{2n} - 3.2^n - n^2 + n + 2 + 2\delta \end{aligned}$$

□

Theorem 2.11. *Let* $G = SM(\sum_n), n \geq 2$ *.* $ww(G) = 3.2^{2n-1} + (n-9).2^{n-1} - 3n^2 + 2n + 3 + 7\delta$.

Proof. We have,

$$\begin{aligned} \sum_{\{u,v\}\subseteq V} (d(u,v))^2 &= 1^2.n.(2^{n-1}-1) + 4\Big[\frac{n(n-1)}{2} + \frac{(2^n-n-2)(2^n-n-1)}{2} - \delta\Big] + \\ &\quad 9\big[n.2^n + 2^n - 2.2^{n-1} - n.2^{n-1} - n^2\big] + 16\delta \\ &= n.2^n + 3n - 5n^2 + 2.2^{2n} + 4 - 6.2^n + 12\delta \\ Therefore, ww(G) &= \frac{1}{2}\Big[w(G) + \sum_{\{u,v\}\subseteq V} (d(u,v))^2\Big] \\ &= 3.2^{2n-1} + (n-9).2^{n-1} - 3n^2 + 2n + 3 + 7\delta. \end{aligned}$$

Hence the theorem. □

3 R-sets, Nandu sequence and Properties related to SM sum graphs

In set theory, usually elements are not repeated in set symbol. So it is confusing that how many times an element is repeated. We are making use of the idea of multiset to represent the repeating elements of a collection in a set by using the following concept of $R-$ *sets*.

Definition 3.1. *If A is a well defined collection of objects , we define R -set on A, A^R as*
$A^R = \{x_{(i)}; x \in A,$ *where x repeats i times*$\}$.

Definition 3.2. *Let $SM(\sum_n)$ be an n^{th} SM sum graph with vertex set $V = \{v_i, 1 \le i \le 2^n - 1\}$. Then Adi set of degrees is defined as follows:*
$A_n = \{deg\ v_i;\ v_i \in V\}$ *for all* $n \ge 2$.

Definition 3.3. *Let $SM(\sum_n)$ with vertex set $V = \{v_i, 1 \le i \le 2^n - 1\}$ be an n^{th} sum graph. The Adi -R-set of degrees, denoted by A_n^R is defined as $A_n^R = \{degv_{i(x)}, 1 \le i \le 2^n - 1\}$, where x is the number of times each $degv_i$ repeats.*

Example 3.4. $A_3^R = \{2_{(3)}, 3_{(4)}\}$ *and*
$A_n^R = \{2_{(\binom{n}{2})}, 3_{(\binom{n}{3})}, \ldots, n_{(\binom{n}{n})}, 2^{n-1} - 1_{(n)}\}$, *for* $n \ge 4$.

Observation 3.5. $A_{n-1} - \{2^{n-2} - 1\} \subset A_n$, *for all* $n > 2$.

We introduce two new sequences, called Nandu sequence and Double Nandu sequence and try to study some of their properties.

Definition 3.6. *Consider the graph $SM(\sum_{n+1})$ with vertex set $V = \{v_i, 1 \le i \le 2^{n+1} - 1\}$. Define the sequence $\{Nt_n\}$ as $Nt_n = \frac{1}{2}\sum_{v\in V} deg\ v$ as Nandu sequence and the sequence $\{DNt_n\}$ as $DNt_n = \sum_{v\in V} deg\ v$ as Double Nandu sequence.*
ie,$\{Nt_n\} = 2, 9, 28, 75, 186, \ldots$ *and* $\{DNt_n\} = 4, 18, 56, 150, 372, \ldots$.

Definition 3.7. *Let $G = SM(\sum_n)$ with vertex set V. The vertex degree polynomial of G is defined as $D_n(V, x) = \sum_{v\in V} |V_{d_m}|.x^m = \sum_{k=2}^{n} \binom{n}{k} x^k + n.x^{2^{n-1}-1}$, where V_{d_m} is the number of vertices of degree m.*

Example 3.8. *For* $n = 6$, $D_n(V, x) = x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x^{31}$.

Theorem 3.9. *Let $G = SM(\sum_n)$ with vertex set V, then*

1. $D_n(V, 1) =$ *number of vertices of* $G = 2^n - 1$.

