

New Approach to Classify Second Order Linear Delay Differential Equations With Constant Coefficients to Solvable Lie Algebra

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Abstract

The quest of developing efficient and accurate classification scheme for solving second order differential equations (DE) with various coefficients to solvable Lie algebra is never ending. We report a novel classification scheme for second order linear delay differential equations (DDEs) of retarded type with constant coefficients to solvable Lie algebra. The absence of an equivalent transformation related with the change in variables is considered to determine the invariant of DDEs in solvable Lie algebra. The infinitesimal generator of DDEs is used to determine the associated symmetry group. The equations are solved and the solvable Lie algebras spanned by these parameters are obtained by satisfying the inclusion property. This method is further extended to classify neutral delay DE and all classes of second order linear DDEs

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with constant coefficients to solvable Lie algebra. The excellent features of the results and successful implementation suggest that our new classifier may constitute a basis for categorizing second order linear DDEs with constant coefficients to solvable Lie Algebra.

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1 Introduction

Undoubtedly, differential equations (DE) has paramount importance in mathematics and other branches of science and engineering. There are several forms of DE such as ordinary, partial, stochastic, integro and delay to cite a few [1]. A number of mathematical models in physics, engineering sciences, biomathematics etc. [2, 3, 4, 5, 6, 7] lead to delay differential equations (DDEs). Often, researchers face difficulty in modeling DDEs due to lack of accurate classification scheme. Recently, the DDEs generated great interests in the mathematical formulation for solving a varieties of real life problem including population dynamics, coexistence of competitive micro-organisms, sugar quantity in blood, models of lasers and metal cutting, infectious disease, control problems, secure communication, traffic control and economics(see [8, 9, 10, 11, 12]). Moreover, the neutral delay type equations contain the derivative of the unknown function. The emergence of new phenomena makes the theory of neutral delay differential equations is even more complicated than non-neutral one. Therefore, it is mandatory to generalize the classification procedure uniquely to neutral delay with constant coefficient.

Group analysis introduced by Sophus Lie (see [13]) is considered to be an accurate method for studying the properties of DE, and the notation of continuous groups is called Lie groups. In the past, group analysis is widely exploited [14, 15, 16]. Ibragimov further applied this concept for classification of ordinary DE in terms of their symmetry groups [17]. Tanthanuch treated on definition of an admitted Lie group for functional DE [18] which helped Prapart Pue-on to introduce group classification for specific cases of second order delay DE of the form

$$x'' = f(t, x, x_\tau, x', x'_\tau),$$

by changing the variables and found the invariance of the Lie algebra in the space of changed variables. To determine the invariants of second order DDEs, consideration of Lie algebra [1] is inevitable because they do not posses an equivalent transformation related to the change of dependent and independent variables. Boyko studied Lie symmetries of a system of second order linear

ordinary DE with constant coefficients in real and complex field to compute the dimension of maximal Lie invariance [19]. Later, Moyo reported the properties of symmetry Lie group of a system of two linear second order ordinary DE with constant coefficients and changed variables for providing group classification. He achieved the general solution of this system [20]. Despite many dedicated efforts the validity, versatility and successful implementation of a generalized classification scheme remain debatable.

We developed a new approach based on group analysis for complete classification of second order DDEs to solvable Lie algebra of the form

$$x'' = f(t, x, x_\tau, x', x'_\tau),$$

where $x = x(t)$, $x_\tau = x(t - \tau)$, $x' = x'(t)$, $x'_\tau = x'(t - \tau)$ and $x'' = x''(t)$. Furthermore, the generalization of this classification is carried out to second order neutral delay DEs of the form

$$x'' = f(t, x, x_\tau, x', x'_\tau, x''_\tau),$$

where $x''_\tau = x''(t - \tau)$.

This paper is organized as follows. Section 2 renders the details of Lie algebra and delay ordinary DE. The classification of second order retarded DDEs and neutral DDEs to solvable Lie algebra and its generalization with main results are described in Section 3.

2 Preliminary Notes

A classification of second order linear delay differential equations with constant coefficients to solvable Lie algebra is proposed. It is customary to provide some striking features of Lie algebra and delay differential equations.

Definition 2.1 [21] *A Lie algebra L is an n -dimensional solvable algebra if there exists a sequence that yields,*

$$L_1 \subset L_2 \subset \dots \subset L_n = L,$$

here L_k is called k -dimensional Lie algebra and L_{k-1} is an ideal of L_k , $k = 1, 2, \dots, n$ in which two-dimensional Lie algebra are solvable.

Definition 2.2 [14] *Let*

$$Q_i = \zeta_s \frac{\partial}{\partial x_s} \text{ and } Q_j = \eta_s \frac{\partial}{\partial x_s}, \quad i, j = 1, 2, \dots, r \text{ and } s = 1, 2, \dots, n$$

be two infinitesimal generators. The commutator $[Q_i, Q_j]$ of Q_i and Q_j is the first order operator

$$[Q_i, Q_j] = Q_i Q_j - Q_j Q_i = \sum_s^n \sum_m^n \left(\zeta_m \frac{\partial \eta_s}{\partial x_m} - \eta_m \frac{\partial \zeta_s}{\partial x_m} \right) \frac{\partial}{\partial x_s}.$$

Example 2.3 [14] Let

$$Q_1 = y \frac{\partial}{\partial x}, \quad Q_2 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y},$$

then

$$\begin{aligned} [Q_1, Q_2] &= Q_1 x^2 \frac{\partial}{\partial x} + Q_1 xy \frac{\partial}{\partial y} - Q_2 y \frac{\partial}{\partial x} \\ &= xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}. \end{aligned}$$

The commutator $[\cdot, \cdot]$ satisfies the properties of Lie algebra.

Definition 2.4 [22] A finite set of infinitesimal generator $\{Q_1, Q_2, \dots, Q_r\}$ is said to be a basis for the Lie algebra L if $Q_i \in L$ and

1. Q_1, Q_2, \dots, Q_r form a basis of the vector space L ,
2. $[Q_i, Q_j] = c_{ijk} Q_k$.

The coefficients c_{ijk} are called the structure constant of the Lie algebra, $i, j, k = 1, 2, \dots, r$.

