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$(2,1)$-Total Labeling of Cycle with Parallel Paths

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Abstract

A $(p,1)$-total labeling of a graph G is an assignment of integers to $V(G) \cup E(G)$ such that

(i) any two adjacent vertices of G receive distinct integers,

(ii) any two adjacent edges of G receive distinct integers, and

(iii) a vertex and an edge incident receive integers that differ by at least p in absolute value.

The span of a $(p,1)$-total labeling is the maximum difference between two labels. The minimum of span of all possible $(p,1)$-total labeling of G is called the $(p,1)$-total number and denoted by $\lambda_p^T(G)$. The well known Havet and Yu Conjecture [6] states that for any connected graph G with $\Delta(G) \leq 3$ and $G \neq K_4$, $\lambda_2^T(G) \leq 5$. In this paper, we determine the $(2,1)$-total number of cycle with parallel paths. This result supports the Havet and Yu conjecture.

Mathematics Subject Classification: 05C15

Keywords: $(2,1)$-total labeling, cycle with parallel paths, Havet and Yu conjecture on $(2,1)$-total labeling

1 Introduction

In the channel/frequency assignment problem we need to assign different frequencies to 'close' transmitters so that they can avoid interference. Motivated by this problem, Griggs and Yeh [3] introduced $L(2,1)$-labeling. Its natural generalization $L(p,1)$-labeling of a graph G is an integer assignment f to the vertex set $V(G)$ such that, $|f(u)-f(v)| \geq p$ if $d(u,v)=1$ and $|f(u)-f(v)| \geq 1$ if $d(u,v)=2$. Whittlesey et al.[14] studied the $L(2,1)$-labeling number of incidence graphs, where the incidence graph of a graph G is the graph obtained from G by replacing each edge (v_i, v_j) with two edges (v_i, v_{ij}) and (v_{ij}, v_j) introducing one new vertex v_{ij}. Observe that an $L(2,1)$-labeling of the incidence graph of a given graph G can be regarded as an assignment f from $V(G) \cup E(G)$ to the set of non-negative integers such that $|f(x)-f(y)| \geq 2$ if x is a vertex and y is an edge incident to x and $|f(x)-f(y)| \geq 1$ if x and y are a pair of adjacent vertices or a pair of adjacent edges for all x, y in $V(G)$ or for all x, y in $E(G)$. Havet and Yu [4] called such a labeling a "$(2,1)$-total labeling of G". A generalization of $(2,1)$-total labeling called $(p,1)$-total labeling is defined in the following manner. Let $p \geq 1$ be an integer. A k-$(p,1)$-total labeling of a graph G is a function f from $V(G) \cup E(G)$ to the set $\{0, 1, \ldots, k\}$ such that $f(u) \neq f(v)$ if u and v are two adjacent vertices, $f(e) \neq f(e')$ if e and e' are two adjacent edges and $|f(u)-f(e)| \geq p$ if a vertex u is incident to an edge e. The $(p,1)$-total number denoted by $\lambda_p^T(G)$, is the smallest integer k such that G has a k-$(p,1)$-total labeling. Note that $(1,1)$-total labeling of G is equivalent to total colouring of G. As a generalization of the Total Colouring Conjecture, Havet and Yu [6] posed the following conjecture on $(p,1)$-total labeling number called $(p,1)$-Total Labeling Conjecture.

$(p,1)$**-Total Labeling Conjecture:** For any graph G, $\lambda_p^T(G) \leq \Delta + 2p - 1$, where $p \geq 1$ and Δ denotes the maximum degree of a vertex in G.

Havet and Yu [6, 7] have obtained some interesting bounds on $\lambda_p^T(G)$ supporting $(p,1)$-total labeling conjecture. Also, $(p,1)$-total number is determined for classes of special graphs. For example, the $(p,1)$-total number is determined for complete graphs [6], planar graphs [1], graphs with a given maximum average degree [8], outer planar graphs [2, 11], etc. The case $p=1$ corresponds to the usual notion of total colouring, which is NP-hard to compute even for cubic bipartite graphs. Havet et al [7] have completely settled the computational complexity of deciding whether $\lambda_p^T(G)$ is equal to $\Delta+p-1$ or $\Delta+p$, for $p \geq 2$, when G is bipartite and the remaining cases are NP-complete. When $p=2$, Havet and Yu [6] posed the following stronger version of the $(p,1)$-total labeling conjecture.

