International Journal of Mathematical Analysis
Vol. 9, 2015, no. 45, 2235 - 2242
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijma.2015.57177

Common Fixed Point Theorems for Compatible Mappings in Dislocated Metric Space

Dinesh Panthi

Department of Mathematics, Valmeeki Campus
Nepal Sanskrit University, Kathmandu, Nepal

Abstract

In this article we establish common fixed point results for two pairs of compatible mappings in dislocated metric space which generalize and extend similar results in the literature.

Mathematics Subject Classification: 47H10, 54H25

Keywords: d-metric space, common fixed point, compatible maps, cauchy sequence

1 Introduction

In 1922, S. Banach [8] established a fixed point theorem for contraction mapping in metric space. Since then a number of fixed point theorems have been proved by many authors and various generalizations of this theorem have been established. In 1976, G. Jungck [3] initiated the concept of commuting maps and generalized it with the concept of compatible maps [4], [5] and established some important fixed point results.
The study of common fixed point of mappings satisfying contractive type conditions has been a very active field of research activity. In 1986, S. G. Matthews [9] introduced the concept of dislocated metric space under the name of metric domains in domain theory. In 2000, P. Hitzler and A. K. Seda [7] generalized the famous Banach Contraction Principle in dislocated metric space. The

study of dislocated metric plays very important role in topology, logic programming and in electronics engineering.

The purpose of this article is to establish a common fixed point theorem for two pairs of compatible mappings in dislocated metric spaces which generalize and improve similar results of fixed point in the literature.

2 Preliminary Notes

We start with the following definitions, lemmas and theorems.

Definition 2.1 *[7] Let X be a non empty set and let $d : X \times X \to [0, \infty)$ be a function satisfying the following conditions:*
(i) $d(x, y) = d(y, x)$
(ii) $d(x, y) = d(y, x) = 0$ implies $x = y$.
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then d is called dislocated metric(or simply d-metric) on X.

Definition 2.2 *[7] A sequence $\{x_n\}$ in a d-metric space (X, d) is called a Cauchy sequence if for given $\epsilon > 0$, there corresponds $n_0 \in N$ such that for all $m, n \geq n_0$, we have $d(x_m, x_n) < \epsilon$.*

Definition 2.3 *[7] A sequence in d-metric space converges with respect to d (or in d) if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.*

Definition 2.4 *[7] A d-metric space (X, d) is called complete if every Cauchy sequence in it is convergent with respect to d.*

Definition 2.5 *[7] Let (X, d) be a d-metric space. A map $T : X \to X$ is called contraction if there exists a number λ with $0 \leq \lambda < 1$ such that $d(Tx, Ty) \leq \lambda\ d(x, y)$.*

Theorem 2.6 *[7] Let (X, d) be a complete d-metric space and let $T : X \to X$ be a contraction mapping, then T has a unique fixed point.*

Definition 2.7 *Let A and S be two self mappings on a set X. Mappings A and S are said to be commuting if $ASx = SAx \quad \forall x \in X$.*

Definition 2.8 *[4] Two mappings S and T from a metric space (X, d) into itself are called compatible if*

$$\lim_{n \to \infty} d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$ for some $x \in X$

Proposition 2.9 *[4] Let S and T be compatible mappings from a metric space (X, d) into itself. Suppose that*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x \quad for \quad some \quad x \in X$$

if S is continuous then $\lim_{n\to\infty} TSx_n = Sx$

3 Main Results

Theorem 3.1 *Let (X, d) be a complete dislocated metric space. Let* $A, B, S, T : X \to X$ *be mappings satisfying*

$$T(X) \subset A(X) \quad and \quad S(X) \subset B(X) \tag{1}$$

$$The \quad pairs \quad (T, B) \quad and \quad (S, A) \quad are \quad compatible \tag{2}$$

$$d(Tx, Sy) \le \alpha[d(Bx, Tx) + d(Ay, Sy)] + \beta d(Bx, Ay) + \gamma d(Bx, Sy) + \delta d(Tx, Ay) \tag{3}$$

If any one of A, B, S and T is continuous for all $x, y \in X$
where $\alpha, \beta, \gamma, \delta \ge 0, \quad 0 \le 2\alpha + \beta + 2\gamma + 2\delta < 1$, *then A, B, S and T have a unique common fixed point.*

Proof :
Using condition 1 we define sequences $\{x_n\}$ and $\{y_n\}$ in X by the rule

$$y_{2n+1} = Ax_{2n+1} = Tx_{2n} \quad and \quad y_{2n} = Bx_{2n} = Sx_{2n-1} \tag{4}$$