Theorem 2.5 (Second fundamental theorem of Lie) [14]

Any two infinitesimal generators of an r -parameter Lie group, satisfy commutation relation of the form $[Q_i, Q_j] = c_{ijk} Q_k$, where i, j and $k = 1, 2, \dots, r$.

The commutator and the Jacobi identity together are capable to form real (or complex) linear combinations of the Q_i , providing these infinitesimal generator the structure of the Lie algebra associated with the Lie group. Furthermore, the given infinitesimal generators Q_1, Q_2, \dots, Q_r form a basis for a Lie algebra.

Theorem 2.6 (Third fundamental theorem of Lie) [14]

The structure constants satisfy the relations

$$c_{ijk} = -c_{jik},$$

$$c_{ijk}c_{klm} + c_{jlk}c_{kim} + c_{lik}c_{kjm} = 0,$$

where i, j, k, l and $m = 1, 2, \dots, r$. In other word, the structure constants determine the Lie algebra, and hence the Lie group.

Now we consider the second order delay differential equations of the form

$$x'' = f(t, x, x_\tau, x', x'_\tau). \tag{1}$$

The infinitesimal generator

$$X = \xi(t, x) \frac{\partial}{\partial t} + \eta(t, x) \frac{\partial}{\partial x}, \tag{2}$$

is admitted by a second order DDE (1). The corresponding canonical Lie-Bäcklund operator has the form

$$X = \zeta(t, x, x') \frac{\partial}{\partial x},$$

where $\zeta = \eta - x'\xi$. In Equation (1) the periodic property of ξ which has to be satisfied for any solution of the Cauchy problem [23], implies the independence of the function ξ on x , i.e., $\xi_x = 0$. The determining equation (1) takes the form

$$\tilde{X} \left(x'' - f(t, x, x_\tau, x', x'_\tau) \right) \Big|_{(1)} = 0, \tag{3}$$

where

$$\tilde{X} = \xi \frac{\partial}{\partial t} + \eta^x \frac{\partial}{\partial x} + \eta^{x_\tau} \frac{\partial}{\partial x_\tau} + \eta^{x'} \frac{\partial}{\partial x'} + \eta^{x'_\tau} \frac{\partial}{\partial x'_\tau} + \eta^{x''} \frac{\partial}{\partial x''}, \tag{4}$$

and

$$\begin{aligned} \eta^x(t, x) &= \eta(t, x), \\ \eta^{x_\tau}(t, x) &= \eta^\tau(t - \tau, x_\tau), \\ \eta^{x'}(t, x, x') &= \eta_1(t, x, x') = \eta_t(t, x) + [\eta_x(t, x) - \xi_t(t, x)]x' - \xi_x(t, x)(x')^2, \\ \eta^{x'_\tau}(t, x_\tau, x'_\tau) &= \eta_1^\tau(t, x_\tau, x'_\tau) = \eta_t^\tau(t - \tau, x_\tau) + [\eta_x^\tau(t - \tau, x_\tau) - \xi_t^\tau(t - \tau, x_\tau)]x'_\tau \\ &\quad - \xi_x^\tau(t - \tau, x_\tau)(x'_\tau)^2, \\ \eta^{x''}(t, x, x', x'') &= \eta_2(t, x, x', x'') = \eta_{tt}(t, x) + [2\eta_{tx}(t, x) - \xi_{tt}(t, x)]x' + [\eta_{xx}(t, x) \\ &\quad - 2\xi_{tx}(t, x)](x')^2 - \xi_{xx}(t, x)(x')^3 + [\eta_x(t, x) - 2\xi_t(t, x)]x'' - 3\xi_x(t, x)x'x'', \end{aligned} \tag{5}$$

3 Second order linear delay differential equation classification to solvable Lie algebra

Here we present the classification of second order linear non-homogenous delay differential equations to solvable Lie algebra.

The general form of second order linear retarded delay differential equation is given by,

$$x''(t) + a(t)x'(t) + b(t)x'(t - \tau) + c(t)x(t) + d(t)x(t - \tau) = 0, \quad (6)$$

where $a(t), b(t), c(t), d(t)$ are arbitrary coefficients. The general infinitesimal generator associated with Equation (6) is of the form

$$X = \xi(t, x) \frac{\partial}{\partial t} + \eta(t, x) \frac{\partial}{\partial x} + \eta^\tau(t - \tau, x_\tau) \frac{\partial}{\partial x_\tau}. \quad (7)$$

From Equation (4) the second order extension

$$X^{(2)} = X + \eta_1(t, x, x') \frac{\partial}{\partial x'} + \eta_2(t, x, x', x'') \frac{\partial}{\partial x''} + \eta_1^\tau(t - \tau, x_\tau, x'_\tau) \frac{\partial}{\partial x'_\tau}. \quad (8)$$

By combining (8) and (6), the following invariance conditions are achieved,

$$\eta_2 + a\eta_1 + b\eta_1^\tau + c\eta + d\eta^\tau = 0,$$

where η_1, η_1^τ are the first extensions of ordinary and delay equations respectively, and the second extension of ODE are η_2 .

Substituting (5), we get

$$\eta_{tt} + [2\eta_{tx} - \xi_{tt}]x' + [\eta_{xx} - 2\xi_{tx}](x')^2 - \xi_{xx}(x')^3 + [\eta_x - 2\xi_t]x'' - 3\xi_x x'x'' + a[\eta_t + [\eta_x - \xi_t]x' - \xi_x(x')^2] + b[\eta_t^\tau + [\eta_x^\tau - \xi_t^\tau]x'_\tau - \xi_x^\tau(x'_\tau)^2] + c\eta + d\eta^\tau = 0.$$

Equating the first and second order coefficients of various monomials of x and x_τ , the following determining equations for the symmetry group of Equation (6) are acquired.