Havet and Yu Conjecture : If G is any connected graph with $\Delta(G) \leq 3$ and $G \neq K_4$, then $\lambda_2^T(G) \leq 5$.

In a graph G, a set of r paths are called parallel paths if the origin and the terminus of each of the r paths must be non-adjacent, all the r paths have no common vertex, no common edge and edges of all the r paths do not cross. In this paper, we determine $\lambda_2^T(C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor \frac{n}{2} \rfloor -1}})) \leq 5$, where $C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor \frac{n}{2} \rfloor -1}})$ denote the graph cycle C_n with $\left\lfloor \frac{n}{2} \right\rfloor - 1$ parallel paths. As $\Delta(C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor \frac{n}{2} \rfloor -1}})) = 3$, our result that $\lambda_2^T(C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor \frac{n}{2} \rfloor -1}})) \leq 5$ supports the Havet Conjecture.

2 Main Result

In this section we prove our main result.

Consider the graph $C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor \frac{n}{2} \rfloor -1}})$, cycle C_n with $\left\lfloor \frac{n}{2} \right\rfloor - 1$ parallel paths P_{k_i}'s. For the convenience, we describe the cycle C_n in the graph $C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor \frac{n}{2} \rfloor -1}})$ as u_0, u_1, …, $u_{\left\lfloor \frac{n}{2} \right\rfloor -1}$, v_0, $v_{\left\lfloor \frac{n}{2} \right\rfloor -1}$, …, v_1, u_0 when n is even and u_0, u_1, …, $u_{\left\lfloor \frac{n}{2} \right\rfloor -1}$, $u_{\left\lfloor \frac{n}{2} \right\rfloor}$, $v_{\left\lfloor \frac{n}{2} \right\rfloor}$, $v_{\left\lfloor \frac{n}{2} \right\rfloor -1}$, …, v_1, u_0 when n is odd and the path P_{k_i} as $u_i, w_{i,1}, w_{i,2}, \ldots, w_{i,k_i-1}, v_i$, for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$, where k_i is the size of the path and $k_i \geq 1$. Note that $w_{i,j}$, for $1 \leq j \leq k_i - 1$ are not the vertices of the cycle C_n. Figure 1 and Figure 2 shows that the cycle C_8 with 3 parallel paths and cycle C_{11} with 4 parallel paths respectively.

