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On a Special Nonhomogeneous Riemann Problem in Generalized Hardy Classes

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Abstract

In this work, we consider one Riemann problem generated by a single exponential system with discontinuous complex coefficients. We find conditions on the jumps of the argument of coefficient which are sufficient for the Noetherness of our problem. We also construct a general solution for nonhomogeneous problem in generalized Hardy classes with the variable rate of summability.

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1 Introduction

When considering equations of mixed or elliptic type, there often arise the systems of sines

$$\{\sin\ (n+\alpha)\ t\}_{n\in N}\,, \tag{1}$$

and cosines

$$\{\cos\ (n+\alpha)\ t\}_{n\in Z_+}\,, \tag{2}$$

where $\alpha : [0,\pi] \to R$ is some real parameter (N is the set of all natural numbers, $Z_+ = \{0\} \bigcup N$). The basis properties of these systems (such as completeness, minimality, basicity) have been studied in various function spaces (see, e.g., [1-11]). In the context of applications to specific problems of mechanics and mathematical physics, there has recently been a growing interest in the study of various problems of analysis in the Lebesgue and Sobolev spaces with the variable exponent of summability. Many research works, review articles, and even one monograph by Cruz-Uribe D.V., A. Fiorenza [12] have extensively treated the problems of harmonic analysis, approximation theory, theory of partial differential equations, etc in the above mentioned spaces. Matters related to approximation in these spaces have been considered in [13-17].

One of the most efficient methods to study the basis properties of systems like

$$a\,(t)\,e^{int} - b\,(t)\,e^{-int},\ n \in N, \tag{3}$$

with complex-valued coefficients $a\,(\cdot)$ and $b\,(\cdot)$, which are the generalizations of systems (1), (2), is the method of boundary value problems for analytic functions dating back to A.V. Bitsadze [1]. This method was further developed in [1-5; 10; 11; 18-29].

In this work, we consider a boundary value Riemann problem of the theory of analytic functions adapted to the system (3). We focus on the case where the argument of the coefficient of problem is piecewise continuous on the unit circle, with the possibility to have an infinite number of discontinuities of the first kind. Under some conditions on the corresponding jumps of the argument, we study the solvability of nonhomogeneous Riemann problem in generalized Hardy classes with the variable rate of summability. It should be noted that Noetherness of the Riemann problem in the weighted generalized Hardy classes is studied in [33].

2 Needful Information

We will use the standard notation. Z will be the set of all integers; R will denote the set of all real numbers; C will be a complex plane; $(\bar{\cdot})$ will denote a complex conjugation, δ_{nk} will be the Kronecker symbol; and $\chi_A\,(\cdot)$ will stand for the characteristic function of the set A.

Let $p : [-\pi,\pi] \to [1,+\infty)$ be some Lebesgue-measurable function. A class of all the Lebesgue-measurable functions on $[-\pi,\pi]$ will be denoted by $\mathscr{L}_0$.

Also denote

$$I_p(f) \overset{def}{\equiv} \int_{-\pi}^{\pi} |f(t)|^{p(t)}\, dt.$$

Let

$$\mathscr{L} \equiv \{f \in \mathscr{L}_0 : I_p(f) < +\infty\}.$$

With $p^+ = \sup\limits_{[-\pi,\pi]} vrai\, p(t) < +\infty$, $\mathscr{L}$ becomes a linear space with the usual linear operations of addition of functions and multiplication by a number. Equipped with a norm

$$\|f\|_{p(\cdot)} \overset{def}{\equiv} \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\},$$

$\mathscr{L}$ becomes a Banach space, denote it by $L_{p(\cdot)}$. Let

$$WL \overset{def}{\equiv} \{p : p(-\pi) = p(\pi); \exists C > 0, \quad \forall t_1, t_2 \in [-\pi,\pi] : |t_1 - t_2| \le \tfrac{1}{2} \Rightarrow \\ \Rightarrow |p(t_1) - p(t_2)| \le \tfrac{C}{-\ln|t_1 - t_2|}\Big\}.$$

Throughout this paper, $q(\cdot)$ will denote a function conjugate to $p(\cdot)$: $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. Let $p^- = \inf\limits_{[-\pi,\pi]} vrai\, p(t)$. The following generalized Hölder's inequality is true

$$\int_{-\pi}^{\pi} |f(t)\, g(t)|\, dt \le c\left(p^-; p^+\right) \|f\|_{p(\cdot)} \|g\|_{q(\cdot)},$$

where $c(p^-; p^+) = 1 + \frac{1}{p^-} - \frac{1}{p^+}$. A property below, to be used in the sequel, follows directly from the definition.