Table 1: The determining equations for the symmetry group of Equation (6).

| monomial | coefficient | number of equation |
|----------|---|--------------------|
| 1 | $\eta_{tt} + a\eta_t + b\eta_t^\tau + c\eta + d\eta^\tau = 0$ | a_1 |
| x' | $2\eta_{tx} - \xi_{tt} + a[\eta_x - \xi_t] = 0$ | a_2 |
| $(x')^2$ | $\eta_{xx} - 2\xi_{tx} - a\xi_x = 0$ | a_3 |
| $(x')^3$ | $\xi_{xx} = 0$ | a_4 |
| $x'x''$ | $\xi_x = 0$ | a_5 |

| monomial | coefficient | number of equation |
|---------------|-----------------------------------|--------------------|
| x'' | $\eta_x - 2\xi_t = 0$ | a_6 |
| x'_τ | $b[\eta_x^\tau - \xi_t^\tau] = 0$ | a_7 |
| $(x'_\tau)^2$ | $b\xi_x^\tau = 0$ | a_8 |

From (a₅) and onward ξ does not depend on x . In (a₃), η is linear in x , meaning $\eta = g(t)x + h(t)$ with $g(t)$ and $h(t)$ are arbitrary functions. In (a₆), $\xi_t = \frac{1}{2}g$. In (a₈), since $b \neq 0$ and $\xi_x^\tau = 0$ implies that ξ^τ does not depend on x . The periodic property achieves $\xi = \xi^\tau$ and $\xi_t = \xi_t^\tau$. In (a₇), $\eta^\tau = \frac{1}{2}g(t)x + k(t - \tau)$ with $k(t - \tau)$ as arbitrary function.

Case 1 Here $a, b, c,$ and d are arbitrary constant not equal zero.

From (a₁), $g_{tt}x + h_{tt} + ag_tx + ah_t + \frac{b}{2}g_tx + bk_t + cgx + ch + \frac{d}{2}gx + dk = 0$. Equating the coefficients of the various monomials one obtains,

$$g_{tt} + (a + \frac{b}{2})g_t + (c + \frac{d}{2})g = 0. \tag{9}$$

$$h_{tt} + ah_t + bk_t + ch + dk = 0.$$

Which means that $g(t), h(t)$ and $k(t - \tau)$ are the solutions of (6). From (9),

$$g = -\frac{g_{tt} + (a + \frac{b}{2})g_t}{c + \frac{d}{2}},$$

but $g = 2\xi_t$,

$$\xi_t = -\frac{\xi_{ttt} + (a + \frac{b}{2})\xi_{tt}}{c + \frac{d}{2}},$$

this implies that

$$\xi = -\frac{\xi_{tt} + (a + \frac{b}{2})\xi_t}{c + \frac{d}{2}} + c_1.$$

Let $a + \frac{b}{2} = B$, and $c + \frac{d}{2} = D$, then,

$$\xi = c_1 - \frac{\frac{1}{2}g_t + \frac{1}{2}Bg}{D}.$$

Now,

$$g = c_2 - \frac{1}{B}g_t - \frac{2D}{B}\xi.$$

This yield $\eta = (c_2 - \frac{1}{B}g_t - \frac{2D}{B}\xi)x + h$ and $\eta^\tau = (c_3 - \frac{1}{2B}g_t - \frac{D}{B}\xi)x + k$, where $c_i, i = 1, 2, 3$ are arbitrary constants.

Recall Equation (7), the general infinitesimal generator for (6) gives,

$$X = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x} + \eta^\tau \frac{\partial}{\partial x_\tau}$$

Let $x_\tau = u$, then

$$X = [c_1 - \frac{g_t + Bg}{2D}] \frac{\partial}{\partial t} + [(c_2 - \frac{1}{B}g_t - \frac{2D}{B}\xi)x + h] \frac{\partial}{\partial x} + [(c_3 - \frac{1}{2B}g_t - \frac{D}{B}\xi)x + k] \frac{\partial}{\partial u}.$$

The Lie algebra of (6) is spanned by the following three infinitesimal generator corresponding to each parameters $c_i, i = 1, 2, 3$.

$$Q_1 = \frac{\partial}{\partial t}, \quad Q_2 = x \frac{\partial}{\partial x}, \quad Q_3 = x \frac{\partial}{\partial u},$$

$$Q_4 = -\left(\frac{1}{2D}g_t + \frac{B}{2D}g\right) \frac{\partial}{\partial t} + \left[h(t) - x\left(\frac{1}{B}g_t + \frac{2D}{B}\xi\right)\right] \frac{\partial}{\partial x} + \left[k(t) - x\left(\frac{1}{2B}g_t + \frac{D}{B}\xi\right)\right] \frac{\partial}{\partial u},$$

Where Q_4 is an infinite dimensional Lie subalgebra. The commutator table is given by,

| | | | |
|-------|-------|--------|-------|
| | Q_1 | Q_2 | Q_3 |
| Q_1 | 0 | 0 | 0 |
| Q_2 | 0 | 0 | Q_3 |
| Q_3 | 0 | $-Q_3$ | 0 |

Here $L_3 = \{Q_1, Q_2, Q_3\}$ is spanned by Q_1, Q_2, Q_3 is the Lie algebra of (6). The subspace $L_1 = \{Q_1\}$, $L_2 = \{Q_1, Q_2\}$ are one, two dimensional Lie subalgebra of L_3 respectively. Furthermore, they obey the inclusion property: $L_1 \subset L_2 \subset L_3$, and hence L_3 is a solvable Lie algebra of (6).

Remark 3.1 *If $c = 0$ or $d = 0$, the same result is obtained.*

Case 2 If $b = 1$, and $a = c = d = 0$. The Equation (6) classified to Lie algebra takes the form,

$$Q_1 = t \frac{\partial}{\partial t} + tx \left(\frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial}{\partial u}\right), \quad Q_2 = \frac{\partial}{\partial t} + x \left(\frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial}{\partial u}\right), \quad Q_3 = \frac{\partial}{\partial u},$$

$$Q_4 = -g \frac{\partial}{\partial t} + (h - x\xi) \frac{\partial}{\partial x} - (h_t + \frac{1}{2}x\xi) \frac{\partial}{\partial u}.$$

Now, Q_4 is an infinite dimensional Lie subalgebra and the corresponding commutator table yields,

| | | | |
|-------|--------|-------|-------|
| | Q_1 | Q_2 | Q_3 |
| Q_1 | 0 | Q_2 | 0 |
| Q_2 | $-Q_2$ | 0 | 0 |
| Q_3 | 0 | 0 | 0 |

Then $L_3 = \{Q_1, Q_2, Q_3\}$ is a solvable Lie algebra of (6).