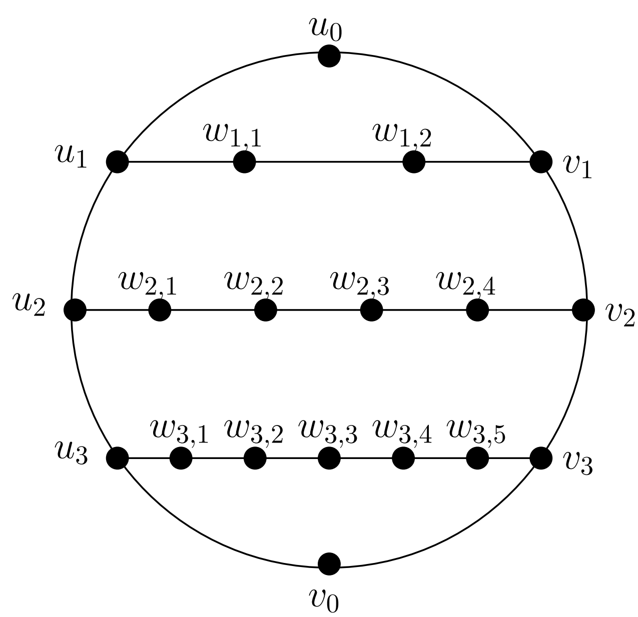

Figure 1: $C_8 \oplus (P_3, P_5, P_6)$

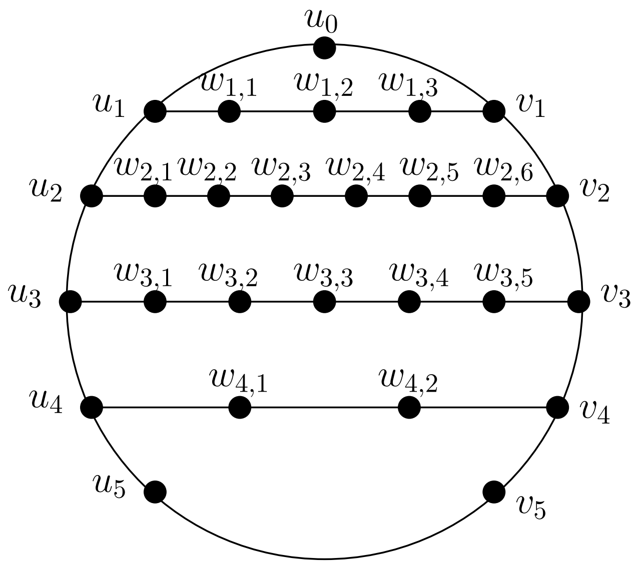

Figure 2: $C_{11} \oplus (P_4, P_7, P_6, P_3)$

We use the following lemma to prove our main result.

Lemma 2.1. *Let $G(V, E)$ be a connected graph with maximum degree $\Delta = 3$. If there exist three vertices u, v, w in G with each of degree Δ such that u is adjacent to v and w, then $\lambda_2^T(G) \geq 5$.*

Proof. Let G be a connected graph with maximum degree $\Delta = 3$. Let u, v and w be three vertices of G with each of degree Δ such that u is adjacent to v and w.
Suppose there exist $(2, 1)$-total labeling of G with $\lambda_2^T(G) = 4$. Let $f : V \cup E \to \{0, 1, 2, 3, 4\}$ be a $(2, 1)$-total labeling of G with $\lambda_2^T(G) = 4$. Then, Table 1 shows all the possible vertex labels for the vertices of G under the $(2, 1)$-total labeling f as well as all the possible labels for the edges incident at a vertex with the corresponding vertex label under f.

Table 1: Possible labels for the vertices and for their incident edges under $(2, 1)$-total labeling f

Vertex label	Labels of the corresponding incident edges
4	$2, 1, 0$
3	$1, 0$
2	$0, 4$
1	$3, 4$
0	$2, 3, 4$

Case 1: Suppose the vertices v and w are adjacent.
Then, the vertices u, v and w are mutually adjacent. From Table 1 observe that the only possible labels for the vertices u, v and w satisfying the property of $(2, 1)$-total labeling are 4 and 0. As the vertices u, v and w are mutually adjacent, it is not possible to assign three distinct labels to the three mutually adjacent vertices u, v and w. (Refer Figure 3).

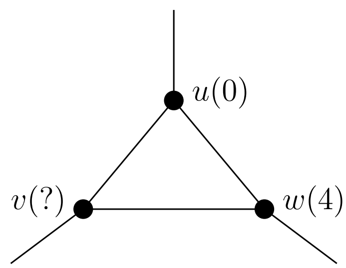

Figure 3: Choice of the labels for the vertices u, v and w under the Case 1

Case 2: Suppose v and w are not adjacent.
Then from Table 1 we observe that 4 and 0 are the only possible labels for the vertices u, v and w under f. Since the vertices v and w are not adjacent, the same label for the vertices v and w could have been assigned under f. Suppose $f(u) = 4$, $f(v) = 0$ and $f(w) = 0$. From Table 1 observe that the only possible edge label for the edge having end vertices labeled 0 and 4 is 2. As uv and uw are adjacent edges, the label 2 cannot be assigned to both the edges uv and uw. Consequently, only one of the edges uv or uw can get the edge label 2. Thus one of the edge uv or uw cannot be assigned any label. (Refer Figure 4).

