Int. Journal of Math. Analysis, Vol. 7, 2013, no. 10, 453 - 460

On Reproducing Kernel Riesz Bases in Model Spaces

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Abstract. Reproducing kernel Riesz bases associated to real points in model subspaces $K_I^2 = H^2 \ominus IH^2$ of the Hardy space H^2 have been described in terms of equality of spaces by A. Baranov [1]. We now answer the question of whether the natural analogue of his result with the real points being replaced by sequences of points located in the whole complex plane holds. It turns out that the analogous conditions are indeed sufficient but not in general necessary. A consequence of this is that larger perturbations of such bases are admissible with sequences of associated points from the upper or lower half-plane than along the real line.

Keywords: Model spaces, Reproducing kernels Riesz bases, Equality of spaces

1. Introduction

The study of systems of reproducing kernel Riesz bases in model subspaces has a long history, beginning with a classical result of Paley and Wiener [10] on nonharmonic fourier series. Since then the problem to describe such bases has been a subject of high interest and remains open despite several partial answers; for instance in [1, 2, 3, 6, 7].

The approach we intend to follow here begins with a problem of R. Duffin and A. Schaeffer [5] in describing real sequences which generate Fourier frames in $L^2(-\pi, \pi)$. Fifty years later, their problem was solved by J. Ortega–Cerdà and K. Seip [8] by equivalently describing the sampling sequences in the Paley–Wiener space

$$PW_\pi = \Big\{ f, \text{ entire} : f(x) = \int_\pi^\pi g(t)e^{itx}, \; g \in L^2(-\pi, \pi) \Big\}$$

in terms of equality of two spaces.

By equality of two spaces we mean equality as a set equipped with equivalent norms. We write $\mathcal{H}_1 = \mathcal{H}_2$ whenever spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ satisfy such a relation. Following the approach in [8], A. Baranov [1] was able to prove more general results in K_I^2 (see below) when I belongs to the class of meromorphic inner functions.

We say that an entire function E belongs to the Hermite–Biehler (HB) class if it has no real zeros and satisfies $|E(z)| > |E(\overline{z})|$ for each $z \in \mathbb{C}_+$. Each such E generates a reproducing kernel Hilbert space $H(E)$ consisting of all entire functions f for which both f/E and f^*/E belong to H^2 where $f^*(z) = \overline{f(\overline{z})}$, and with norm

$$\|f\|_{H(E)}^2 = \int_{\mathbb{R}} |f(t)|^2 |E(t)|^{-2} dt.$$

Each meromorphic inner function I admits the representation $I = E^*/E$ for some E in HB class. Such an E is unique up to an entire function factor which is zero free on both the upper and the lower half-planes and real valued on the real line.

We will need the fact that the reproducing kernel for K_I^2 at some point $\lambda \in \mathbb{C}_+$ is

$$k_\lambda(z) = \frac{i}{2\pi} \frac{1 - \overline{I(\lambda)} I(z)}{z - \overline{\lambda}}.$$

This formula extends to points on the real line when I is a meromorphic inner function, and hence every member of K_I^2 admits analytic continuation across the real axis.

Theorem 1.1 (Baranov, [1])**.** *Let E be an HB class function, $I = E^*/E$ and (t_n) be a sequence of real points for which $\big(\|k_{t_n}\|_{K_I^2}^{-1} k_{t_n}\big)$ constitutes a frame for K_I^2. Then there exist entire functions E_1 in HB class and E_2 either in HB class or a constant such that*

(i) $H(E) = H(E_1)$,
(ii) *the sequence (t_n) constitutes a zero set for the function $E_1E_2 - E_1^*E_2^*$ and*
(iii) $1 - I_1 I_2 \notin L^2(\mathbb{R})$ *with $I_1 = E_1^*/E_1$ and $I_2 = E_2^*/E_2$.*

These conditions are really about the lower frame inequality, and require the Bessel property in order to be sufficient. In particular when E_2 is a constant, the next stronger result holds.