Case 3 If b are arbitrary constants not equal zero, and $a = c = d = 0$ then the Equation (6) classified to Lie algebra of the form

$$Q_1 = t \frac{\partial}{\partial t}, \quad Q_2 = \frac{\partial}{\partial t}, \quad Q_3 = tx \left(\frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial}{\partial u} \right),$$

$$Q_4 = x \left(\frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial}{\partial u} \right), \quad Q_5 = \frac{\partial}{\partial u},$$

$$Q_6 = -\frac{1}{b}g \frac{\partial}{\partial t} + [h - bx\xi] \frac{\partial}{\partial x} - \left[\frac{1}{b}h_t + \frac{b}{2}x\xi \right] \frac{\partial}{\partial u}.$$

where Q_6 is an infinite dimensional Lie subalgebra. The commutator table is

| | | | | | |
|-------|--------|--------|-------|-------|-------|
| | Q_1 | Q_2 | Q_3 | Q_4 | Q_5 |
| Q_1 | 0 | $-Q_2$ | Q_3 | 0 | 0 |
| Q_2 | Q_2 | 0 | Q_4 | 0 | 0 |
| Q_3 | $-Q_3$ | $-Q_4$ | 0 | 0 | 0 |
| Q_4 | 0 | 0 | 0 | 0 | 0 |
| Q_5 | 0 | 0 | 0 | 0 | 0 |

Then $L_4 = \{Q_1, Q_2, Q_3, Q_4, Q_5\}$ is a solvable Lie algebra of (6).

3.1 Second order neutral delay differential equations (NDDEs)

Here we render a classification scheme for second order neutral delay differential equations of the form

$$x'' = f(t, x, x_\tau, x', x'_\tau, x''_\tau), \tag{10}$$

to solvable Lie algebra.

The following lemma is used to complete the classification.

Lemma 3.2 *The second order neutral delay differential equations (10), containing the infinitesimal generator ξ that obeys periodic property is given by,*

$$\xi(t, x) = \xi^\tau(t - \tau, x_\tau).$$

Proof

First the canonical Lie-Bäcklund operator to the seven-dimensional space of variables $(t, x, x_\tau, x', x'_\tau, x'', x''_\tau)$ is searched, which yields

$$\tilde{X}_\beta = \zeta^x \frac{\partial}{\partial x} + \zeta^{x_\tau} \frac{\partial}{\partial x_\tau} + \zeta^{x'} \frac{\partial}{\partial x'} + \zeta^{x'_\tau} \frac{\partial}{\partial x'_\tau} + \zeta^{x''} \frac{\partial}{\partial x''} + \zeta^{x''_\tau} \frac{\partial}{\partial x''_\tau},$$

where $\zeta = \eta - x'\xi$, then

$$\zeta^x(t, x, x') = \eta(t, x) - x'\xi(t, x).$$

$$\zeta^{x_\tau}(t, x_\tau, x'_\tau) = \eta^\tau(t - \tau, x_\tau) - x'_\tau \xi^\tau(t - \tau, x_\tau).$$

$$\zeta^{x'}(t, x, x', x'') = D(\zeta_1) = \eta_t(t, x) + [\eta_x(t, x) - \xi_t(t, x)]x' - \xi_x(t, x)(x'')^2 - \xi(t, x)x''.$$

$$\zeta^{x'_\tau}(t, x_\tau, x'_\tau, x''_\tau) = D(\zeta_1^\tau) = \eta_t^\tau(t - \tau, x_\tau) + [\eta_x^\tau(t - \tau, x_\tau) - \xi_t^\tau(t - \tau, x_\tau)]x'_\tau - \xi_x^\tau(t - \tau, x_\tau)(x''_\tau)^2 - \xi^\tau(t - \tau, x_\tau)x''_\tau.$$

$$\zeta^{x''}(t, x, x', x'', x''') = D(\zeta_2) = \eta_{tt}(t, x) + [2\eta_{tx}(t, x) - \xi_{tt}(t, x)]x' + [\eta_{xx}(t, x) - 2\xi_{tx}(t, x)](x'')^2 - \xi_{xx}(t, x)(x''')^3 + [\eta_x(t, x) - 2\xi_t(t, x)]x'' - 3\xi_x(t, x)x'x'' - \xi(t, x)x'''.$$

$$\zeta^{x''_\tau}(t, x_\tau, x'_\tau, x''_\tau, x'''_\tau) = D(\zeta_2^\tau) = \eta_{tt}^\tau(t - \tau, x_\tau) + [2\eta_{tx}^\tau(t - \tau, x_\tau) - \xi_{tt}^\tau(t - \tau, x_\tau)]x'_\tau + [\eta_{xx}^\tau(t - \tau, x_\tau) - 2\xi_{tx}^\tau(t - \tau, x_\tau)](x''_\tau)^2 - \xi_{xx}^\tau(t - \tau, x_\tau)(x'''_\tau)^3 + [\eta_x^\tau(t - \tau, x_\tau) - 2\xi_t^\tau(t - \tau, x_\tau)]x''_\tau - 3\xi_x^\tau(t - \tau, x_\tau)x'_\tau x''_\tau - \xi^\tau(t - \tau, x_\tau)x'''_\tau.$$

D is the total derivative with respect to t . The determining equation of (10) is

$$\tilde{X}_\beta = \left(x'' - f(t, x, x_\tau, x', x'_\tau, x''_\tau) \right) \Big|_{(10)} = 0. \tag{11}$$

Note that (11) have to be satisfied for any solution of (10).