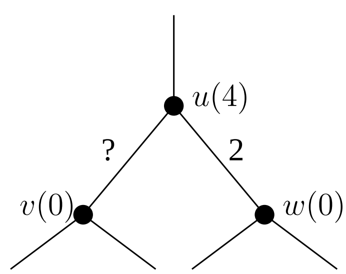

Figure 4: Choice of the labels for the edges uw and uv under the Case 2

Similar argument hold good if the labels $f(u) = 0$, $f(v) = 4$ and $f(w) = 4$. This implies that there cannot be any possible $(2, 1)$-total labeling with span 4. Thus $\lambda_2^T(G) > 4$. Hence $\lambda_2^T(G) \geq 5$. □

Theorem 2.2. *Let n be an integer and $n \geq 8$, then*
$\lambda_2^T(C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor \frac{n}{2} \rfloor -1}})) = \Delta + p = 5.$

Proof. Consider $C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor \frac{n}{2} \rfloor -1}})$ with $n \geq 8$. Let V and E denote the vertex set and edge set of $C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor \frac{n}{2} \rfloor -1}})$ respectively.
Define $f : V \cup E \to \{0, 1, 2, 3, 4, 5\}$ in the following way. First we define f for all the vertices as given below.

I. Labeling the vertices of C_n

Define $f(u_0) = 3$,
$f(v_0) = 3$, if n is even,
$f(u_i) = 5$, $f(u_{i+1}) = 4$, $f(v_i) = 4$, $f(v_{i+1}) = 5$, for $i = 1, 3, \ldots, \alpha$,
where

$$\alpha = \begin{cases} \lfloor \frac{n}{2} \rfloor - 2, & \text{if } n = 2l,\ l \text{ odd and } l \geq 5 \\ \lfloor \frac{n}{2} \rfloor - 3, & \text{if } n = 2l,\ l \text{ even and } l \geq 4 \\ \lfloor \frac{n}{2} \rfloor - 2, & \text{if } n = 2l+1,\ l \text{ odd and } l \geq 5 \\ \lfloor \frac{n}{2} \rfloor - 1, & \text{if } n = 2l+1,\ l \text{ even and } l \geq 4 \end{cases}$$

$f(u_{\lfloor \frac{n}{2} \rfloor -1}) = 5$, $f(v_{\lfloor \frac{n}{2} \rfloor -1}) = 4$, if $n = 2l$, l even and $l \geq 4$
$f(u_{\lfloor \frac{n}{2} \rfloor}) = 5$, $f(v_{\lfloor \frac{n}{2} \rfloor}) = 4$, if $n = 2l+1$, l odd and $l \geq 5$.

II. Labeling the vertices of P_{k_i} with $i = 1, 3, \ldots, \alpha$,
where

$$\alpha = \begin{cases} \lfloor \frac{n}{2} \rfloor - 2, & \text{if } n = 2l,\ l \text{ odd and } l \geq 5 \\ \lfloor \frac{n}{2} \rfloor - 1, & \text{if } n = 2l,\ l \text{ even and } l \geq 4 \\ \lfloor \frac{n}{2} \rfloor - 2, & \text{if } n = 2l+1,\ l \text{ odd and } l \geq 5 \\ \lfloor \frac{n}{2} \rfloor - 1, & \text{if } n = 2l+1,\ l \text{ even and } l \geq 4 \end{cases}$$

Define $f(w_{i,k_i-1}) = 0$, if k_i is even and $k_i \geq 2$,
$f(w_{i,j}) = 4$, $f(w_{i,j+1}) = 5$, for $j = 1, 3, \ldots, \beta$, where

$$\beta = \begin{cases} k_i - 3, & \text{if } k_i \text{ is even and } k_i \geq 4 \\ k_i - 2, & \text{if } k_i \text{ is odd and } k_i \geq 3 \end{cases}$$

III. Labeling the vertices of P_{k_i} with $i = 2, 4, \ldots, \alpha$,
where

$$\alpha = \begin{cases} \lfloor \frac{n}{2} \rfloor - 1, & \text{if } n = 2l,\ l \text{ odd and } l \geq 5 \\ \lfloor \frac{n}{2} \rfloor - 2, & \text{if } n = 2l,\ l \text{ even and } l \geq 4 \\ \lfloor \frac{n}{2} \rfloor - 1, & \text{if } n = 2l+1,\ l \text{ odd and } l \geq 5 \\ \lfloor \frac{n}{2} \rfloor - 2, & \text{if } n = 2l+1,\ l \text{ even and } l \geq 4 \end{cases}$$