Theorem 1.2 (Baranov, [1])**.** *Let E be an HB class function, $I = E^*/E$ and (t_n) be a sequence of real points. Then $\left(\|k_{t_n}\|^{-1}_{K^2_I} k_{t_n}\right)$ is a Riesz basis in K^2_I if and only if there exists an HB class function E_1 such that*

(i) *$H(E) = H(E_1)$ and*

(ii) *the sequence (t_n) is the zero set of the function $I_1 - 1$ and $I_1 - 1 \notin L^2(\mathbb{R})$ where $I_1 = E^*_1/E_1$.*

From the two results, we observe that the overcompleteness of a frame system comes from the existence of a second entire function E_2 in HB class. A natural question has been that of whether the analog of Theorem 1.2 holds when we associate the kernel functions with sequences of points located in the whole complex plane $\mathbb{C}$. We now answer the question negatively, namely that the analogues conditions are indeed sufficient but not in general necessary. Since all the arguments are more or less similar whether the sequence of points are from $\mathbb{C}_+, \mathbb{C}_-, \mathbb{C} \setminus \mathbb{R}, \mathbb{C}_+ \cup \mathbb{R}$ or $\mathbb{C}_- \cup \mathbb{R}$, we will restrict ourselves to the upper half plane $\mathbb{C}_+$ and first prove the following positive part.

Theorem 1.3. *Let E be an HB class function, $I = E^*/E$ and $(\lambda_n) \subset \mathbb{C}_+$. Then $\left(\|k_{\lambda_n}\|^{-1}_{K^2_I} k_{\lambda_n}\right)$ is a Riesz basis in K^2_I if there exists an HB class function E_1 with an associated interpolating Blaschke product B such that*

(i) *$H(E) = H(E_1)$ and*

(ii) *the sequence (λ_n) constitutes the zero set of $B = E^*_1/E_1$.*

We may first remark that by Theorem 1 in [7], and since the Carleson condition implies the Blaschke condition, (ii) is always necessary in all model spaces. Thus the requirement B to be an interpolating Blaschke product in the theorem is not really a pre imposed restriction. On the contrary, condition (i) fails to be a necessity in general. We will construct counterexamples for this in the next section. The phenomena with (i) seems rather more natural since the requirement equality of spaces is so strong to be a necessity. As evidence of fact, the condition remains a necessity even for the weaker frame property whenever the associated points are from the real line. An important consequence of Theorems 1.2 and 1.3 is that larger perturbations of reproducing kernel Riesz bases are admissible with sequences of associated points from the complex plane than along the real line.

Proof of Theorem 1.3. We will extend the arguments used by Baranov [1]. Instead of the de Branges orthonormal basis, we will argue this time using Riesz basis of normalized reproducing kernels associated with sequence of points from the upper half-plane. Denote by K_z, k_z, K^1_z and k^1_z the reproducing kernels of the spaces $H(E)$, K^2_I, $H(E_1)$ and K^2_B respectively at the point z. If (λ_n) is the zero set of an interpolating Blaschke product B, then by a result of Shapiro and Shields [9], the system $(k^1_{\lambda_n}/\|k^1_{\lambda_n}\|_{K^2_B})$ constitutes a Riesz basis in

K_B^2. In view of the unitary isomorphism $f \mapsto E_1 f$ from K_B^2 onto $H(E_1)$, which in particular maps reproducing kernels onto reproducing kernels, the Shapiro – Shields result holds if and only if

$$\left\{ \frac{E_1 k_{\lambda_n}^1}{\|k_{\lambda_n}^1\|_{K_B^2}} \right\} = \left\{ \frac{K_{\lambda_n}^1}{\|k_{\lambda_n}^1\|_{K_B^2} \overline{E_1(\lambda_n)}} \right\} = \left\{ \frac{K_{\lambda_n}^1}{\|K_{\lambda_n}^1\|_{H(E_1)}} \right\}$$

constitutes a Riesz basis in $H(E_1)$. Observe that the equalities above are due to

$$K_z^1(w) = E_1(w)\overline{E_1(z)} \frac{i}{2\pi} \left(\frac{1 - \overline{I_1(z)} I_1(w)}{w - \overline{z}} \right) = \overline{E_1(z)} E_1(w) k_z^1(w) \tag{1.1}$$

for points $z, \ w \in \mathbb{C}_+$. Equivalently, it means that the interpolation problem $f(\lambda_n) = a_n$ has a unique solution f in $H(E_1)$ whenever the admissibility condition

$$\sum_{n=1}^{\infty} |a_n|^2 \|K_{\lambda_n}^1\|_{H(E_1)}^{-2} < \infty$$

holds. By duality and the hypothesis, we have that

$$\begin{aligned} \|K_z^1\|_{H(E_1)} &= \sup_{\substack{g \in H(E_1) \\ \|g\|_{H(E_1)} = 1}} |\langle g, K_z^1 \rangle|_{H(E_1)} = \sup_{\substack{g \in H(E_1) \\ \|g\|_{H(E_1)} = 1}} |g(z)| \\ &\simeq \sup_{\substack{g \in H(E) \\ \|g\|_{H(E)} = 1}} |g(z)| = \|K_z\|_{H(E)} \end{aligned}$$

for each point z in $\mathbb{C}_+$ and in particular for the $\lambda_n' s$[1]. It follows that for each sequence c_n satisfying

$$\sum_{n=1}^{\infty} |c_n|^2 \|K_{\lambda_n}\|_{H(E)}^{-2} \simeq \sum_{n=1}^{\infty} |c_n|^2 \|K_{\lambda_n}^1\|_{H(E_1)}^{-2} < \infty,$$

there exists a unique function f in $H(E) = H(E_1)$ such that $f(\lambda_n) = c_n$. This proves that (λ_n) is a complete interpolating sequence for $H(E)$ and so is for K_I^2.

