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Comparison of Differences between Power Means[1]

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Abstract

We show that the differences of power means associated to distinct sequences of weights are comparable, with constants that depend on the smallest and largest quotients of the weights. The obtained results are then utilized to generalize the operator arithmetic-geometric-harmonic mean inequalities.

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1 Introduction

The power means are defined by

$$M_t(\mathbf{x},\alpha) = \left(\sum_{i=1}^{n} \alpha_i x_i^t\right)^{1/t},\ t \neq 0$$

and

$$M_0(\mathbf{x},\alpha) = \prod_{i=1}^{n} x_i^{\alpha_i},$$

where α_i, x_i are positive numbers with $\sum\limits_{i=1}^{n} \alpha_i = 1$.

It follows from Jensen inequality that if $s \leq t$, then $M_s(\mathbf{x},\alpha) \leq M_t(\mathbf{x},\alpha)$. In particular, we have the general arithmetic-geometric inequality

$$\prod_{i=1}^{n} x_i^{\alpha_i} \leq \sum_{i=1}^{n} \alpha_i x_i.$$

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Recently, Aldaz [1] obtained the self-bounds of differences between arithmetic and geometric means in the following manner:

$$\min_{k=1,\cdots,n}\{\frac{\alpha_k}{\beta_k}\}(\sum_{i=1}^{n}\beta_i x_i-\prod_{i=1}^{n}x_i^{\beta_i})\leq\sum_{i=1}^{n}\alpha_i x_i-\prod_{i=1}^{n}x_i^{\alpha_i}$$
$$\leq\max_{k=1,\cdots,n}\{\frac{\alpha_k}{\beta_k}\}(\sum_{i=1}^{n}\beta_i x_i-\prod_{i=1}^{n}x_i^{\beta_i}),$$

where $n\geq 2$, $x_i\geq 0(i=1,\dots,n)$, and $\alpha_i,\beta_i>0$ satisfying $\sum_{i=1}^{n}\alpha_i=\sum_{i=1}^{n}\beta_i=1$. See [2, 3] for more recent developments of the improved arithmetic and geometric means inequality. See also [6] for a further generalization concerning the precision in Jensen-Steffensen inequality.

In this paper we consider the comparison of differences between power means associated to different sequences of weights and its application to operator means inequalities.

2 Comparison of Differences between Power Means

In the following, we need the Jensen-Mercer inequality[5].

Lemma 2.1. *Let $[a,b]$ be an interval in $\mathbb{R}$, $x_1,\cdots,x_n\in[a,b]$ and $\alpha_1,\cdots,\alpha_n$ positive real numbers such that $\sum_{i=1}^{n}\alpha_i=1$. If $f:[a,b]\to R$ is convex on $[a,b]$, then*

$$f(a+b-\sum_{i=1}^{n}\alpha_i x_i)\leq f(a)+f(b)-\sum_{i=1}^{n}\alpha_i f(x_i).$$

Also, a natural generalizations of Jensen-Mercer inequality to convex functions defined on $\mathbb{R}^k$ was obtained in [4].

Now, we present precise self-bounds on differences between power means.

Theorem 2.2. *If $\alpha_i,\beta_i>0$ such that $\sum_{i=1}^{n}\alpha_i=\sum_{i=1}^{n}\beta_i=1$ and $x_i\geq 0(i=1,2\cdots,n)$, then, for $n\geq 2$, $s\leq t$ and $0<t\leq 1$, we have*

$$\min_{k=1,\cdots,n}\left\{\frac{\alpha_k}{\beta_k}\right\}^{\frac{1}{t}}(M_t(\mathbf{x},\beta)-M_s(\mathbf{x},\beta))\leq M_t(\mathbf{x},\alpha)-M_s(\mathbf{x},\alpha)$$
$$\leq\max_{k=1,\cdots,n}\left\{\frac{\alpha_k}{\beta_k}\right\}^{\frac{1}{t}}(M_t(\mathbf{x},\beta)-M_s(\mathbf{x},\beta)).$$