Let $x''' = f_t + x'f_x + x'_\tau f_{x_\tau} + x''f_{x'} + x''_\tau f_{x'_\tau}$, $x''_\tau = f'_t + x'f'_x + x'_\tau f'_{x_\tau} + x''f'_{x'} + x''_\tau f'_{x'_\tau}$, $x'' = f$ and $x''_\tau = f_\tau$, then (11) become

$$\begin{aligned} &\xi_x^\tau f_{f_\tau} (x'_\tau)^3 - \xi_{xx} (x')^3 + [\eta_{xx} - 2\xi_{tx} + \xi_x f_{x'}] (x')^2 + [\xi_x^\tau f_{x'_\tau} + (2\xi_{tx}^\tau - \eta_{xx}^\tau) f_{f_\tau}] (x'_\tau)^2 + \\ &[2\eta_{tx} - \xi_{tt} + (\xi_t - \eta_x) f_{x'} - 3\xi_x f + \xi^\tau f_x^\tau f_{f_\tau}] x' + [(\xi_t^\tau - \eta_x^\tau) f_{x'_\tau} + (\xi_{tt}^\tau - 2\eta_{tx}^\tau) f_{f_\tau} + 3\xi_x^\tau f f_{f_\tau} + \xi^\tau f_{x_\tau}^\tau f_{f_\tau}] x'_\tau \\ &+ \eta_{tt} + (\eta_x - 2\xi_t) f - \xi f_t - \eta f_x - \eta^\tau f_{x_\tau} - \eta_t f_{x'} - \eta_t^\tau f_{x'_\tau} - \xi f f_x + \xi f f_{x'} - \eta_{tt}^\tau f_{f_\tau} + (2\xi_t^\tau - \eta_x^\tau) f_\tau f_{f_\tau} + \\ &\xi^\tau f_t^\tau f_{f_\tau} + \xi^\tau f f_{x'_\tau}^\tau f_{f_\tau} + \xi^\tau f_\tau f_{x'_\tau}^\tau f_{f_\tau} + (\xi^\tau - \xi) f_{x_\tau} x'_\tau + (\xi^\tau - \xi) f_\tau f_{x'_\tau} = 0. \end{aligned} \tag{12}$$

Where $f_\tau = f(t - \tau, x_\tau, x_{2\tau}, x'_\tau, x'_{2\tau})$, $x_{2\tau} = x(t - 2\tau)$ and $x'_{2\tau} = x'(t - 2\tau)$. By virtue of the Cauchy problem (see [23]), one can account the variables $t, x, x_\tau, x', x'_\tau, x_{2\tau}$ and $x'_{2\tau}$ in (12) as arbitrary variables.

For the case $f_{x'_\tau} \neq 0$, we can split the determining equation (12) with respect to $x'_{2\tau}$. This implies that $\xi = \xi^\tau$.

If $f_{x'_\tau} = 0$, then the assumption of DDE implies f must depend on the delay terms, i.e. $f_{x_\tau} \neq 0$. Splitting (12) with respect to x'_τ , we also get $\xi = \xi^\tau$. This shows the periodic property of ξ .

Lemma (3.2) allows one to rewrite (11) in the form

$$\tilde{X} \left(x'' - f(t, x, x_\tau, x', x'_\tau, x''_\tau) \right) \Big|_{(10)} = 0.$$

Where

$$\tilde{X} = \tilde{X}_\beta + \xi D = \xi \frac{\partial}{\partial t} + \eta^x \frac{\partial}{\partial x} + \eta^{x_\tau} \frac{\partial}{\partial x_\tau} + \eta^{x'} \frac{\partial}{\partial x'} + \eta^{x'_\tau} \frac{\partial}{\partial x'_\tau} + \eta^{x''} \frac{\partial}{\partial x''} + \eta^{x''_\tau} \frac{\partial}{\partial x''_\tau}, \tag{13}$$

and

$$\begin{aligned} \eta^{x''_\tau} (t, x_\tau, x'_\tau, x''_\tau) &= \eta_{2\tau}^\tau (t, x_\tau, x'_\tau, x''_\tau) = \eta_{tt}^\tau (t - \tau, x_\tau) + [2\eta_{tx}^\tau (t - \tau, x_\tau) - \xi_{tt}^\tau (t - \tau, x_\tau)] x'_\tau \\ &+ [\eta_{xx}^\tau (t - \tau, x_\tau) - 2\xi_{tx}^\tau (t - \tau, x_\tau)] (x'_\tau)^2 - \xi_{xx}^\tau (t - \tau, x_\tau) (x'_\tau)^3 + [\eta_x^\tau (t - \tau, x_\tau) - \\ &2\xi_t^\tau (t - \tau, x_\tau)] x''_\tau - 3\xi_x^\tau (t - \tau, x_\tau) x'_\tau x''_\tau. \end{aligned} \tag{14}$$

The difference between the generator \tilde{X} and \tilde{X}_β is, the generator \tilde{X} acts in the space of variables $(t, x, x_\tau, x', x'_\tau, x'', x''_\tau)$, whereas the latter coefficients of the operator include the derivatives x''_τ and x''' .

Now, using the invariant manifold theorem, one obtains the generator \tilde{X} [?]. Consequently, the equation (11) readily implies that the manifold defined by (10) is an invariant manifold of \tilde{X} .

The general form of second order neutral delay differential equation is given by,

$$x''(t) + a(t)x''(t - \tau) + b(t)x'(t) + c(t)x'(t - \tau) + d(t)x(t) + e(t)x(t - \tau) = 0, \tag{15}$$

where a, b, d, e are arbitrary coefficients.