Property 2.1 ([12]) *If $p(\cdot) : 1 < p^- \le p^+ < +\infty$, then the class $C_0^\infty(-\pi,\pi)$ (class of finite and infinitely differentiable functions) is everywhere dense in $L_{p(\cdot)}$.*

Denote by S the singular integral

$$Sf = \frac{1}{2\pi i} \int_\Gamma \frac{f(\tau)}{\tau - t} d\tau, \quad t \in \Gamma,$$

where $\Gamma \subset C$ is some piecewise Hölder curve in C. Define the weighted class

$$L_{p(\cdot),\rho(\cdot)} \overset{def}{\equiv} \left\{f : \quad \rho f \in L_{p(\cdot)}\right\},$$

equipped with the norm $\|f\|_{p(\cdot),\rho(\cdot)} \overset{def}{\equiv} \|\rho f\|_{p(\cdot)}$. The statement below was proved in [32].

Statement 2.2 ([32]) *Let $p \in WL$, $1 < p^-$. Then the singular operator S acts boundedly from $L_{p(\cdot),\rho(\cdot)}$ to $L_{p(\cdot),\rho(\cdot)}$ only when the inequalities*

$$-\frac{1}{p\left(\tau_k\right)}<\alpha_k<\frac{1}{q\left(\tau_k\right)}, \quad k=\overline{0,m}, \tag{4}$$

hold, where the weight function $\rho\left(\cdot\right)$ is defined by

$$\rho\left(t\right)=\prod_{k=0}^{m}\left|t-t_k\right|^{\alpha_k},$$

$$\left\{t_k\right\}_0^m\subset\left[-\pi,\pi\right),\ \left\{\alpha_k\right\}_0^m\subset R.$$

By $H_{p_0}^{+}$ we denote the usual Hardy class, where $p_0\in[1,+\infty)$ is some number. Let

$$H_{p(\cdot),\rho}^{\pm}\equiv\left\{f\in H_1^{+}: f^{+}\in L_{p(\cdot),\rho}(\partial\omega)\right\},$$

where $\omega=\{z\in C:\ |z|<1\}$ and f^{+} are nontangential boundary values of f on $\partial\omega$. The theorem below was proved in [16].

Theorem 2.3 *Let $p\in WL, p^{-}>1$, and the inequalities (4) be satisfied. If $F\in H_{p(\cdot),\rho}^{+}$, then $F^{+}\in L_{p(\cdot),\rho}$:*

$$F(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}K_z(t)F^{+}\left(e^{it}\right)dt\ , \tag{5}$$

where $F^{+}\left(\cdot\right)$ are nontangential boundary values of the function $F\left(\cdot\right)$ on $\partial\omega$ and $K_z(t)\equiv\frac{1}{1-ze^{-it}}$ is the Poisson kernel. Conversely, if $F^{+}\in L_{p(\cdot),\rho}$, then the function F, defined by (5), belongs to the class $H_{p(\cdot),\rho}^{+}$.

Following the classics, it is not difficult to define the weighted Hardy class ${}_mH_{p(\cdot),\rho}^{-}$ of analytic functions in $C\backslash\bar{\omega}$ ($\bar{\omega}=\omega\bigcup\partial\omega$) of order $m_0\leq m$ at infinity. Let $f\left(z\right)$ be an analytic function in $C\backslash\bar{\omega}$ of finite order $m_0\leq m$ at infinity, i.e.

$$f(z)=f_1(z)+f_2(z),$$

where $f_1(z)$ is a polynomial of degree $m_0\leq m$ ($f_1(z)\equiv 0$ for $m_0<0$, $f_2(z)$ is the principal part of Laurent decomposition of the function $f(z)$ at infinity. If the function $\varphi(z)\equiv\overline{f_2\left(\frac{1}{\bar{z}}\right)}$ belongs to the class $H_{p(\cdot),\rho}^{+}$, then we will say that the function $f\left(z\right)$ belongs to the class ${}_mH_{p(\cdot),\rho}^{-}$.

Absolutely similar to the classical case, one can prove the theorem below.