Proof. The second inequality is equivalent to

$$(\sum_{i=1}^{n}\beta_i x_i^s)^{\frac{1}{s}} \le (\sum_{i=1}^{n}\beta_i x_i^t)^{\frac{1}{t}} - \min_{k=1,\cdots,n}\left\{\frac{\beta_k}{\alpha_k}\right\}^{\frac{1}{t}}(\sum_{i=1}^{n}\alpha_i x_i^t)^{\frac{1}{t}}$$
$$+ \min_{k=1,\cdots,n}\left\{\frac{\beta_k}{\alpha_k}\right\}^{\frac{1}{t}}(\sum_{i=1}^{n}\alpha_i x_i^s)^{\frac{1}{s}}.$$

First of all, we put

$$A=\sum_{i=1}^{n}\beta_i x_i^t,\ B=\min_{k=1,\cdots,n}\left\{\frac{\beta_k}{\alpha_k}\right\}(\sum_{i=1}^{n}\alpha_i x_i^s)^{\frac{t}{s}},\ C=\min_{k=1,\cdots,n}\left\{\frac{\beta_k}{\alpha_k}\right\}\sum_{i=1}^{n}\alpha_i x_i^t.$$

Since $s \le t$ and $0 < t \le 1$, Jensen inequality implies that

$$(\sum_{i=1}^{n}\alpha_i x_i^s)^{\frac{t}{s}} \le \sum_{i=1}^{n}\alpha_i (x_i^s)^{\frac{t}{s}} = \sum_{i=1}^{n}\alpha_i x_i^t,$$

so that

$$B \le C \le A.$$

Then, by Jensen-Mercer inequality, for $t \in (0,1]$, we have

$$(A+B-C)^{\frac{1}{t}} \le A^{\frac{1}{t}} + B^{\frac{1}{t}} - C^{\frac{1}{t}}.$$

i.e.

$$\Big(\textstyle\sum_{i=1}^{n}(\beta_i -$$

$$\le \Big(\textstyle\sum_{i=1}^{n}\beta_i x_i^t\Big)^{\frac{1}{t}} - \text{mi}$$

On the other hand, Jensen's inequality implies that

$$\left(\sum_{i=1}^{n}\beta_i x_i^s\right)^{\frac{1}{s}} = \left(\left(\sum_{i=1}^{n}(\beta_i - \min_{k=1,\cdots,n}\left\{\frac{\beta_k}{\alpha_k}\right\}\alpha_i)x_i^s + \min_{k=1,\cdots,n}\left\{\frac{\beta_k}{\alpha_k}\right\}\sum_{i=1}^{n}\alpha_i x_i^s\right)^{\frac{t}{s}}\right)^{\frac{1}{t}}$$
$$\le\left(\sum_{i=1}^{n}(\beta_i - \min_{k=1,\cdots,n}\{\frac{\beta_k}{\alpha_k}\}\alpha_i)(x_i^s)^{\frac{t}{s}} + \min_{k=1,\cdots,n}\{\frac{\beta_k}{\alpha_k}\}(\sum_{i=1}^{n}\alpha_i x_i^s)^{\frac{t}{s}}\right)^{\frac{1}{t}}$$
$$=\left(\sum_{i=1}^{n}(\beta_i - \min_{k=1,\cdots,n}\left\{\frac{\beta_k}{\alpha_k}\right\}\alpha_i)x_i^t + \min_{k=1,\cdots,n}\left\{\frac{\beta_k}{\alpha_k}\right\}(\sum_{i=1}^{n}\alpha_i x_i^s)^{\frac{t}{s}}\right)^{\frac{1}{t}}.$$

Hence the second inequality is proved.

To obtain the first inequality, we multiply both sides of the second inequality by $\min\limits_{k=1,\cdots,n}\{\frac{\beta_k}{\alpha_k}\}^{\frac{1}{t}}$, and note this is just the first inequality with the roles of the $\alpha's$ and the $\beta's$ interchanged.

We use theorem 2.2 as the following form since $\lim_{s\to 0} M_s(\mathbf{x},\alpha) = M_0(\mathbf{x},\alpha)$.