The general infinitesimal generator associated with (15) can be expressed as,

$$X = \xi(t, x) \frac{\partial}{\partial t} + \eta(t, x) \frac{\partial}{\partial x} + \eta^\tau(t - \tau, x_\tau) \frac{\partial}{\partial x_\tau}. \tag{16}$$

Using Equation (13) the second order extension yields,

$$\begin{aligned} X^{(2)} = X + \eta_1(t, x, x') \frac{\partial}{\partial x'} + \eta_2(t, x, x', x'') \frac{\partial}{\partial x''} + \eta_1^\tau(t - \tau, x_\tau, x'_\tau) \frac{\partial}{\partial x'_\tau} \\ + \eta_2^\tau(t - \tau, x_\tau, x'_\tau, x''_\tau) \frac{\partial}{\partial x''_\tau}. \end{aligned} \tag{17}$$

Substituting (17) to (15), one recovers the invariance conditions as,

$$\eta_2 + a\eta_2^\tau + b\eta_1 + c\eta_1^\tau + d\eta + e\eta^\tau = 0$$

Substituting (14), we get

$$\begin{aligned} \eta_{tt} + [2\eta_{tx} - \xi_{tt}]x' + [\eta_{xx} - 2\xi_{tx}](x')^2 - \xi_{xx}(x')^3 + [\eta_x - 2\xi_t]x'' - 3\xi_x x'x'' + a[\eta_{tt}^\tau + [2\eta_{tx}^\tau - \xi_{tt}^\tau]x'_\tau \\ + [\eta_{xx}^\tau - 2\xi_{tx}^\tau](x'_\tau)^2 - \xi_{xx}^\tau(x'_\tau)^3 + [\eta_x^\tau - 2\xi_t^\tau]x''_\tau - 3\xi_x^\tau x'_\tau x''_\tau] + b[\eta_t + [\eta_x - \xi_t]x' - \xi_x(x')^2] + c[\eta_t^\tau + \\ [\eta_x^\tau - \xi_t^\tau]x'_\tau - \xi_x^\tau(x'_\tau)^2] + d\eta + e\eta^\tau = 0. \end{aligned}$$

Equating the first and second order coefficients of the various monomials of x and x_τ , the following determining equations for the symmetry group of (15) are obtained.

Table 2: The determining equations for the symmetry group of (15).

| monomial | coefficient | number of equation |
|----------|---|--------------------|
| 1 | $\eta_{tt} + a\eta_{tt}^\tau + b\eta_t + c\eta_t^\tau + d\eta + e\eta^\tau = 0$ | a_1 |
| x' | $2\eta_{tx} - \xi_{tt} + b[\eta_x - \xi_t] = 0$ | a_2 |
| $(x')^2$ | $\eta_{xx} - 2\xi_{tx} - b\xi_x = 0$ | a_3 |
| $(x')^3$ | $\xi_{xx} = 0$ | a_4 |
| $x'x''$ | $\xi_x = 0$ | a_5 |

| monomial | coefficient | number of equation |
|--------------------|--|--------------------|
| x'' | $\eta_x - 2\xi_t = 0$ | a_6 |
| x'_τ | $a[2\eta_{tx}^\tau - \xi_{tt}^\tau] + c[\eta_x^\tau - \xi_t^\tau] = 0$ | a_7 |
| $(x'_\tau)^2$ | $a[\eta_{xx}^\tau - 2\xi_{tx}^\tau] - c\xi_x^\tau = 0$ | a_8 |
| $(x')^3$ | $a\xi_{xx}^\tau = 0$ | a_9 |
| x''_τ | $a[\eta_x^\tau - 2\xi_t^\tau] = 0$ | a_{10} |
| $x'_\tau x''_\tau$ | $a\xi_x^\tau = 0$ | a_{11} |

In (a_5) , ξ does not depend on x . In (a_3) , η is linearly vary with x , following $\eta = g(t)x + h(t)$, where $g(t)$ and $h(t)$ are arbitrary functions. In (a_6) , $\xi_t = \frac{1}{2}g$.

In (a_{11}) , ξ^τ is independent of x , In lemma (3.2), $\xi = \xi^\tau$ then $\xi_t = \xi_t^\tau$, which implies that $\xi_t^\tau = \frac{1}{2}g(t)$. In (a_{10}) , $\eta_x^\tau = 2\xi_t^\tau$, and $\eta_x^\tau = g(t)$, thus $\eta^\tau = g(t)x + k(t - \tau)$, with $k(t - \tau)$ is arbitrary function.

Case 1 If a, b, c, d and e are arbitrary constants not equal to zero. Then using (a_1) one gets $g_{tt}x + h_{tt} + ag_{tt}x + ak_{tt} + bg_t x + bh_t + cg_t x + ck_t + dgx + dh + egx + ek = 0$. Equating the coefficients of the various monomials one obtains,

$$(1 + a)g_{tt} + (b + c)g_t + (d + e)g = 0. \tag{18}$$

$$h_{tt} + ak_{tt} + bh_t + ck_t + dh + ek = 0.$$

This implies that $g(t), h(t)$ and $k(t - \tau)$ are the solutions of (15). From (18), and since $g = 2\xi_t$, then

$$\xi_t = -\frac{1 + a}{d + e}\xi_{ttt} - \frac{b + c}{d + e}\xi_{tt},$$

then

$$\xi = -\frac{1 + a}{d + e}\xi_{tt} - \frac{b + c}{d + e}\xi_t + c_1.$$

Let $1 + a = A, b + c = C$ and $d + e = E$, then,

$$\xi = c_1 - \frac{A}{2E}g_t - \frac{C}{2E}.$$

Now,

$$g = c_2 - \frac{A}{C}g_t - \frac{2E}{C}\xi.$$

From reduce to $\eta = (c_2 - \frac{A}{C}g_t - \frac{2E}{C}\xi)x + h$ and $\eta^\tau = (c_2 - \frac{A}{C}g_t - \frac{2E}{C}\xi)x + k$, where $c_i, i = 1, 2$ are arbitrary constants.