2. *The derivative of $D_n(V,x)$ at $x=1$ is DNt_n, the $(n-1)th$ term of the double Nandu sequence of $G = SM(\sum_n)$.*

Proof. 1. $D_n(V,1) = \sum_{v\in V} |V_{d_m}|.1 = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{2} + n$
$= 2^n - 1.$

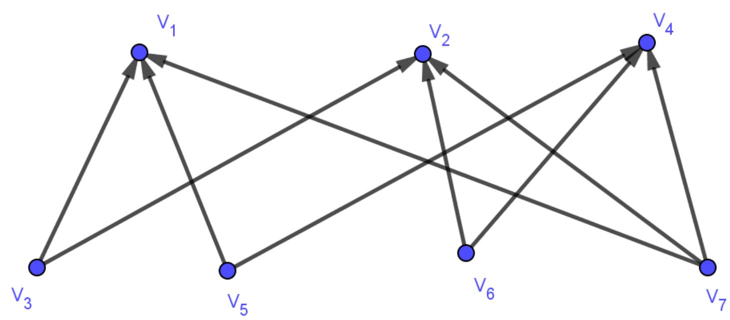
2. $D'_n(V,x) = n(x+1)^{n-1} - n + n.(2^{n-1}-1).x^{2^{n-1}-2}$.
$Hence, D'_n(V,1) = n.2^{n-1} - n + n.(2^{n-1}-1) = DNt_n$.

□

4 $n^{th}SMD$ sum graphs

Definition 4.1. *For a fixed $n \geq 2$, define a simple digraph $SMD(\sum_n)$, called n^{th} sum digraph, with vertex set $\{v_1, v_2, \ldots, v_{2^n-1}\}$ and adjacency of vertices defined by v_i is adjacent to v_j if j is an additive component of i.*

Example 4.2. *The graph $SMD(\sum_3)$ is given below.*



Observation 4.3. *Let $SMD(\sum_n)$ be the n^{th} sum digraph with vertex set V. Then*

1. *The indegree of each vertex $v_i, i = 2^r, 0 \leq r \leq n-1$ is $2^{n-1}-1$.*

2. $\sum_{v\in V}$ *deg* $v_- = n(2^{n-1}-1)$ *and* $\sum_{v\in V}$ *deg* $v_+ = n(2^{n-1}-1)$.

Definition 4.4. *Let $G = SMD(\sum_n)$ and $P = \{2^m : 0 \leq m \leq n-1\}$. For each vertex v_i , $i \notin P$ the subgraph induced by $\{v_i\}$ is a rooted tree with root v_i denoted by $K_i(G)$. Define $K-graph$ by $K(G) = \{K_i(G), i \in P^c\}$.*

Proposition 4.5. 1. *Let $G = SMD(\sum_n)$ with vertex set V. Then $|V| = |K(G)| + n$ for all $n \geq 2$.*

2. *The number of K_i subgraphs which is a binary tree is H_L.*

Proposition 4.6. *The graph $SM(\sum_n)$ is not Eulerian, for all $n \geq 2$.*

Proof. The result is clear when $n = 2$. For $n > 2$, we have
$A_n^R = \{2_{(\binom{n}{2})}, 3_{(\binom{n}{3})}, \ldots, n_{(\binom{n}{n})}, 2^{n-1} - 1_{(n)}\}$. In A_n^R, there are two or more vertices of odd degree. So it is not an Eulerian graph. □

Conclusion

The n^{th} SM sum graphs are triangle free graphs. These graphs are correlated with the cardinality of the set $P = \{2^m : 0 \leq m \leq n-1\}$. Since the domination number of this graph is given by $\gamma(SM(\sum_n)) = n - 1$, it will help in network graphs to solve many problems. The set $D = \{v_i : i \in P\}$ is a 2 - dominating set for the n^{th} SM sum graphs and the 2- domination number is n. Since the construction of these graphs are based on different combinations of powers of 2, it will be useful in database, counting techniques, computer Science and IT related fields. A comprehensive study of these graphs may help in solving many real life network problems or other graph theory problems and can be used in sorting algorithms. The Adi- R- sets and the Nandu sequences may help to study different graph theoretic parameters. Further studies can be done on the relationship between Wiener indices with the terms of Nandu sequence of these graphs.

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