Define $f(w_{i,k_i-1}) = 0$, if k_i is even and $k_i \geq 2$,
$f(w_{i,j}) = 5,\ f(w_{i,j+1}) = 4$, for $j = 1, 3, \ldots, \beta$, where

$$\beta = \begin{cases} k_i - 3, & \text{if } k_i \text{ is even and } k_i \geq 4 \\ k_i - 2, & \text{if } k_i \text{ is odd and } k_i \geq 3 \end{cases}$$

Now we label the edges of $C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor \frac{n}{2} \rfloor -1}})$.

I. Labeling the edges of C_n

Define $f(u_0u_1) = 1,\ f(u_0v_1) = 0$,
$f(u_{\lfloor \frac{n}{2} \rfloor} v_{\lfloor \frac{n}{2} \rfloor}) = 2$, if n is odd,
$f(v_0u_{\lfloor \frac{n}{2} \rfloor -1}) = 0,\ f(v_0v_{\lfloor \frac{n}{2} \rfloor -1}) = 1$, if $n = 2l$, l even and $l \geq 4$,
$f(v_0u_{\lfloor \frac{n}{2} \rfloor -1}) = 1,\ f(v_0v_{\lfloor \frac{n}{2} \rfloor -1}) = 0,\ f(u_{\lfloor \frac{n}{2} \rfloor -1}u_{\lfloor \frac{n}{2} \rfloor -2}) = 0$,
$f(v_{\lfloor \frac{n}{2} \rfloor -1}v_{\lfloor \frac{n}{2} \rfloor -2}) = 1$, if $n = 2l$, l odd and $l \geq 5$,
$f(u_{\lfloor \frac{n}{2} \rfloor -1}u_{\lfloor \frac{n}{2} \rfloor}) = 0,\ f(v_{\lfloor \frac{n}{2} \rfloor -1}v_{\lfloor \frac{n}{2} \rfloor}) = 1$, if $n = 2l+1$, l even and $l \geq 4$,
$f(u_iu_{i+1}) = 0,\ f(u_{i+1}u_{i+2}) = 1,\ f(v_iv_{i+1}) = 1,\ f(v_{i+1}v_{i+2}) = 0$,
for $i = 1, 3, \ldots, \alpha$, where

$$\alpha = \begin{cases} \lfloor \frac{n}{2} \rfloor - 4, & \text{if } n = 2l,\ l \text{ odd and } l \geq 5 \\ \lfloor \frac{n}{2} \rfloor - 3, & \text{if } n = 2l,\ l \text{ even and } l \geq 4 \\ \lfloor \frac{n}{2} \rfloor - 2, & \text{if } n = 2l+1,\ l \text{ odd and } l \geq 5 \\ \lfloor \frac{n}{2} \rfloor - 3, & \text{if } n = 2l+1,\ l \text{ even and } l \geq 4 \end{cases}$$

II. Labeling the edges of P_{k_i} with $i = 1, 3, \ldots, \alpha$,
where

$$\alpha = \begin{cases} \lfloor \frac{n}{2} \rfloor - 2, & \text{if } n = 2l,\ l \text{ odd and } l \geq 5 \\ \lfloor \frac{n}{2} \rfloor - 1, & \text{if } n = 2l,\ l \text{ even and } l \geq 4 \\ \lfloor \frac{n}{2} \rfloor - 2, & \text{if } n = 2l+1,\ l \text{ odd and } l \geq 5 \\ \lfloor \frac{n}{2} \rfloor - 1, & \text{if } n = 2l+1,\ l \text{ even and } l \geq 4 \end{cases}$$