Recalling the general infinitesimal generator for (15) we get

$$X = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x} + \eta^\tau \frac{\partial}{\partial x_\tau}.$$

Let $x_\tau = u$, then

$$X = [c_1 - \frac{A}{2E}g_t - \frac{C}{2E}] \frac{\partial}{\partial t} + [(c_2 - \frac{A}{C}g_t - \frac{2E}{C}\xi)x + h] \frac{\partial}{\partial x} + [(c_2 - \frac{A}{C}g_t - \frac{2E}{C}\xi)x + k] \frac{\partial}{\partial u}.$$

The Lie algebra of (15) is spanned by the following three infinitesimal generator corresponding to each parameters and is given by $c_i, i = 1, 2$

$$Q_1 = \frac{\partial}{\partial t}, \quad Q_2 = x \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u} \right),$$

$$Q_3 = - \left(\frac{A}{2E}g_t + \frac{C}{2E}g \right) \frac{\partial}{\partial t} + \left[h(t) - x \left(\frac{A}{C}g_t + \frac{2E}{C}\xi \right) \right] \frac{\partial}{\partial x} + \left[k(t) - x \left(\frac{A}{C}g_t + \frac{2E}{C}\xi \right) \right] \frac{\partial}{\partial u},$$

Where Q_3 is an infinite dimensional Lie subalgebra, $L_2 = \{Q_1, Q_2\}$ is a solvable Lie algebra of (15).

Remark 3.3 *The same result is obtained for d , or e equal to zero.*

Case 2 For a, b, c are arbitrary constants and $d = e = 0$, the Lie algebra of (15) is spanned by

$$Q_1 = tx \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u} \right), \quad Q_2 = x \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u} \right), \quad Q_3 = t \frac{\partial}{\partial t}, \quad Q_4 = \frac{\partial}{\partial t}, \quad Q_5 = \frac{\partial}{\partial u},$$

with an infinite dimensional Lie subalgebra gives,

$$Q_6 = \left(\frac{-A}{2C}g \right) \frac{\partial}{\partial t} + [h - \frac{2C}{A}\xi x] \frac{\partial}{\partial x} - \left[\frac{1}{c}h_t + \frac{a}{c}k_t + \frac{b}{c}h + \frac{2C}{A}\xi x \right] \frac{\partial}{\partial u}.$$

The commutator table can be written as,

| | | | | | |
|-------|-------|-------|--------|--------|-------|
| | Q_1 | Q_2 | Q_3 | Q_4 | Q_5 |
| Q_1 | 0 | 0 | $-Q_1$ | $-Q_2$ | 0 |
| Q_2 | 0 | 0 | 0 | 0 | 0 |
| Q_3 | Q_1 | 0 | 0 | $-Q_4$ | 0 |
| Q_4 | Q_2 | 0 | Q_4 | 0 | 0 |
| Q_5 | 0 | 0 | 0 | 0 | 0 |

Now $L_4 = \{Q_1, Q_2, Q_3, Q_4, Q_5\}$ is a solvable Lie algebra of Equation (15).

Remark 3.4 If a, c are arbitrary constant not equal to one, and $b = d = e = 0$, we get the same result of Case 2.

The infinite dimensional Lie subalgebra is casted as,

$$Q = -\frac{-A}{2c}g\frac{\partial}{\partial t} + (h - \frac{2c}{A}\xi x)\frac{\partial}{\partial x} - (\frac{1}{c}h_t + \frac{a}{c}k_t + x\xi)\frac{\partial}{\partial u}.$$

Case 3 If $a = b = c = d = 1$, and $e = 0$ or $a = b = c = e = 1$, and $d = 0$, the solvable Lie algebra of (15) is spanned by

$$Q_1 = \frac{\partial}{\partial t} + x(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}),$$

$$Q_2 = -(g_t + g)\frac{\partial}{\partial t} + [h - (g_t + \xi)x]\frac{\partial}{\partial x} + [k - (g_t + \xi)x]\frac{\partial}{\partial u}.$$

Where Q_2 is an infinite dimensional Lie subalgebra.

Case 4 If $a = c = 1$, and $b = d = e = 0$, the Lie algebra of (15) is spanned by

$$Q_1 = t\frac{\partial}{\partial t} + xt(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}), \quad Q_2 = \frac{\partial}{\partial t} + x(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}), \quad Q_3 = \frac{\partial}{\partial u},$$

$$Q_4 = -g\frac{\partial}{\partial t} + (h - \xi x)\frac{\partial}{\partial x} - (h_t + k_t + \xi x)\frac{\partial}{\partial u}.$$

Where Q_4 is an infinite dimensional Lie subalgebra. The commutator table is represented as,

| | | | |
|-------|-------|--------|-------|
| | Q_1 | Q_2 | Q_3 |
| Q_1 | 0 | $-Q_2$ | 0 |
| Q_2 | Q_2 | 0 | 0 |
| Q_3 | 0 | 0 | 0 |

Then $L_2 = \{Q_1, Q_2, Q_3\}$ is a solvable Lie algebra of (15).

Based on the above results we give the following algorithm

3.2 Algorithm

The procedure of classifying second order linear delay differential equations with constant coefficients to solvable Lie algebra is summarized as follows:

1. Write the delay differential equation in the solved form.
2. Select the general infinitesimal generator of DDEs.
3. Extend the infinitesimal generator acting on second order.
4. Apply the extended infinitesimal generator to the given delay differential equation to obtain invariance condition.
5. Substitute Equations (5) or (14) in the invariance condition conform to an infinitesimal symmetry.
6. Split up invariance conditions of the coefficients of the various monomials in the first and second orders of x and x_τ for determining equations of the infinitesimal symmetry group.
7. Solve the determining equations for the infinitesimals ξ , η , and η^τ .
8. Substitute the infinitesimals ξ , η , and η^τ in the generator.
9. Thus the Lie algebra of the given equation is spanned by the three infinitesimals generators corresponding to each parameter c_i , $i = 1, 2, \dots, n$ with c_i as arbitrary constant.
10. Compute the commutator table on the basis of Lie algebra.
11. Satisfy the inclusion property to achieve solvable Lie algebra.