Theorem 2.4 *Let $p\in WL, p^{-}>1$, and the inequalities (4) be satisfied. If $f\in H_{p(\cdot),\rho}^{+}$, then*

$$\left\|f(re^{it})-f^{+}(e^{it})\right\|_{p(\cdot),\rho}\to 0,\ r\to 1-0,$$

where f^{+} are nontangential boundary values of f on $\partial\omega$.

The following theorem is also true.

Theorem 2.5 *Let $p \in WL, p^- > 1$, and the inequalities* (4) *be satisfied. If $f \in {}_mH^-_{p(\cdot),\rho}$, then*

$$\left\| f(re^{it}) - f^-(e^{it}) \right\|_{p(\cdot),\rho} \to 0, \ r \to 1+0,$$

where f^- are nontangential boundary values of f on $\partial\omega$ from outside ω.

Let's show the validity of an analog of the classical Smirnov theorem. Assume that $p \in WL, p^- > 1$, and the inequalities (4) are satisfied. Let $u \in H_1^+$ and $u^+ \in L_{p(\cdot),\rho}$, where u^+ are nontangential values of u on $\partial\omega$. Then it is known that $\exists\, f \in L_1(\partial\omega)$:

$$u(z) = \frac{1}{2\pi i}\int_{\partial\omega} \frac{f(\tau)}{\tau - z} d\tau.$$

Consequently, $u(re^{i\theta}) \to f(e^{i\theta})$ a.e. on $(-\pi, \pi)$ as $r \to 1-0$. It directly follows that $f \in L_{p(\cdot),\rho}$. Then from Theorem 2.3 we obtain $u \in H^+_{p(\cdot),\rho}$. So the following theorem is true.

Theorem 2.6 *Let $p \in WL, p^- > 1$, and the inequalities* (4) *be satisfied. If $u \in H_1^+$ and $u^+ \in L_{p(\cdot),\rho}$, then $u \in H^+_{p(\cdot),\rho}$.*

Consider the following Riemann problem in the classes $H^+_{p(\cdot),\rho} \times {}_mH^-_{p(\cdot),\rho}$:

$$F^+(\tau) - G(\tau)F^-(\tau) = f(\tau), \ \tau \in \partial\omega, \tag{6}$$

where $f \in L_{p(\cdot),\rho}$ is some function. By the solution of problem (6) we mean a pair of analytic functions $(F^+(z); F^-(z)) \in H^+_{p(\cdot),\rho} \times {}_mH^-_{p(\cdot),\rho}$ whose boundary values satisfy the equality (6) a.e. on $\partial\omega$.

3 Main Results

We consider the Riemann boundary value problem adapted to the system (3). Namely, let $A(t)$, $B(t)$ be complex-valued functions on $[0, \pi]$ satisfying the following conditions:

i) $A^{\pm 1}(\cdot)\,; B^{\pm 1}(\cdot) \in L_\infty(0, \pi)$;

ii) $\alpha(\cdot)\,; \beta(\cdot)$ are piecewise continuous functions on $(0, \pi)$ with the discontinuities $\{t_k\}_{k\in N}$ and $\{\tau_k\}_{k\in N}$, respectively. Assume that the set $\{\tilde{s}_k\} \equiv \{t_k\} \bigcup \{\tau_k\}$ can only have one limit point $\tilde{s}_0 \in (0, \pi)$ and the function $\tilde{\theta}(t) \equiv \beta(t) - \alpha(t)$ has finite limits from the right and left at the point $\tilde{s}_0$.

iii) $\sum_{k=1}^{\infty} |h(\tilde{s}_k)| < +\infty$, where $h(\tilde{s}_k) = \tilde{\theta}(\tilde{s}_k - 0) - \tilde{\theta}(\tilde{s}_k + 0)$ are the jumps of the function $\tilde{\theta}(\cdot)$ at the points $\tilde{s}_k$.