Corollary 2.3. *If $\alpha_i, \beta_i > 0$ such that $\sum\limits_{i=1}^{n}\alpha_i = \sum\limits_{i=1}^{n}\beta_i = 1$ and $x_i \geq 0 (i = 1, 2\cdots, n)$, then, for $n \geq 2$ and $t \in (0,1]$, we have*

$$\begin{aligned}\min_{k=1,\cdots,n}\left\{\frac{\alpha_k}{\beta_k}\right\}^{\frac{1}{t}}(M_t(\mathbf{x},\beta) - M_0(\mathbf{x},\beta)) &\leq M_t(\mathbf{x},\alpha) - M_0(\mathbf{x},\alpha)\\ &\leq \max_{k=1,\cdots,n}\left\{\frac{\alpha_k}{\beta_k}\right\}^{\frac{1}{t}}(M_t(\mathbf{x},\beta) - M_0(\mathbf{x},\beta)).\end{aligned}$$

Furthermore, the case $t = 1$ in the above formula is simplified to the following one, which was shown by Aldaz, see [1, Theorem 2.1].

Corollary 2.4. *If $\alpha_i, \beta_i > 0$ such that $\sum\limits_{i=1}^{n}\alpha_i = \sum\limits_{i=1}^{n}\beta_i = 1$ and $x_i \geq 0 (i = 1, 2\cdots, n)$, then, for $n \geq 2$, we have*

$$\begin{aligned}\min_{k=1,\cdots,n}\left\{\frac{\alpha_k}{\beta_k}\right\}(\sum_{i=1}^{n}\beta_i x_i - \prod_{i=1}^{n}x_i^{\beta_i}) &\leq \sum_{i=1}^{n}\alpha_i x_i - \prod_{i=1}^{n}x_i^{\alpha_i}\\ &\leq \max_{k=1,\cdots,n}\left\{\frac{\alpha_k}{\beta_k}\right\}(\sum_{i=1}^{n}\beta_i x_i - \prod_{i=1}^{n}x_i^{\beta_i}).\end{aligned}$$

3 Applications to Operator A-G-H Means Inequalities

In this section, we use the following notations: $A\nabla_\nu B = (1-\nu)A + \nu B$, $A\sharp_\nu B = A^{1/2}(A^{-1/2}BA^{-1/2})^\nu B^{1/2}$, where A, B are positive operators on a Hilbert space.

Next, utilizing our preceding results we also consider the comparison of differences between operator arithmetic means and operator geometric means associated to distinct sequences of weights, where the operator arithmetic and geometric means inequality is specialized to just two terms.

Theorem 3.1. *Let A, B be two positive operators, then, for $\mu, \nu \in (0,1)$*

$$\begin{aligned}\min\{\frac{1-\nu}{1-\mu}, \frac{\nu}{\mu}\}(A\nabla_\mu B - A\sharp_\mu B) &\leq A\nabla_\nu B - A\sharp_\nu B\\ &\leq \max\{\frac{1-\nu}{1-\mu}, \frac{\nu}{\mu}\}(A\nabla_\mu B - A\sharp_\mu B).\end{aligned}$$

Proof. By Corollary 2.4, the following inequalities

$$\min\{\frac{1-\nu}{1-\mu},\frac{\nu}{\mu}\}[(1-\mu)a+\mu b-a^{1-\mu}b^{\mu}] \le (1-\nu)a+\nu b-a^{1-\nu}b^{\nu}$$
$$\le \max\{\frac{1-\nu}{1-\mu},\frac{\nu}{\mu}\}[(1-\mu)a+\mu b-a^{1-\mu}b^{\mu}]$$

hold for $a \ge 0$, $b > 0$. Let $t = a/b$, then

$$\min\{\frac{1-\nu}{1-\mu},\frac{\nu}{\mu}\}[(1-\mu)t+\mu-t^{1-\mu}] \le (1-\nu)t+\nu-t^{1-\nu}$$
$$\le \max\{\frac{1-\nu}{1-\mu},\frac{\nu}{\mu}\}[(1-\mu)t+\mu-t^{1-\mu}].$$

Hence, for the positive operator $T \ge 0$, we have

$$\min\{\frac{1-\nu}{1-\mu},\frac{\nu}{\mu}\}[(1-\mu)T+\mu-T^{1-\mu}] \le (1-\nu)T+\nu-T^{1-\nu}$$
$$\le \max\{\frac{1-\nu}{1-\mu},\frac{\nu}{\mu}\}[(1-\mu)T+\mu-T^{1-\mu}]$$

by standard operational calculus. Putting $T = B^{-1/2}AB^{-1/2}$ and then multiplying $B^{1/2}$ from the both sides, we obtain the desired operator inequalities.

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