Table 3: Classification of second order linear delay differential equations with constant coefficients to solvable Lie algebra.

| Second order LDDEs with constant coefficients | Basis of Lie algebra | Solvable Lie algebra |
|---|---|----------------------|
| $x'' + ax' + bx'_\tau + cx + dx_\tau = 0$ or if $c = 0$ or $d = 0$ | $Q_1 = \frac{\partial}{\partial t}, Q_2 = x \frac{\partial}{\partial x}, Q_3 = x \frac{\partial}{\partial u},$ with Lie subalgebra $Q_4 = -(\frac{1}{2D}g_t + \frac{B}{2D})\frac{\partial}{\partial t} + (h(t) - x(\frac{1}{B}g_t + \frac{D}{B}\xi))\frac{\partial}{\partial x} + (k - x(\frac{1}{2B}g_t + \frac{D}{2B}\xi))\frac{\partial}{\partial u}$ | L_3^1 |
| $x'' + x'_\tau = 0$ | $Q_1 = t \frac{\partial}{\partial t} + xt(\frac{\partial}{\partial x} \frac{1}{2} \frac{\partial}{\partial u}), Q_2 = \frac{\partial}{\partial t} + x(\frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial}{\partial u}), Q_3 = \frac{\partial}{\partial u},$ with Lie subalgebra $Q_4 = -g \frac{\partial}{\partial t} + (h - x\xi) \frac{\partial}{\partial x} - (h_t + \frac{1}{2}x\xi) \frac{\partial}{\partial u}$ | L_3^2 |
| $x'' + ax''_\tau + bx' + cx'_\tau = 0$ Or $x'' + ax''_\tau + cx'_\tau = 0$ | $Q_1 = tx(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}), Q_2 = x(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}), Q_3 = t \frac{\partial}{\partial t}, Q_4 = \frac{\partial}{\partial t}, Q_5 = \frac{\partial}{\partial u},$ with Lie subalgebra $Q_6 = -\frac{A}{2C}g \frac{\partial}{\partial t} + [h - \frac{2C}{A}x\xi] \frac{\partial}{\partial x} - [\frac{1}{c}h_t + \frac{a}{c}k_t + \frac{b}{c}h + \frac{2C}{A}x\xi] \frac{\partial}{\partial u}$ | L_5^3 |

| Second order LDDEs with constant coefficients | Basis of Lie algebra | Solvable Lie algebra |
|---|---|----------------------|
| $x'' + bx'_\tau = 0$ | $Q_1 = t\frac{\partial}{\partial t}$, $Q_2 = \frac{\partial}{\partial t}$, $Q_3 = tx(\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u})$, $Q_4 = x(\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u})$, $Q_5 = \frac{\partial}{\partial u}$, with Lie subalgebra $Q_6 = -\frac{1}{b}g\frac{\partial}{\partial t} + [h - bx\xi]\frac{\partial}{\partial x} - [\frac{1}{b}h_t - \frac{b}{2}x\xi]\frac{\partial}{\partial u}$ | L_5^4 |
| $x'' + ax''_\tau + bx' + cx'_\tau + dx + ex_\tau = 0$ or if $d = 0$ or $e = 0$ | $Q_1 = \frac{\partial}{\partial t}$, $Q_2 = x(\frac{\partial}{\partial x} + \frac{\partial}{\partial u})$ with Lie subalgebra $Q_3 = -(\frac{A}{2E}g_t + \frac{C}{2E}g)\frac{\partial}{\partial t} + [h - (\frac{A}{C}g_t + \frac{2E}{C}\xi)x]\frac{\partial}{\partial x} + [k - (\frac{A}{C}g_t + \frac{2E}{C}\xi)x]\frac{\partial}{\partial u}$ | L_2^5 |
| $x'' + x''_\tau + x' = 0$ | $Q_1 = t\frac{\partial}{\partial t} + xt(\frac{\partial}{\partial x} + \frac{\partial}{\partial u})$, $Q_2 = \frac{\partial}{\partial t} + x(\frac{\partial}{\partial x} + \frac{\partial}{\partial u})$, $Q_3 = \frac{\partial}{\partial u}$, with Lie subalgebra $Q_4 = -g\frac{\partial}{\partial t} + (h - \xi x)\frac{\partial}{\partial x} - (h_t + k_t + \xi x)\frac{\partial}{\partial u}$ | L_2^6 |
| $x'' + x''_\tau + x' + x'_\tau + x = 0$ $x'' + x''_\tau + x' + x'_\tau + x_\tau = 0$ | $Q_1 = \frac{\partial}{\partial t} + x(\frac{\partial}{\partial x} + \frac{\partial}{\partial u})$, with Lie subalgebra $Q_2 = -(g_t + g)\frac{\partial}{\partial t} + [h - (g_t + \xi)x]\frac{\partial}{\partial x} + [k - (g_t + \xi)x]\frac{\partial}{\partial u}$ | L_1^7 |

Here $g = g(t)$, $h = h(t)$, and $k = k(t - \tau)$ are arbitrary solutions of the given equation, $A = 1 + a$, $B = a + \frac{b}{2}$, $C = b + c$, $D = c + \frac{d}{2}$, $E = d + e$. And $x = x(t)$, $x_\tau = x(t - \tau)$, $x' = x'(t)$, $x'_\tau = x'(t - \tau)$, $x'' = x''(t)$ and $x''_\tau = x''(t - \tau)$. In L_m^n , m is dimension of solvable Lie algebra and n is the

number in the table.

4 Conclusion

We developed a new approach for classifying second order linear DDEs with constant coefficients of the form

$$x'' = f(t, x, x_\tau, x', x'_\tau),$$

to solvable Lie algebra. The method is further productively applied to second order neutral DDEs of the form

$$x'' = f(t, x, x_\tau, x', x'_\tau, x''_\tau).$$

Interestingly, the change in the space variables are not considered because the DDEs do not acquire an equivalent transformation related to this change. Furthermore, the general infinitesimal generator of DDEs is used to determine the equations for the same symmetry group. The equations are successfully solved to span the solvable Lie algebras by satisfying the inclusion property. This unique classification scheme is validated using the emerging neutral DDEs which is due to the failure of retarded DDEs in explaining problems related to natural phenomena such as motion of radiating electrons, population growth, the spread of epidemics and in networks containing lossless transmission lines. The flexibility of the proposed method suggests that it may further be generalized to classify second order linear DDE with functional coefficients, as well as non-linear DDEs.

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