Define

$$G\left(e^{it}\right)=\left\{\begin{array}{l} B\left(t\right)A^{-1}\left(t\right),\ 0<t<\pi\,, \\ A\left(-t\right)B^{-1}\left(-t\right)\ ,\ -\pi<t<0\,. \end{array}\right.$$

Let $f\in L_{p(\cdot)}\left(0,\pi\right)$ be some function, and let

$$g\left(t\right)=\left\{\begin{array}{l} f\left(t\right)A^{-1}\left(t\right),\ 0<t<\pi, \\ -f\left(-t\right)B^{-1}\left(-t\right),-\pi<t<0. \end{array}\right.$$

Consider the following Riemann boundary value problem in the classes $H^{+}_{p(\cdot)}\times_m H^{-}_{p(\cdot)}$:

$$F^{+}\left(\tau\right)+G\left(\tau\right)F^{-}\left(\tau\right)=0,\ \tau\in\partial\omega. \tag{7}$$

Denote $\theta\left(t\right)=\arg G\left(e^{it}\right)$. We have

$$\theta\left(t\right)=\left\{\begin{array}{l} \beta\left(t\right)-\alpha\left(t\right),\ t\in\left(0,\pi\right)\ , \\ \alpha\left(-t\right)-\beta\left(-t\right),\ t\in\left(-\pi,0\right)\ . \end{array}\right.$$

It is clear that $\{s_0=0\}\bigcup\{-\tilde{s}_k\}$ are also the discontinuities of the function $\theta\left(\cdot\right)$ on $(-\pi,\pi)$.

Consider the following jumping function $\theta_1\left(\cdot\right)$:

$$\theta_1\left(-\pi\right)=0,$$

$$\theta_1\left(s\right)=\left[\theta\left(-\pi+0\right)-\theta\left(-\pi\right)\right]+$$

$$+\sum_{-\pi<s_k<s}h\left(s_k\right)+\left[\theta\left(s\right)-\theta\left(s-0\right)\right],-\pi<s\leq\pi, \tag{8}$$

where $\{s_k\}\equiv\{-\tilde{s}_k\}\bigcup\{s_0\}\bigcup\{\tilde{s}_k\}$, and $h\left(s_k\right)=\theta\left(s_k+0\right)-\theta\left(s_k-0\right)$.

In [30], the following lemma was proved.

Lemma 3.1 *Let the condition iii) be true, and the function $\theta_1\left(\cdot\right)$ be defined by* (8). *Then $\theta\left(s\right)=\theta_0\left(s\right)+\theta_1\left(s\right)$, $\forall s\in\left[-\pi,\,\pi\right]$, where $\theta_0\in C\left[-\pi,\,\pi\right]$.*

This lemma allows to apply the method developed by I.I. Daniliuk [31] to solve the homogeneous Riemann problem (7) in the classes $H^{+}_{p(\cdot)}\times_m H^{-}_{p(\cdot)}$. Assume that the inequalities

$$-\frac{2\pi}{p\left(s_i\right)}<\tilde{h}_i<\frac{2\pi}{q\left(s_i\right)},i=\overline{1,\,\infty}, \tag{9}$$

hold. Using the notations of [31], we have

$$h_0^{(0)}=\theta_0\left(\pi\right)-\theta_0\left(-\pi\right),h_0^{(i)}=\theta_1\left(-\pi+0\right)-\theta_1\left(\pi-0\right);$$

$$h_0=h_0^{(i)}-h_0^{(0)}=\theta\left(-\pi+0\right)-\theta\left(\pi-0\right)=2\left(\alpha\left(\pi\right)-\beta\left(\pi\right)\right).$$

The jump of the function $\theta_1(s)$ at the point $s=0$ is equal to

$$h(0)=\theta(+0)-\theta(-0)=2\theta(+0)=2(\beta(0)-\alpha(0)).$$

We first require that the inequality

$$-\frac{\pi}{p(0)}<\alpha(0)-\beta(0)<\frac{\pi}{q(0)}, \tag{10}$$

hold. Before proceeding further, let's show the validity of one result of [31] for $L_{p(\cdot)}$. Let $\{s_k\}\subset[-\pi,\,\pi]$ be an arbitrary, no more than countable set, and $\{\delta_k\}\subset(0,\,+\infty)$ be an arbitrary totality of positive numbers, but of the same cardinality. We will assume that

$$\sum_k \delta_k<+\infty;\delta_{k+1}\le\delta_k,\forall k\in N. \tag{11}$$

Consider the following infinite product

$$\varphi(s)=\prod_k\left\{\sin\left|\frac{s-s_k}{2}\right|\right\}^{-\delta_k}.$$

