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A Recursive Sequence of Sums of Consecutive Embedded Coalitions

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Abstract

The set of players in a cooperative game may be divided into various coalitions forming partitions with different coalition structures. The well-known Bell (1934) number is used to obtain the number of partitions in a $n-$person cooperative game. The number of embedded coalitions in a partition is the number of subsets formed in that partition. The total number of embedded coalitions in a $n-$person game is the sum of the numbers of embedded coalitions in different partitions of the game. This article presents a recursive sequence yielding the total sum of the embedded coalitions from a $1-person$ game to a $n-person$ game.

Keywords: Recursive sequence, partitions, embedded coalitions, cooperative games

1 Introduction

Lucas and Thrall (1963) introduced the formulation for the theory of cooperative games in terms of partition functions. In particular, the set of players may be divided into various coalitions forming partitions with different coalition structures. Let $N = \{1, 2, \cdots, n\}$ be a finite set of n players. The subsets of N are coalitions. A partition Λ in a $n-$person game is formed by disjoint non-empty subsets of N representing a way that these n players are joined. Given a partition Λ and a coalition $S \subset N$, the pair (S, Λ) is called an embedded coalition, that is the coalition S embedded in partition Λ. The n^{th} Bell (1934) number $\beta(n)$ is used to obtain the number of partitions in a $n-$person game. The number of embedded coalitions in a partition is the number of subsets formed in that partition. The number of embedded coalitions in a $n-$person game is the sum of the numbers of embedded coalitions in different partitions of the game. The number of embedded coalitions $Y(n)$ in a $n-$person game is given by Yeung (2008).

For example, in a 3-person game there are 5 partitions (each embraced by a pair of square brackets) and 10 embedded coalitions (each embraced by a pair of curly brackets) as shown in Figure 1.1. In a 4-person game there are 15 partitions (each embraced by a pair of square brackets) and 37 embedded coalitions (each embraced by a pair of curly brackets) as shown in Figure 1.2.

$$[\ \{P_1, P_2, P_3\}\ ]\ \ [\{P_1, P_2\}\{P_3\}]\ \ [\{P_1, P_3\}\{P_2\}]\ \ [\{P_2, P_3\}\{P_1\}]\ \ [\{P_1\}\{P_2\}\{P_3\}]$$

Figure 1.1. Partitions and embedded coalitions in a 3-person game

$$\begin{array}{lll}
[\ \{P_1, P_2, P_3, P_4\}\ ] & [\{P_1, P_2, P_3\}\ \{P_4\}] & [\{P_1, P_2, P_4\}\ \{P_3\}] \\
[\{P_1, P_3, P_4\}\ \{P_2\}] & [\{P_2, P_3, P_4\}\ \{P_1\}] & [\{P_1, P_2\}\ \{P_3, P_4\}] \\
[\{P_1, P_3\}\ \{P_2, P_4\}] & [\{P_1, P_4\}\ \ \{P_2, P_3\}] & [\{P_1, P_2\}\{P_3\}\{P_4\}] \\
[\{P_1, P_3\}\{P_2\}\{P_4\}] & [\{P_1, P_4\}\{P_2\}\{P_3\}] & [\{P_2, P_3\}\{P_1\}\{P_4\}] \\
[\{P_2, P_4\}\{P_1\}\{P_3\}] & [\{P_3, P_4\}\{P_1\}\ \{P_2\}] & [\{P_1\}\{P_2\}\{P_3\}\{P_4\}]
\end{array}$$

Figure 1.2. Partitions and embedded coalitions in a 4-person game

In particular, the Bell (1934) number can be obtained from the recursive sequence:

$$\beta(0) = 1,$$

$$\beta(n+1) = \sum_{t=0}^{n} \binom{n}{t} \beta(t), \text{ for } n \geq 1. \tag{1.1}$$

The number of embedded coalitions in a n-person game can be obtained from the recursive sequence (Yeung 2008):

$$\mathrm{Y}(1) = \sum_{t=0}^{0}\binom{1}{t} = \binom{1}{0} = 1, \qquad \text{for } n = 1;$$

$$\mathrm{Y}(2) = \sum_{t=0}^{1}\binom{2}{t} = \binom{2}{1} + \binom{2}{0} = 3, \qquad \text{for } n = 2;$$

$$\mathrm{Y}(n) = \sum_{t=2}^{n-1}\binom{n}{t}\left(\sum_{k=1}^{t-1}\mathrm{Y}(k)\right) + \sum_{t=0}^{n-1}\binom{n}{t}, \text{ for } n \geq 3. \tag{1.2}$$

In this article, we consider the sum of the embedded coalitions in a $1-person$ game, the embedded coalitions in a $2-person$ game and up to the embedded coalitions in a $n-person$ game.