Define $f(u_i v_i) = 2$, if $k_i = 1$,

If k_i is even and $k_i \geq 2$, define
$f(w_{i,k_i-1} v_i) = 2$,
$f(u_i w_{i,k_i-1}) = 3$, if $k_i = 2$,
$f(w_{i,k_i-1} w_{i,k_i-2}) = 3$, if $k_i \geq 4$,
$f(u_i w_{i,1}) = 2,\ f(w_{i,1} w_{i,2}) = 1$,
$f(w_{i,j} w_{i,j+1}) = 2,\ f(w_{i,j+1} w_{i,j+2}) = 1$, for $j = 2, 4, \ldots, k_i - 4$,

If k_i is odd and $k_i \geq 3$, define
$f(u_i w_{i,1}) = 2,\ f(v_i w_{i,k_i-1}) = 2,\ f(w_{i,k_i-1} w_{i,k_i-2}) = 1$,
$f(w_{i,j} w_{i,j+1}) = 1,\ f(w_{i,j+1} w_{i,j+2}) = 2$, for $j = 1, 3, \ldots, k_i - 4$.

III. Labeling the edges of P_{k_i} with $i = 2, 4, \ldots, \alpha$,
where

$$\alpha = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor - 1, & \text{if } n = 2l,\ l \text{ odd and } l \geq 5 \\ \left\lfloor \frac{n}{2} \right\rfloor - 2, & \text{if } n = 2l,\ l \text{ even and } l \geq 4 \\ \left\lfloor \frac{n}{2} \right\rfloor - 1, & \text{if } n = 2l+1,\ l \text{ odd and } l \geq 5 \\ \left\lfloor \frac{n}{2} \right\rfloor - 2, & \text{if } n = 2l+1,\ l \text{ even and } l \geq 4 \end{cases}$$

Define $f(u_i v_i) = 2$, if $k_i = 1$,

If k_i is even and $k_i \geq 2$, define
$f(w_{i,k_i-1} v_i) = 3$,
$f(u_i w_{i,k_i-1}) = 2$, if $k_i = 2$,
$f(w_{i,k_i-1} w_{i,k_i-2}) = 2$, if $k_i \geq 4$,
$f(u_i w_{i,1}) = 2,\ f(w_{i,1} w_{i,2}) = 1$,
$f(w_{i,j} w_{i,j+1}) = 2,\ f(w_{i,j+1} w_{i,j+2}) = 1$, for $j = 2, 4, \ldots, k_i - 4$,

If k_i is odd and $k_i \geq 3$, define
$f(u_i w_{i,1}) = 2,\ f(v_i w_{i,k_i-1}) = 2,\ f(w_{i,k_i-1} w_{i,k_i-2}) = 1$,
$f(w_{i,j} w_{i,j+1}) = 1,\ f(w_{i,j+1} w_{i,j+2}) = 2$, for $j = 1, 3, \ldots, k_i - 4$.
From the definition of f it follows that adjacent vertices get distinct labels and adjacent edges get distinct labels.

Table 2: Possible labels for the vertices and for their incident edges under $(2,1)$-total labeling f

Vertex label	Labels of the corresponding incident edges
5	$3, 2, 1, 0$
4	$2, 1, 0$
3	$1, 0$

Table 2 we observe that the labels of the incident elements have the difference at least 2. Thus f satisfies all the condition of $(2,1)$-total labeling. Therefore, $\lambda_2^T(C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor \frac{n}{2} \rfloor -1}})) \leq 5$.
By Lemma 2.1, $\lambda_2^T(C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor \frac{n}{2} \rfloor -1}})) \geq 5$.
Hence, $\lambda_2^T(C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor \frac{n}{2} \rfloor -1}})) = 5$. □

Theorem 2.3. *For* $n = 6, 7$.
$\lambda_2^T(C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor \frac{n}{2} \rfloor -1}})) \leq \Delta + p = 5$.

Proof. Let V and E denote the vertex set and edge set of $C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor \frac{n}{2} \rfloor -1}})$ respectively, where $n = 6$ or 7.
Define $f : V \cup E \to \{0, 1, 2, 3, 4, 5\}$ in the following way.