I.I. Daniliuk [31] proved the following

Lemma 3.2 ([31]) *Let* $\{s_k\}\subset[-\pi,\,\pi]$ *be different points and* $\{\delta_k\}\subset(0,\,+\infty)$ *satisfy the condition* (11)*. If* $q_0=\inf\left\{\frac{1}{\delta_k}\right\}$*, then the infinite product* $\varphi(s)$ *belongs to the space* $L_{q_0-\varepsilon}(-\pi,\,\pi)$ *for* $\forall\varepsilon>0$ *(i.e.* $\varphi(\cdot)\in L_{q_0-0}(-\pi,\pi)$*), and does not belong to* $L_q(-\pi,\,\pi)$ *for* $q\ge q_0$*.*

So, let the condition *iii*) hold and the inequalities (9), (10) be true. Then let's show that the infinite product

$$\psi(s)=\prod_{k=0}^{\infty}\left\{\sin\left|\frac{s-\tilde{s}_k}{2}\right|\right\}^{\frac{h(\tilde{s}_k)}{2\pi}},s_0=0,$$

belongs to the space $L_{p(\cdot)}(-\pi,\,\pi)$. In fact, it is clear that for $\forall m\in N$ the function

$$\psi_m(s)=\prod_{k=0}^{m}\left\{\sin\left|\frac{s-\tilde{s}_k}{2}\right|\right\}^{\frac{h(\tilde{s}_k)}{2\pi}},$$

belongs to the space $L_{p(\cdot)}(-\pi,\,\pi)$. Let

$$h_k^-=\begin{cases}-\frac{1}{2\pi}h(\tilde{s}_k), & for\ \ h(\tilde{s}_k)<0,\\ 0, & for\ \ h(\tilde{s}_k)\ge 0.\end{cases}$$

It suffices to show that $\psi_m^-(s) \in L_{p(\cdot)}(-\pi, \pi)$, where

$$\psi_m^-(s) = \prod_{k=m+1}^{m} \left\{ \sin \left| \frac{s - \tilde{s}_k}{2} \right| \right\}^{-h_k^-}.$$

If $card\left(\left\{h_k^-\right\}\right) < +\infty$, then it is clear that $\psi_m^-(\cdot) \in L_{p(\cdot)}(-\pi, \pi)$. Let $card\left(\left\{h_k^-\right\}\right) = +\infty$. Obviously, $\lim_{k\to\infty} h_k^- = 0$. Therefore, $\lim_{m\to\infty} \inf_{k\geq m} \left\{\frac{1}{h_k^-}\right\} = +\infty$. Let $p^+ = \sup_{[-\pi,\pi]} vrai\, p(t)$. Consider $m_0 \in N : p_{m_0} = \inf_{k\geq m_0} \left\{\frac{1}{h_k^-}\right\} \geq p^+$. From Lemma 2 [18] it follows that $\psi_{m_0}^-(\cdot) \in L_{p_{m_0}}(-\pi, \pi)$. Then the continuous embeddings

$$L_{p_{m_0}}(-\pi, \pi) \subset L_{p^+}(-\pi, \pi) \subset L_{p(\cdot)}(-\pi, \pi),$$

imply $\psi_{m_0}^-(\cdot) \in L_{p(\cdot)}(-\pi, \pi)$. So the following lemma is true.

Lemma 3.3 *Let $\{\tilde{s}_k\} \subset [-\pi, \pi]$ be a set of different points with at most one limit point $\tilde{s}_0 \in (-\pi, \pi)$, the set of numbers $\left\{\tilde{h}_k\right\}$ satisfy the condition iii) and the inequalities* (9), (10). *Then the infinite product $\psi(\cdot)$ belongs to the space $L_{p(\cdot)}(-\pi, \pi)$.*

Now we proceed to solve the problem (7). We will solve this problem by the scheme offered by I.I. Daniliuk [31]. Consider the following functions $X_1(\cdot)$ analytic inside (with the sign +) and outside (with the sign -) the unit circle ω:

$$X_1(z) = \exp\left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln\left|G\left(e^{it}\right)\right| \frac{e^{it}+z}{e^{it}-z} dt \right\},$$

$$X_2(z) = \exp\left\{ \frac{i}{4\pi} \int_{-\pi}^{\pi} \theta(t) \frac{e^{it}+z}{e^{it}-z} dt \right\}.$$