Assume also that $y_{2n} \ne y_{2n+1}$ for all n
Now

$$\begin{aligned} d(y_{2n+1}, y_{2n}) &= d(Tx_{2n}, Sx_{2n-1}) \\ &\le \alpha[d(Bx_{2n}, Tx_{2n}) + d(Ax_{2n-1}, Sx_{2n-1})] + \beta d(Bx_{2n}, Ax_{2n-1}) \\ &+ \gamma d(Bx_{2n}, Sx_{2n-1}) + \delta d(Tx_{2n}, Ax_{2n-1}) \\ &\le \alpha[d(y_{2n}, y_{2n+1}) + d(y_{2n-1}, y_{2n})] + \beta d(y_{2n}, y_{2n-1}) \\ &+ \gamma d(y_{2n}, y_{2n}) + \delta d(y_{2n+1}, y_{2n-1}) \end{aligned}$$

Hence

$$d(y_{2n+1}, y_{2n}) \le \frac{(\alpha + \beta + 2\gamma + \delta)}{1 - \alpha - \delta} d(y_{2n}, y_{2n-1})$$

$$d(y_{2n+1}, y_{2n}) \le h d(y_{2n}, y_{2n-1}) \quad where \quad h = \frac{(\alpha + \beta + 2\gamma + \delta)}{1 - \alpha - \delta} < 1$$

This shows that

$$d(y_{n+1}, y_n) \leq hd(y_n, y_{n-1}) \leq ... \leq h^n d(y_1, y_0)$$

For every integer $q > 0$ we have

$$\begin{aligned} d(y_{n+q}, y_n) &\leq d(y_{n+q}, y_{n+q-1}) + ... + d(y_{n+2}, y_{n+1}) + d(y_{n+1}, y_n) \\ &\leq (h^{q-1} + ... + h^2 + h + 1)d(y_{n+1}, y_n) \\ &\leq (h^{q-1} + ... + h^2 + h + 1)h^n d(y_1, y_0) \end{aligned}$$

Since $h < 1$, so $h^n \to 0$ as $n \to \infty$
Therefore $d(y_{n+q}, y_n) \to 0$ as $n \to \infty$. This implies that $\{y_n\}$ is a cauchy sequence.
Since X is complete, so there exists a point $z \in X$ such that $\{y_n\} \to z$. Consequently subsequences

$$\{Tx_{2n}\}, \{Bx_{2n}\}, \{Sx_{2n-1}\} \quad and \quad \{Ax_{2n+1}\} \quad \to \quad z \in X. \tag{5}$$

Let us assume that B is continuous. Since the pair (T, B) is compatible on X so by proposition 2.9 we have

$$B^2x_{2n} \quad \& \quad TBx_{2n} \to Bz \quad as \quad n \to \infty. \tag{6}$$

Now consider

$$\begin{aligned} d(TBx_{2n}, Sx_{2n-1}) &\leq \alpha[d(B^2x_{2n}, TBx_{2n}) + d(Ax_{2n-1}, Sx_{2n-1})] \\ &+ \beta d(B^2x_{2n}, Ax_{2n-1}) + \gamma d(B^2x_{2n}, Sx_{2n-1}) \\ &+ \delta d(TBx_{2n}, Ax_{2n-1}) \end{aligned}$$

Now taking limit as $n \to \infty$ and using conditions 5 and 6 we have

$$\begin{aligned} d(Bz, z) &\leq \beta d(Bz, z) + \gamma d(Bz, z) + \delta d(Bz, z) \\ &= (\beta + \gamma + \delta)d(Bz, z) \end{aligned}$$

which is a contradiction, since $(\beta + \gamma + \delta) \neq 1$. Hence,

$$d(Bz, z) = 0$$

implies $Bz = z$ Now we show that z is fixed point of T . For this consider

$$\begin{aligned} d(Tz, Sx_{2n-1}) &\leq \alpha[d(Bz, Tz) + d(Ax_{2n-1}, Sx_{2n-1})] + \beta d(Bz, Ax_{2n-1}) \\ &+ \gamma d(Bz, Sx_{2n-1}) + \delta d(Tz, Ax_{2n-1}) \\ &= \alpha[d(z, Tz) + d(Ax_{2n-1}, Sx_{2n-1})] + \beta d(z, Ax_{2n-1}) \\ &+ \gamma d(z, Sx_{2n-1}) + \delta d(Tz, Ax_{2n-1}) \end{aligned}$$