I. Labeling the vertices and edges of C_n

Define $f(u_0) = 3$,
$f(u_1) = 5$, $f(u_2) = 4$, $f(v_1) = 4$, $f(v_2) = 5$,
$f(v_0) = 3$, if $n = 6$,
$f(u_3) = 5$, $f(v_3) = 4$, if $n = 7$

Define $f(u_0u_1) = 1$, $f(u_0v_1) = 0$,
$f(u_1u_2) = 0$, $f(v_1v_2) = 1$,
$f(u_2v_0) = 1$, $f(v_2v_0) = 0$, if $n = 6$,
$f(u_2u_3) = 1$, $f(v_2v_3) = 0$, $f(u_3v_3) = 2$, if $n = 7$.

II. Labeling the vertices and edges of P_{k_1}

Define $f(w_{1,k_1-1}) = 0$, if k_1 is even and $k_1 \geq 2$,
$f(w_{1,j}) = 4$, $f(w_{1,j+1}) = 5$, for $j = 1, 3, \ldots, \beta$, where

$$\beta = \begin{cases} k_1 - 3, & \text{if } k_1 \text{ is even and } k_1 \geq 4 \\ k_1 - 2, & \text{if } k_1 \text{ is odd and } k_1 \geq 3 \end{cases}$$

Define $f(u_1v_1) = 2$, if $k_1 = 1$,

If k_1 is even and $k_1 \geq 2$, define
$f(w_{1,k_1-1}v_1) = 2$,
$f(u_1w_{1,k_1-1}) = 3$, if $k_1 = 2$,
$f(w_{1,k_1-1}w_{1,k_1-2}) = 3$, if $k_1 \geq 4$,
$f(u_1w_{1,1}) = 2$, $f(w_{1,1}w_{1,2}) = 1$,
$f(w_{1,j}w_{1,j+1}) = 2$, $f(w_{1,j+1}w_{1,j+2}) = 1$, for $j = 2, 4, \ldots, k_1 - 4$,

If k_1 is odd and $k_1 \geq 3$, define
$f(u_1w_{1,1}) = 2$, $f(v_1w_{1,k_1-1}) = 2$, $f(w_{1,k_1-1}w_{1,k_1-2}) = 1$,
$f(w_{1,j}w_{1,j+1}) = 1$, $f(w_{1,j+1}w_{1,j+2}) = 2$, for $j = 1, 3, \ldots, k_1 - 4$.

III. Labeling the vertices and edges of P_{k_2}
Define $f(w_{2,k_2-1}) = 0$, if k_2 is even and $k_2 \geq 2$,
$f(w_{2,j}) = 5$, $f(w_{2,j+1}) = 4$, for $j = 1, 3, \ldots, \beta$, where

$$\beta = \begin{cases} k_2 - 3, & \text{if } k_2 \text{ is even and } k_2 \geq 4 \\ k_2 - 2, & \text{if } k_2 \text{ is odd and } k_2 \geq 3 \end{cases}$$

Define $f(u_2v_2) = 2$, if $k_2 = 1$,
If k_2 is even and $k_2 \geq 2$, define
$f(w_{2,k_2-1}v_2) = 3$,
$f(u_2w_{2,k_2-1}) = 2$, if $k_2 = 2$,
$f(w_{2,k_2-1}w_{2,k_2-2}) = 2$, if $k_2 \geq 4$,
$f(u_2w_{2,1}) = 2$, $f(w_{2,1}w_{2,2}) = 1$,
$f(w_{2,j}w_{2,j+1}) = 2$, $f(w_{2,j+1}w_{2,j+2}) = 1$, for $j = 2, 4, \ldots, k_2 - 4$,
If k_2 is odd and $k_2 \geq 3$, define
$f(u_2w_{2,1}) = 2$, $f(v_2w_{2,k_2-1}) = 2$, $f(w_{2,k_2-1}w_{2,k_2-2}) = 1$,
$f(w_{2,j}w_{2,j+1}) = 1$, $f(w_{2,j+1}w_{2,j+2}) = 2$, for $j = 1, 3, \ldots, k_2 - 4$.