2 A Sequence of the Sums of Consecutive Embedded Coalitions

Using the number of embedded coalitions in a n-person game in (1.2) the sum of the embedded coalitions in a $1-person$ game, the embedded coalitions in a $2-person$ game and up to the embedded coalitions in a $n-person$ game can be obtained as:

$$\psi_1(n) = \sum_{t=1}^{n} \mathrm{Y}(t). \tag{2.1}$$

A recursive sequence representing the number $\psi_1(n)$ in (2.1) can be obtained as follows.

Theorem 2.1.
The sum of consecutive embedded coalitions from a 1-person game to a $n-$person game can be obtained from the recursive sequence:

$$\psi_1(n) = \sum_{t=1}^{n}\binom{n}{t}[\psi_1(t-1)+1], \qquad \text{for } n \geq 1,$$

$$\psi_1(0) = 0. \tag{2.2}$$

Proof.
Invoking Theorem 1 in Yeung (2008) the number of embedded coalitions in a n-person game can also be expressed as:

$$\mathrm{Y}(n) = \beta(n+1) - \beta(n), \qquad \text{for } n \geq 1. \tag{2.3}$$

where $\beta(n)$ and $\beta(n+1)$ are Bell numbers.

As shown in Yeung (2008) in the case of a n-person game, the number of partitions of this set of n players is $\beta(n)$ while the number of embedded coalitions is $\mathrm{Y}(n)$. If one additional player is added to a n-person game the number of players becomes $n+1$. This additional player (that is the $(n+1)^{\text{th}}$ player) could been placed as i). a single-person coalition in each of the existing $\beta(n)$ partitions from a n-person game to form $\beta(n)$ partitions each with $n+1$ players; and

ii). a new additional player in each one of the $\Upsilon(n)$ coalitions in a n-person game to form $\Upsilon(n)$ partitions with $n+1$ players.

From the results in (i) and (ii) above one can readily observe that there are $\beta(n)$ plus $\Upsilon(n)$ partitions in a game with $n+1$ players. Since the number of partitions in a game with $n+1$ players is $\beta(n+1)$, we have

$$\Upsilon(n)+\beta(n)=\beta(n+1),$$

which yields (2.3).

In addition, one can also see that (2.3) can be expressed as (1.2) in Yeung (2008).

We then use (2.3) to express the sum of consecutive embedded coalitions from a 1-person game to an $n-$person game as:

$$\psi_1(n)=\sum_{t=1}^{n}\Upsilon(t)=\sum_{t=1}^{n}[\beta(t+1)-\beta(t)]=\beta(n+1)-\beta(1)$$
$$=\beta(n+1)-1, \qquad \text{for } n\geq 1. \tag{2.4}$$

With $\beta(n+1)=\sum_{t=0}^{n}\binom{n}{t}\beta(t)$ from the Bell number, we have

$$\psi_1(n)=\beta(n+1)-1=\sum_{t=0}^{n}\binom{n}{t}\beta(t)-1=\sum_{t=1}^{n}\binom{n}{t}\beta(t). \tag{2.5}$$

Since $\beta(n+1)=\psi_1(n)+1$, we can express (2.5) as

$$\psi_1(n)=\sum_{t=1}^{n}\binom{n}{t}[\psi_1(t-1)+1]$$
$$\psi_1(0)=0. \tag{2.6}$$

Hence Theorem 2.1 follows. *Q.E.D.*

Using Theorem 2.1 we can obtain the sum of consecutive embedded coalitions in from a 1-person game to an $n-$person game for $n\in\{1,2,\cdots,5\}$ as:

$$\psi_1(1)=\sum_{t=1}^{1}\binom{1}{t}[\psi_1(t-1)+1]=\binom{1}{0}(1)=1,$$

$$\psi_1(2)=\sum_{t=1}^{2}\binom{2}{t}[\psi_1(t-1)+1]=\binom{2}{1}[\psi_1(0)+1]+\binom{2}{2}[\psi_1(1)+1]=2(1)+1(2)=4,$$

$$\psi_1(3)=\sum_{t=1}^{3}\binom{3}{t}[\psi_1(t-1)+1]=\binom{3}{1}[\psi_1(0)+1]+\binom{3}{2}[\psi_1(1)+1]+\binom{3}{3}[\psi_1(2)+1]$$
$$=3(1)+3(2)+1(5)=14,$$