Taking limit as $n \to \infty$ we have,

$$d(Tz, z) \leq \alpha d(z, Tz) + \delta d(Tz, z) = (\alpha + \delta)d(Tz, z)$$

which is a contradiction since $(\alpha+\delta) \neq 1$. Therefore $d(Tz, z) = 0$ implies $Tz = z$. As $T(X) \subset A(X)$, so there exists a point $u \in X$ such that $z = Tz = Au$. Consider

$$\begin{aligned} d(z, Su) = d(Tz, Su) &\leq \alpha[d(Bz, Tz) + d(Au, Su)] + \beta d(Bz, Au) \\ &+ \gamma d(Bz, Su) + \delta d(Tz, Au) \\ &= \alpha[d(z, z) + d(z, Su)] + \beta d(z, z) \\ &+ \gamma d(z, Su) + \delta d(z, z) \\ &= (3\alpha + 2\beta + \gamma + 2\delta)d(z, Su) \end{aligned}$$

which is a contradiction since $(3\alpha + 2\beta + \gamma + 2\delta) \neq 1$.
Hence $d(z, Su) = 0$ implies $z = Su$. By above relations we obtain

$$z = Tz = Bz = Au = Su$$

Since the pair (S, A) is compatible on X, so $d(SAu, ASu) = 0$ implies $SAu = ASu$. Hence, $Sz = Az$. Now we show that z is the fixed point of A. For this consider

$$\begin{aligned} d(z, Az) = d(Tz, Sz) &\leq \alpha[d(Bz, Tz) + d(Az, Sz)] + \beta d(Bz, Az) \\ &+ \gamma d(Bz, Sz) + \delta d(Tz, Az) \\ &= \alpha[d(z, z) + d(Az, Az)] + \beta d(z, Az) \\ &+ \gamma d(z, Az) + \delta d(z, Az) \\ &\leq \alpha[2d(z, Az) + 2d(Az, z)] + \beta d(z, Az) \\ &+ \gamma d(z, Az) + \delta d(z, Az) \\ &= (4\alpha + \beta + \gamma + \delta)d(z, Az) \end{aligned}$$

which is a contradiction since $(4\alpha+\beta+\gamma+\delta) \neq 1$. Hence. $d(z, Az) = 0$ implies $Az = z$.Therefore $Az = Bz = Sz = Tz = z$. Hence, z is the common fixed point of the mappings A, B, S and T.

Uniqueness: Let z and v be two common fixed point of the mappings A, B, S and T. Now by condition 3 we have

$$\begin{aligned} d(z, v) = d(Tz, Sv) &\leq \alpha[d(Bz, Tz) + d(Av, Sv)] + \beta d(Bz, Av) \\ &+ \gamma d(Bz, Sv) + \delta d(Tz, Av) \\ &= \alpha[d(z, z) + d(v, v)] + \beta d(z, v) \\ &+ \gamma d(z, v) + \delta d(z, v) \\ &\leq (4\alpha + \beta + \gamma + \delta)d(z, v) \end{aligned}$$

which is a contradiction since $(4\alpha + \beta + \gamma + \delta) \neq 1$. So $d(z, v) = 0$ implies $z = v$. Thus z is the unique common fixed point of the mappings A, B, S and T. This completes the proof of the theorem.

Now we have the following corollaries.
If we put $B = A$ in the above theorem 3.1, then the theorem is reduced to the following corollary

Corollary 3.2 *Let (X, d) be a complete dislocated metric space. Let $A, S, T : X \to X$ be mappings satisfying*

$$T(X) and \quad S(X) \subset A(X) \tag{7}$$

$$The \quad pairs \quad (T, A) \quad and \quad (S, A) \quad are \quad compatible \tag{8}$$

$$\begin{aligned} d(Tx, Sy) &\leq \alpha[d(Ax, Tx) + d(Ay, Sy)] + \beta d(Ax, Ay) \\ &+ \gamma d(Ax, Sy) + \delta d(Tx, Ay) \end{aligned} \tag{9}$$

If any one of A, S and T is continuous for all $x, y \in X$ where $\alpha, \beta, \gamma, \delta \geq 0$, $0 \leq 2\alpha + \beta + 2\gamma + 2\delta < 1$, then A, S and T have a unique common fixed point.

If we put $T = S$ in theorem 3.1, then we obtain the following corollary.

Corollary 3.3 *Let (X, d) be a complete dislocated metric space. Let $A, B, S : X \to X$ be mappings satisfying*

$$S(X) \subset A(X) \quad and \quad S(X) \subset B(X) \tag{10}$$

$$The \quad pairs \quad (S, B) \quad and \quad (S, A) \quad are \quad compatible \tag{11}$$

$$\begin{aligned} d(Sx, Sy) &\leq \alpha[d(Bx, Sx) + d(Ay, Sy)] + \beta d(Bx, Ay) \\ &+ \gamma d(Bx, Sy) + \delta d(Sx, Ay) \end{aligned} \tag{12}$$

If any one of A, B and S is continuous for all $x, y \in X$ where $\alpha, \beta, \gamma, \delta \geq 0$, $0 \leq 2\alpha + \beta + 2\gamma + 2\delta < 1$, then A, B and S have a unique common fixed point.