From the definition of f it follows that adjacent vertices get distinct labels and adjacent edges get distinct labels and also labels of the incident elements have the difference at least 2. Thus f satisfies all the condition of $(2, 1)$-total labeling.
Therefore, $\lambda_2^T(C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor \frac{n}{2} \rfloor -1}})) \leq \Delta + p = 5$. □

Corollary 2.4. *Havet and Yu conjecture is true for the family cycles with parallel paths.*

Proof. Let $G = C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor \frac{n}{2} \rfloor -1}})$. Then, by Theorem 2.2 and Theorem 2.3, $\lambda_2^T(G) \leq \Delta + 2 = 5$, since $\Delta(G) = 3$. Hence, Havet and Yu conjecture is true for $G = C_n \oplus (P_{k_1}, P_{k_2}, \ldots, P_{k_{\lfloor \frac{n}{2} \rfloor -1}})$. □

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References

[1] F. Bazzaro, M. Montassier, A. Raspaud, $(d, 1)$-total labeling of planar graphs with large girth and high maximum degree, *Discrete Mathematics*, **307** (2007), 2141–2151. http://dx.doi.org/10.1016/j.disc.2005.12.059

[2] Dong Chen, Weifan Wang, (2, 1)-Total labeling of outerplanar graphs, *Discrete Applied Mathematics*, **155** (2007), 2585–2593. http://dx.doi.org/10.1016/j.dam.2007.07.016

[3] J. R. Griggs, R. K. Yeh, Labelling graphs with a condition at distance two, *SIAM Journal of Discrete Mathematics*, **5** (1992), 586–595. http://dx.doi.org/10.1137/0405048

[4] F. Havet, M.-L. Yu, $(d, 1)$-Total labeling of graphs, *Technical Report*, **4650**, INRIA (2002).

[5] F. Havet, $(d, 1)$-*Total Labelling of Graphs*, in: Workshop on Graphs and Algorithms, Dijon, France, 2003.

[6] F. Havet, M.-L. Yu, $(p, 1)$-Total labeling of graphs, *Discrete Mathematics*, **308** (2008), 496–513. http://dx.doi.org/10.1016/j.disc.2007.03.034

[7] F. Havet, S. Thomasse, Complexity of $(p, 1)$-total labeling, *Discrete Applied Mathematics*, **157** (2009), no. 13, 2859–2870. http://dx.doi.org/10.1016/j.dam.2009.03.021

[8] M. Montassier, A. Raspaud, $(d, 1)$-total labelling of graphs with a given maximum average degree, *Journal of Graph Theory*, **51** (2006), 93–109. http://dx.doi.org/10.1002/jgt.20124

[9] Nasreen Khan, Madhumangal Pal and Anita Pal, (2, 1)-total labeling of cactus graphs, *Journal of Information and Computing Science*, **5** (2010), no. 4, 243–260.

[10] Sumei Zhang, Oiaoling Ma, Jihui Wang, The (2, 1)-total labeling of $S_{n+1} \vee P_m$ and $S_{n+1} \times P_m$, *Applied Mathematics*, **1** (2010), 366–369. http://dx.doi.org/10.4236/am.2010.15048

[11] Toru Hasunuma, Toshimasa Ishii, H. Ono, Y. Uno, A tight upper bound on the (2, 1)-total labeling number of outerplanar graphs, *Journal of Discrete Algorithms*, **14** (2012), 189–206. http://dx.doi.org/10.1016/j.jda.2011.12.020

[12] Toru Hasunuma, Toshimasa Ishii, H. Ono, A. Uno, The (p, q)-total labeling problem of trees, *Discrete Mathematics*, **312** (2012), 1407–1420. http://dx.doi.org/10.1016/j.disc.2012.01.007

[13] Weifan Wang, Jing Huang, Sun Haina, and Danjun Hung, (2, 1)-total number of joins of paths and cycles, *Taiwanese Journal of Mathematics*, **16** (2012), 605–619.

[14] M. A. Whittlesey, J. P., Georges, D. W. Mauro, On the λ-number of Q_n and related graphs, *SIAM Discrete Mathematics*, **8** (1995), 499–506. http://dx.doi.org/10.1137/s0895480192242821

[15] R. K. Yeh, A survey on labeling graphs with a condition at distance two, *Discrete Mathematics*, **306** (2006), 1217–1231. http://dx.doi.org/10.1016/j.disc.2005.11.029

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