$$\psi_1(4)=\sum_{t=1}^{4}\binom{4}{t}[\psi_1(t-1)+1]=\binom{4}{1}[\psi_1(0)+1]+\binom{4}{2}[\psi_1(1)+1]+\binom{4}{3}[\psi_1(2)+1]$$

$$+\binom{4}{4}[\psi_1(3)+1]=4(1)+6(2)+4(5)+1(15)=51,$$

$$\psi_1(5)=\sum_{t=1}^{5}\binom{5}{t}[\psi_1(t-1)+1]=\binom{5}{1}[\psi_1(0)+1]+\binom{5}{2}[\psi_1(1)+1]+\binom{5}{3}[\psi_1(2)+1]$$

$$+\binom{5}{4}[\psi_1(3)+1]+\binom{5}{5}[\psi_1(4)+1]=5(1)+10(2)+10(5)+5(15)+1(52)=202,$$

In Figure 2.1 the values of $\beta(n)$, $\Upsilon(n)$ and $\psi_1(n)$ up to $n=25$ are provided.

Figure 2.1. Values of $\beta(n)$, $\Upsilon(n)$ and $\psi_1(n)$ for $n\le 25$

n	$\beta(n)$	$\Upsilon(n)$	$\psi_1(n)$
0	1	--	0
1	1	1	1
2	2	3	4
3	5	10	14
4	15	37	51
5	52	151	202
6	203	674	876
7	877	3,263	4,139
8	4,140	17,007	21,146
9	21,147	94,828	115,974
10	115,975	562,595	678,569
11	678,570	3,535,027	4,213,596
12	4,213,597	23,430,840	27,644,436
13	27,644,437	163,254,885	190,899,321
14	190,899,322	1,192,059,223	1,382,958,544
15	1,382,958,545	9,097,183,602	10,480,142,146
16	10,480,142,147	72,384,727,657	82,864,869,803
17	82,864,869,804	599,211,936,355	682,076,806,158
18	682,076,806,159	5,150,665,398,898	5,832,742,205,056
19	5,832,742,205,057	45,891,416,030,315	51,724,158,235,371
20	51,724,158,235,372	423,145,657,921,379	474,869,816,156,750
21	474,869,816,156,751	4,031,845,922,290,572	4,506,715,738,447,322
22	4,506,715,738,447,323	39,645,290,116,637,023	44,152,005,855,084,345
23	44,152,005,855,084,346	401,806,863,439,720,943	445,958,869,294,805,288
24	445,958,869,294,805,289	4,192,631,462,935,194,064	4,638,590,332,229,999,352
25	4,638,590,332,229,999,353	44,992,656,191,388,756,921	49,631,246,523,618,756,273

3 Conclusion

The number $\psi_1(n)$ in Theorem 2.1 is a recursive sequence counting the sum of consecutive embedded coalitions from a 1-person game to an $n-$person game. Similar to the Bell number $\beta(n)$, the number $\psi_1(n)$ is a Fibonacci type number obtained from a linear function of the $n-1$ preceding terms along with an initial term $\psi_1(0)=0$ (while the initial term of $\beta(n)$ is $\beta(0)=1$).

In addition, using Theorem 2.1 a recursive sequence counting the sum of consecutive embedded coalitions from a k-person game to a $n-$person game, for $n>k$, can be obtained as:

$$\psi_k(n)=\sum_{t=k}^{n}\binom{n}{t}[\psi_k(t-1)+1], \qquad \text{for } n>k>0,$$

$$\psi_k(k-1)=\psi_1(k-1)=\sum_{t=1}^{k-1}\binom{k-1}{t}[\psi_1(t-1)+1] \text{ and } \psi_1(0)=0. \tag{3.1}$$

Finally this article presents a novel recursive sequence (and some subsequences) which enumerate meaning numbers of large numerical value.

References

[1] E.T. Bell, Exponential numbers, *Amer. Math. Monthly,* **41** (1934), 411-419. http://dx.doi.org/10.2307/2300300

[2] W. Lucas, R. Thrall, N-person games in partition function form, *Naval Research Logistics Quarterly*, **10** (1963), 281-298. http://dx.doi.org/10.1002/nav.3800100126

[3] D.W.K. Yeung, Recursive Sequences Identifying the Number of Embedded Coalitions, *Int. Game Theory Rev.*, **10** (2008), 129-136. http://dx.doi.org/10.1142/s0219198908001819

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