Define

$$Z_k(z) \equiv \begin{cases} X_k(z), & |z| < 1, \\ \left[X_k(z)\right]^{-1}, & |z| > 1,\ k = 1, 2, \end{cases}$$

and let

$$Z(z) = Z_1(z) Z_2(z).$$

$Z(\cdot)$ is called a canonical solution of homogeneous problem. Let

$$u_0(t) \equiv \left\{ \sin \left| \frac{t-\pi}{2} \right| \right\}^{-\frac{h_0^{(0)}}{2\pi}} \exp\left\{ -\frac{1}{4\pi} \int_{-\pi}^{\pi} \theta_0(\tau)\, ctg \frac{t-\tau}{2} d\tau \right\}.$$

Divide the set $\{h_k\}$ into two sets: the positive part $\left\{h_k^+\right\}$ and the absolute values of the negative part $\left\{h_k^-\right\}$. Denote

$$u^{\pm}(t) = \prod_k \sin\left|\frac{t-s_k^{\pm}}{2}\right|^{\frac{h_k^{\pm}}{2\pi}}.$$

By virtue of [31], $\left|Z_2^-(\tau)\right|$ is expressed by the formula

$$\left|Z_2^-\left(e^{it}\right)\right| = u_0(t)\left[u^+(t)\right]^{-1} u^-(t)\left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{-\frac{h_0}{2\pi}}.$$

It follows directly from Sokhotskii-Plemelj formula that

$$\sup_{(-\pi,\pi)} vrai\left\{\left|Z_2^-\left(e^{it}\right)\right|^{\pm 1}\right\} < +\infty.$$

Thus, for $\left|Z^-\left(e^{it}\right)\right|^{-1}$ we have the representation

$$\left|Z^-\left(e^{it}\right)\right|^{-1} = \left|Z_2^-\left(e^{it}\right)\right|^{-1} \left|u_0(t)\right|^{-1}\left|u^+(t)\right| \left|u^-(t)\right|^{-1}\left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{\frac{h_0}{2\pi}}.$$

Based on this representation, the following theorem was proved in [30].

Theorem 3.4 *Let the conditions i) $-$ iii) be fulfilled with respect to the coefficient $G\left(e^{it}\right)$ of the problem (7), $p \in WL$, $1 < p^- \leq p^+ < +\infty$, and the jumps $\{h_k\}_0^{\infty}$ of the function* $\arg G\left(e^{it}\right)$ *satisfy the inequalities*

$$\left.\begin{array}{l} -\frac{1}{q(s_k)} < \frac{h_k}{2\pi} < \frac{1}{p(s_k)},\ k \in N; \\ -\frac{1}{q(\pi)} < \frac{h_0}{2\pi} < \frac{1}{p(\pi)}. \end{array}\right\} \tag{12}$$

Then the general solution of the homogeneous Riemann problem (7) in the classes $\left(H^+_{p(\cdot)};_m H^-_{p(\cdot)}\right)$ has the form $F(z) \equiv Z(z)P_m(z)$, where $Z(z)$ is a canonical solution of homogeneous problem, and $P_m(z)$ is an arbitrary polynomial of degree $\leq m$.

This theorem has the following direct corollary.

Corollary 3.5 *Let all the conditions of Theorem 3.4 be satisfied. Then for $F^-(\infty) = 0$ the homogeneous Riemann problem (7) has only the trivial solution $F^{\pm}(z) \equiv 0$ in the classes $\left(H^+_{p(\cdot)};_{-1} H^-_{p(\cdot)}\right)$.*

Consider the following nonhomogeneous Riemann problem

$$\begin{cases} F^{+}(\tau) + G(\tau) F^{-}(\tau) = \Psi(\tau), & \tau \in \gamma, \\ F(\infty) = 0, \end{cases} \tag{13}$$

where

$$G\left(e^{it}\right) = \begin{cases} B(t) A^{-1}(t), & 0 < t < \pi, \\ A(-t) B^{-1}(-t), & -\pi < t < 0, \end{cases}$$

$$\Psi(t) = \begin{cases} A^{-1}(t) \psi(t), & t \in (0, \pi), \\ -B^{-1}(-t) \psi(-t), & t \in (-\pi, 0), \end{cases}$$

and $\psi \in L_{p(\cdot)}(0, \pi)$ is some function.