1.1. **Necessity fails.** We now construct counterexamples showing that the natural analog of Theorem 1.2 fails to hold in general. We will first exhibit an example in the class of model spaces generated by infinitely many component inner functions. We may note that each E in HB class admits the factorization $E(z) = S(z)P(z)$ with S an entire function which assumes real values on the real line and can have only real zeros, and

[1] The notation $U(z) \simeq V(z)$ means that there is a constant C such that $U(z) = CV(z)$ holds for all z in the set in question, which may be a Hilbert space, a set of complex numbers, or a suitable index set.

$$P(z) = \alpha e^{-aiz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\overline{z_n}}\right) e^{z\Re(1/z_n)} \tag{1.2}$$

where $a \geq 0$, $\alpha \in \mathbb{C}$ with $|\alpha| = 1$, and the sequence z_n in $\mathbb{C}_+$ satisfies the Blaschke condition. If I is a meromorphic inner function identified by such E, then for each z in $\mathbb{C}_+$, we have

$$I(z) = \frac{E^*}{E}(z) = \frac{\overline{\alpha}}{\alpha} e^{2aiz} \prod_{n=1}^{\infty} \frac{1 - z/z_n}{1 - z/\overline{z_n}}$$

which is always independent of the parameter S. In other words, the inner function $I = E^*/E = P^*/P$ acquires all of its structures only from the product factor P. This simple notion will be effectively used to construct our counterexamples in what follows.

We consider a model subspace K_B^2 with B a Blaschke product with simple zeros at the points $z_n = \gamma_n + i$, $\gamma_n > 0$, indexed by the positive integers and satisfying the exponential growth condition

$$\inf_{n \geq 1} \gamma_{n+1}/\gamma_n > 1. \tag{1.3}$$

The system (k_{z_n}) constitutes a Riesz basis in K_B^2. It follows that the norm equivalence

$$\|f\|_{K_B^2}^2 \simeq \sum_{n=1}^{\infty} |\langle f, k_{z_n}\rangle|^2 = \sum_{n=1}^{\infty} |f(z_n)|^2 \tag{1.4}$$

holds for functions f in K_B^2.
We now give a basic lemma which gives us an explicit estimate for the norm of the reproducing kernels.

Lemma 1.1. *If $Z = (z_n)$ and λ is a point in the upper half-plane, then there exists a positive integer m such that*

$$\|k_\lambda\|_{K_B^2}^2 \simeq \max\left\{m|\lambda|^{-2},\ \mathrm{dist}^{-2}(\lambda, Z)\right\}.$$

Proof. By (1.4), we have

$$\|k_\lambda\|_{K_B^2}^2 \simeq \sum_{n=1}^{\infty} |\langle k_\lambda, k_{z_n}\rangle|^2 \simeq \sum_{n=1}^{\infty} \frac{1}{|\lambda - \overline{z_n}|^2}.$$

Using the growth condition (1.3), we get from this

$$\|k_\lambda\|_{K_B^2}^2 \simeq \frac{m}{|\lambda|^2} + \frac{1}{|\lambda - \overline{z_m}|^2} + \sum_{n=m+1} \frac{1}{|\lambda - \overline{z_n}|^2} \tag{1.5}$$

where m is the positive integer such that

$$\frac{\gamma_{m-1} + \gamma_m}{2} \leq |\lambda| < \frac{\gamma_m + \gamma_{m+1}}{2}$$

for $m \geq 2$, and we take $m = 1$ if λ is in the set $\big\{z \in \mathbb{C} : |z| \leq (\gamma_1 + \gamma_2)/2\big\}$. It remains for us to show that the last term on the right-hand side of (1.5) is bounded by a constant times the first two terms. But using again condition (1.3), we find that the term is comparable to γ_{m+1}^{-2} and so the assertion in the lemma follows. $\square$