If we put $T = S$ and $B = A$ in the above theorem 3.1 then we obtain the following corollary

Corollary 3.4 *Let (X, d) be a complete dislocated metric space. Let $A, S : X \to X$ be mappings satisfying*

$$S(X) \subset A(X) \tag{13}$$

$$The \quad pair \quad (S, A) \quad is \quad compatible \quad and \tag{14}$$

$$\begin{aligned} d(Sx, Sy) &\leq \alpha[d(Ax, Sx) + d(Ay, Sy)] + \beta d(Ax, Ay) \\ &+ \gamma d(Ax, Sy) + \delta d(Sx, Ay) \end{aligned} \tag{15}$$

If any one of A and S is continuous for all $x, y \in X$ where $\alpha, \beta, \gamma, \delta \geq 0, \quad 0 \leq 2\alpha + \beta + 2\gamma + 2\delta < 1$, then A and S have a unique common fixed point.

If we put $A = B = I$ in the above theorem 3.1, then the theorem is reduced to the following corollary.

Corollary 3.5 *Let (X, d) be a complete dislocated metric space. Let $S, T, I : X \to X$ be mappings satisfying*

$$T(X) \quad and \quad S(X) \subset X \tag{16}$$

$$The \quad pairs \quad (T, I) \quad and \quad (S, I) \quad are \quad compatible \tag{17}$$

$$\begin{aligned} d(Tx, Sy) &\leq \alpha[d(x, Tx) + d(y, Sy)] + \beta d(x, y) \\ &+ \gamma d(x, Sy) + \delta d(Tx, y) \end{aligned} \tag{18}$$

If any one of S and T is continuous for all $x, y \in X$ where $\alpha, \beta, \gamma, \delta \geq 0, \quad 0 \leq 2\alpha + \beta + 2\gamma + 2\delta < 1$, then S and T have a unique common fixed point.

Now we have the following theorem.

Theorem 3.6 *Let (X, d) be a complete dislocated metric space. Let $A, B, S, T : X \to X$. Suppose that any one of A, B, S and T is continuous for all $x, y \in X$ and for some positive integers p, q, r and s which satisfy the following conditions*

$$T^s(X) \subset A^p(X) \quad and \quad S^r(X) \subset B^q(X) \tag{19}$$

$$The \quad pairs \quad (T, B) \quad and \quad (S, A) \quad are \quad compatible \tag{20}$$

$$\begin{aligned} d(T^s x, S^r y) &\leq \alpha[d(B^q x, T^s x) + d(A^p y, S^r y)] + \beta d(B^q x, A^p y) \\ &+ \gamma d(B^q x, S^r y) + \delta d(T^s x, A^p y) \end{aligned} \tag{21}$$

where $\alpha, \beta, \gamma, \delta \geq 0, \quad 0 \leq 2\alpha + \beta + 2\gamma + 2\delta < 1$, then A, B, S and T have a unique common fixed point.

Proof :
The proof of the theorem is similar to the above theorem 3.1

References

[1] B. Fisher, Common Fixed Point of Four Mappings, *Bull. Inst. of Math. Academia, Sinicia*, **11** (1983), 103-113.

[2] D. Panthi, K. Jha, P. K. Jha and P. S. Kumari, A Common Fixed Point Theorem for Two Pairs of Mappings in Dislocated Metric Space, *American Journal of Computational Mathematics,* **5** (2015), 106-112. http://dx.doi.org/10.4236/ajcm.2015.52009

[3] G. Jungck, Commuting mappings and fixed points, *Amer. Math. Monthly,* **83** (1976), 261-263. http://dx.doi.org/10.2307/2318216

[4] G. Jungck, Compatible mappings and common fixed points, *International Journal of Mathematics and Mathematical Sciences,* **9** (1986), 771-779. http://dx.doi.org/10.1155/s0161171286000935

[5] G. Jungck, Compatible mappings and common fixed points (2), *International Journal of Mathematics and Mathematical Sciences,* **11** (1988), 285-288. http://dx.doi.org/10.1155/s0161171288000341

[6] K. Jha and D. Panthi, A common fixed point theorem in dislocated metric space, *Applied Mathematical Sciences*, **6** (2012), no. 91, 4497-4503.

[7] P. Hitzler and A. K. Seda, Dislocated Topologies, *J. Electr. Eng.*, **51** (2000), no. 12/s, 3-7.

[8] S. Banach, Sur les operations dans les ensembles abstraits et leur applications aux equations integrales, *Fundamental Mathematicae*, **3** (1922), no. 7, 133-181.

[9] S. G. Matthews, *Metric Domains for Completeness, Technical Report 76 [Ph.D. Thesis]*, Department of Computer Science, University of Warwick, Coventry, UK, 1986.

Received: July 19, 2015; Publsihed: September 18, 2015