We will solve this problem by the method developed in [31]. We first require that the inequality

$$-\frac{\pi}{p(0)} < \alpha(0) - \beta(0) < \frac{\pi}{q(0)}, \tag{14}$$

hold. Consider the following Cauchy type integral

$$F(z) = \frac{Z(z)}{2\pi} \int_{-\pi}^{\pi} \frac{\Psi(\sigma)}{Z^{+}(e^{i\sigma})} \frac{d\sigma}{1 - e^{-i\sigma} z}. \tag{15}$$

Applying Sokhotskii-Plemelj formulas, we obtain

$$F^{\pm}\left(e^{is}\right) = \pm \frac{1}{2} \Psi(s) + \frac{Z^{\pm}(e^{is})}{2\pi} \int_{-\pi}^{\pi} \frac{\Psi(\sigma)}{Z^{+}(e^{i\sigma})} \frac{d\sigma}{1 - e^{i(s-\sigma)}}. \tag{16}$$

Taking into account that $Z(\cdot)$ is a canonical solution, i.e. taking into account that it satisfies the homogeneous equation, it is not difficult to see that $F(\cdot)$ satisfies the boundary condition (13). Let's verify that the function $F(\cdot)$ belongs to the Hardy classes $H^{+}_{p(\cdot)} \times_{-1} H^{-}_{p(\cdot)}$. It is not difficult to see that $F(\infty) = 0$. We have

$$\left|Z^{-}\left(e^{it}\right)\right| = \left|Z_1^{-}\left(e^{it}\right)\right| \left|u_0(t)\right| \prod_k \left|\sin \frac{t - s_k}{2}\right|^{-\frac{h_k}{2\pi}},$$

where

$$u_0(t) = \left|\sin \frac{t - \pi}{2}\right|^{-\frac{h_0^{(0)}}{2\pi}} \exp\left\{-\frac{1}{4\pi} \int_{-\pi}^{\pi} \theta_0(\tau)\, ctg \frac{t - \tau}{2} d\tau\right\}.$$

It is clear that

$$\left|Z^{+}\left(e^{it}\right)\right| \sim \left|Z^{-}\left(e^{it}\right)\right|, \; t \in (-\pi, \pi).$$

Consider the following singular operator S_ρ :

$$(S_\rho g)(\tau) = \frac{\rho(\arg\tau)}{2\pi}\int_{-\pi}^{\pi}\frac{g(t)\,dt}{\rho(t)(e^{it}-\tau)},\ \ \tau\in\partial\omega.$$

The following statement is true.

Statement 3.6 ([32]) *Let $p\in WL_\pi\wedge p^->1$, and the weight $\rho(\cdot)$ be defined by*

$$\rho(t)=\prod_k\left|\sin\frac{t-t_k}{2}\right|^{\alpha_k},\ t\in[-\pi,\pi],$$

where $\sum_k|\alpha_k|<+\infty$, and $\{t_k\}\subset[-\pi,\pi)$ are different points. Then the singular operator S_ρ acts boundedly in $L_{p(\cdot)}$ only when the inequalities

$$-\frac{1}{p(t_k)}<\alpha_k<\frac{1}{q(t_k)},\ \ k=\overline{1,r},$$

hold.

Now, let

$$\tilde\rho(t)=Z^+\left(e^{it}\right),\ \ t\in[-\pi,\pi].$$

We have

$$F^+\left(e^{it}\right)=\frac{1}{2}\Psi(t)-(S_{\tilde\rho}\Psi)(t).$$

It is clear that $\Psi(\cdot)\in L_{p(\cdot)}$. We have

$$|\tilde\rho(t)|\sim\left|Z_1^-\left(e^{it}\right)\right|\,|u_0(t)|\prod_k\left|\sin\frac{t-s_k}{2}\right|^{-\frac{h_k}{2\pi}}.$$

We first assume that the following inequalities are true

$$-\frac{1}{q(s_k)}<\frac{h_k}{2\pi}<\frac{1}{p(s_k)},\ \forall k; \tag{17}$$

$$-\frac{1}{p(\pi)}<\frac{\beta(\pi)-\alpha(\pi)}{\pi}<\frac{1}{q(\pi)}. \tag{18}$$

Then, applying Statement 3.6 to the operator $S_{\tilde\rho}$, we obtain that it acts boundedly in $L_{p(\cdot)}$. As a result, from (16) we obtain that the function $F^+\left(e^{it}\right)$ belongs to $L_{p(\cdot)}$. Then, by Smirnov theorem (see, e.g., [16]), we have $F^+\in H^+_{p(\cdot)}$. It is not difficult to see that $F(\infty)=0$. Similar reasoning shows that $F^-\in_{-1}H^-_{p(\cdot)}$. So the following statement is true.