The reason such a simple estimate holds for $\|k_\lambda\|_{K_B^2}$ is the "minimal" interaction between the zeros of B implied by our a priori growth assumption (1.3): Geometrically, this almost lack of interaction is reflected in the (essential) lack of intersection between the disks

$$D_n = \Big\{z \in \mathbb{C}_+ : |z - z_n| \lesssim |z_n|/\sqrt{n}\Big\}.$$

We now pick a sequence (λ_n) which would lead us to the desired conclusion. In doing so, we set

$$(\lambda_n) = \left(\gamma_n\left(1 + \frac{1}{\sqrt{n}\log(n+1)}\right) + i\right).$$

It is easily seen that λ_n belongs to the disc D_n for each n. By Theorem 5.1 in [3], we observe that the sequence of the kernel functions in K_B^2 associated to such a sequence constitutes a reproducing kernel Riesz basis in K_B^2. Set

$$E(z) = \prod_{m=1}^{\infty}\left(1 - \frac{z}{\overline{z_m}}\right), \quad E_1(z) = \prod_{m=1}^{\infty}\left(1 - \frac{z}{\overline{\lambda_m}}\right), \quad I = E_1^*/E_1 \ \text{ and } \quad \varrho_n = \prod_{k=1}^{n}\frac{|z_k|^2}{|\lambda_k|^2},$$

and claim that $\|K^1_{\lambda_n}\|_{H(E_1)} \not\simeq \|K_{\lambda_n}\|_{H(E)}$. Were it not, then

$$\|K^1_{\lambda_n}\|^2_{H(E_1)} = |E_1(\lambda_n)|^2\|k^1_{\lambda_n}\|^2_{K_I^2} \simeq \frac{\Im\lambda_n|\lambda_n|^{2n-2}}{\prod_{k=1}^n|\lambda_k|^2} \tag{1.6}$$

and applying Lemma 1.1, we also have

$$\|K_{\lambda_n}\|^2_{H(E)} = |E(\lambda_n)|^2\|k_{\lambda_n}\|^2_{K_I^2} \simeq \frac{|\lambda_n|^{2n-2}}{\prod_{k=1}^n|z_k|^2}. \tag{1.7}$$

Invoking equality would imply that $1 = \Im\lambda_n \simeq 1/\varrho_n \to \infty$ when $n \to \infty$ and yields a contradiction.

1.2. **Component of inner functions and equality.** The Blaschke product B constructed in the previous counterexample is an infinitely many component inner function. i.e., the set $\{z \in \mathbb{C}_+ : |I(z)| < \delta\}$ is not connected for any δ in $(0, 1)$. It is easily seen that an inner function has either one-component or infinitely many components. Results valid for model subspaces generated by the class of one-component inner functions may in general fail when the generating inner function has infinitely many components and viceversa; for example see [1, 2, 3, 4]. A pertinent question along this is wether the necessity of condition (i) in Theorem 1.3 still fails when the inner function has more components. It turns out that this is again the case as seen in the next counterexample. This justifies that the property of equality spaces depends more on the arguments of the points associated to the kernel functions than the components of the inner functions generating the respective model spaces. Our next counterexample with one-component inner function appears first in [1] where it was used in a different context.

Let $E(z) = \exp(-\pi i z)$ and

$$E_1(z) = \lim_{R\to\infty} \prod_{|\lambda|<R} \left(1 - \frac{z}{\overline{\lambda_n}}\right) \quad \text{with}$$

$$\lambda_n = \begin{cases} n+i, & n \le 0 \\ n+\delta+i, & n > 0 \end{cases}$$

where $0 < \delta < 1/4$. Then $I(z) = E^*/E(z) = \exp(2\pi i z)$ and $B = E_1^*/E_1$. By Kadets 1/4 theorem, $(e^{i\lambda_n t})$ is a Riesz basis in $L^2(0, 2\pi)$.

Recall that for each meromorphic inner function Θ, there exists an increasing C^∞ function ψ (increasing branch of its argument along the real line), such that $\Theta(t) = e^{i\psi(t)}$ for each real t, from which we also have

$$2\pi \|k_t\|^2 = \psi'(t) = |\Theta'(t)|. \tag{1.8}$$

We now claim that $H(E) \neq H(E_1)$. Were it not, then setting φ and φ_1 respectively as increasing branches of the arguments of I and B, we have $|E(t)|^2\varphi'(t) \simeq |E_1(t)|^2\varphi_1'(t)$ for each $t \in \mathbb{R}$ where we have used (1.8). But since $\varphi' \simeq \varphi_1' \simeq 1$, $\quad |E(x)|/|E_1(x)| \simeq |x|^\delta \to \infty$ as $|x| \to \infty$ and gives a contradiction.

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Received: September, 2012