Statement 3.7 ([32]) *Let $p \in WL$, $p^{-1} > 1$, and the functions $A(\cdot)$ and $B(\cdot)$ satisfy the conditions i), ii). If the inequalities* (14), (17) *and* (18) *are true, then the function* (15) *is a solution of the problem* (13).

Consider the case where at least one of the following inequalities is satisfied:

$$-\frac{1}{p(0)} - 2 < \frac{\beta(0) - \alpha(0)}{\pi} < \frac{1}{p(0)} - 1,$$

$$1 - \frac{1}{p(\pi)} < \frac{\beta(\pi) - \alpha(\pi)}{\pi} < -\frac{1}{p(\pi)} + 2. \tag{19}$$

Let

$$(T\psi)(\varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\Psi(\sigma)}{Z_0^+(e^{i\sigma})} \frac{d\sigma}{1 - e^{i(\varphi-\sigma)}}.$$

We have

$$(T\psi)(\varphi) = -\frac{1}{4\pi} \int_0^{\pi} \frac{\cos\frac{\theta}{2}}{\cos\frac{\varphi}{2}} e^{-I(\theta)} \frac{\psi(\theta)}{\sqrt{A(\theta) B(\theta)}} \left[\frac{1}{\sin\frac{\theta-\varphi}{2}} + \frac{1}{\sin\frac{\theta+\varphi}{2}}\right] d\theta.$$

The relation

$$(T\psi)(\varphi) = -\frac{i}{4\pi} \int_0^{\pi} \frac{\sin\frac{\sigma}{2}}{\sin\frac{\varphi}{2}} \frac{\psi(\sigma)}{e^{I(\sigma)}\sqrt{A(\sigma) B(\sigma)}} \left[\frac{1}{\sin\frac{\sigma-\varphi}{2}} - \frac{1}{\sin\frac{\sigma+\varphi}{2}}\right] d\sigma,$$

is also true, where

$$I(\sigma) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\ln G(s)}{1 - e^{i(\sigma-s)}} ds = \frac{1}{2\pi} \int_0^{\pi} \left[\frac{1}{1 - e^{i(\sigma-\theta)}} - \frac{1}{1 - e^{i(\sigma+\theta)}}\right] \ln\frac{B(\theta)}{A(\theta)} d\theta =$$

$$= -\frac{i}{4\pi} \int_0^{\pi} \frac{\cos\frac{\theta}{2}}{\cos\frac{\sigma}{2}} \left[\frac{1}{\sin\frac{\theta-\sigma}{2}} + \frac{1}{\sin\frac{\theta+\sigma}{2}}\right] \ln\frac{B(\theta)}{A(\theta)} d\theta.$$

In the last relation, we used the following formula

$$\frac{1}{1 - e^{i(\theta-\varphi)}} - \frac{1}{1 - e^{i(\theta+\varphi)}} = -\frac{i}{2} \frac{\cos\frac{\varphi}{2}}{\cos\frac{\theta}{2}} \left[\frac{1}{\sin\frac{\varphi-\theta}{2}} + \frac{1}{\sin\frac{\varphi+\theta}{2}}\right].$$

Considering these relations in the expression for the singular operator S_ρ, absolutely similar to the previous case we obtain that the function (15) is a solution of the problem (13) if the inequalities (17) and (19) are satisfied. Uniting these two cases, we arrive at the following conclusion.

Theorem 3.8 *Let $p \in WL \wedge p^- > 1$ and the functions $A(\cdot), B(\cdot)$ satisfy the conditions i), ii). If the inequalities* (17) *are fulfilled and the relations*

$$\frac{1}{p(0)} - 2 < \frac{\beta(0) - \alpha(0)}{\pi} < \frac{1}{p(0)},$$

$$-\frac{1}{p(\pi)} < \frac{\beta(\pi) - \alpha(\pi)}{\pi} < -\frac{1}{p(\pi)} + 2,$$

are true, then the Cauchy type integral

$$F(z) = \frac{Z(z)}{2\pi} \int_{-\pi}^{\pi} \frac{\Psi(\sigma)}{Z^+(e^{i\sigma})} \frac{d\sigma}{1 - e^{i\sigma} z},$$

is a solution of the Riemann problem (13) *in the classes $H^+_{p(\cdot)} \times_{-1} H^-_{p(\cdot)}$.*

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