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Exact Geometric Solutions and Approximating Analytic Solutions of Functional Equations

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Abstract. We recall some of our earlier results on the construction of a mapping defined implicitly, without using the implicit function theorem. Then we mention the analyticity of the function defined implicitly, under certain hypothesis. We deduce applications to a concrete functional and operatorial equation. Finally, an approximating formula for the analytic form of the solution is considered.

Keywords: constructive solutions, implicit function, analyticity, approximations

1 Introduction

Obviously, the equation

$$g = g \circ f,$$

where g is given, while f is the unknown function, always has the trivial solution

$$f(x) = x, \quad \forall x \in D,$$

where D is the domain of definition for f.

These equations appeared firstly in [4]. Under some assumptions on g, there exists a unique decreasing solution of the same equation, which has many supplementary general qualities (see Theorem 2.1). For concrete given functions g, one obtains special qualities of the corresponding solutions f.

The present approach allows the construction of the solutions of such functional and operatorial equations, without using the implicit function theorem. In section 4, we apply these general-type results to a concrete functional equation and to the corresponding operatorial equation. In the operatorial case (Theorem 4.2), the solution F is a function of $U \in D \subset X$, where X is the commutative algebra of selfadjoint operators (1). We essentially use the fact that X is also an order-complete vector lattice, with respect to the natural order relation on $\mathrm{A}(H)$.

This work continues theorems published in [3]-[6], and contains new results too. The background is partially contained in [1], [2], [7].

2 General results

Theorem 2.1. (see also [3]-[6]). *Let* $u, v \in \overline{\mathrm{R}}$, $u < v$, $\alpha \in]u, v[$ *and let* $g :]u, v[\to \mathrm{R}$ *be a continuous function. Assume that*

(a) $\lim\limits_{x \downarrow u} g(x) = \lim\limits_{x \uparrow v} g(x) = w \in \overline{\mathrm{R}}$,

(b) *g is strictly decreasing on* $]u, \alpha[$ *and strictly increasing on* $[\alpha, v[$.

Then there exists $f :]u, v[\to]u, v[$ *such that*

$$g(x) = g(f(x)), \quad \forall x \in]u, v[$$

and f has the following qualities:

(i) *f is strictly decreasing on* $]u, v[$ *and we have*

$$\lim_{x \downarrow u} f(x) = v, \quad \lim_{x \uparrow v} f(x) = u;$$

(ii) α *is the unique fixed point of f;*

(iii) *we have* $f^{-1} = f$ *on* $]u, v[$;

(iv) *f is continuous on* $]u, v[$;

(v) *if* $g \in C^n(]u, v[\setminus \{\alpha\})$, $n \in \mathrm{N} \cup \{\infty\}$, $n \geq 1$, *then* $f \in C^n(]u, v[\setminus \{\alpha\})$;

(vi) *if g is derivable on* $]u, v[\setminus \{\alpha\}$, *so is* f;

(vii) *if* $g \in C^2(]u, v[)$, $g''(\alpha) \neq 0$ *and there exists* $\rho_1 := \lim\limits_{x \to \alpha} f'(x) \in \overline{\mathrm{R}}$

then $f \in C^1(]u, v[) \cap C^2(]u, v[\setminus \{\alpha\})$ *and* $f'(\alpha) = -1$;

(viii) *if* $g \in C^3(]u, v[)$, $g''(\alpha) \neq 0$ *and there exist*

$\rho_1 := \lim_{x\to\alpha} f'(x) \in \overline{\mathrm{R}}$ and $\rho_2 := \lim_{x\to\alpha} f''(x) \in R,$ then $f \in C^2(]u,v[) \cap C^3(]u,v[\setminus\{\alpha\})$ and

$$f''(\alpha) = \rho_2 = -\frac{2}{3}\cdot\frac{g'''(\alpha)}{g''(\alpha)};$$

(ix) *if* g *is analytic at* α, *then* f *is derivable at* α *and* $f'(\alpha) = -1;$

(x) *let* $g_l := g|_{]u,\alpha[}$, $g_r := g|_{]\alpha,v[}$; *then we have*

$$f(x_0) = (g_r^{-1} \circ g_l)(x_0) = \sup\{x \in [\alpha, v[\,; g_r(x) \le g_l(x_0)\}\ \forall x_0 \in]u,\alpha]$$

and

$$f(x_0) = (g_l^{-1} \circ g_r)(x_0) = \inf\{x \in]u,\alpha]; g_l(x) \le g_r(x_0)\}\ \forall x_0 \in [\alpha, v[$$

The proof of this theorem is similar to that of Theorem 1.1. [4] (see Figure 1 [4], p. 62).

Let $\alpha \in]0,\infty[$ and denote by G the set of all continuous functions $g :]0,\infty[\to]0,\infty[$ which are decreasing on $]0,\alpha]$ and increasing on $[\alpha,\infty[$, such that

$$\lim_{x\downarrow 0} g(x) = \lim_{x\uparrow\infty} g(x) = \infty,\ g(\alpha) = 0.$$

For $g, h \in G,$ an interesting problem is the following one: find necessary and sufficient conditions on g, h for the equality:

$$f_g = f_h,$$

where f_g, f_h are the corresponding functions attached to $g,$ respectively to h by Theorem 2.1. The following statement is giving the answer.

Theorem 2.2. (Theorem 1.4 [4]). *Let* $g, h \in G,\ \lambda \in]0,\infty[.$ *Then* $g+h,\ \lambda g,\ g\cdot h$ *are also elements of* G *and the following statements are equivalent*

(a) $f_g = f_h;$

(b) $h_r \circ g_r^{-1} = h_l \circ g_l^{-1},\ g_l := g|_{]0,\alpha]},\ g_r := g|_{[\alpha,\infty[},\quad g \in G;$

(c) *there exists* $\varphi : [0,\infty[\to [0,\infty[$ *such that* $\varphi(0) = 0,$ φ *is continuous and increasing, verifying the relation*

$$h = \varphi \circ g.$$

Next we consider the abstract operatorial version of Theorem 2.1. In the sequel, X will be an order-complete vector lattice, and $Izom_+(X)$ will be the set of all vector space isomorphisms $T : X \to X$ which apply X_+ onto itself

Theorem 2.3. *Let* X *be an order-complete vector lattice,* X_+ *its positive cone,* $\alpha \in X$, D_l *a convex subset such that*

$$\alpha \in D_l \subset \{x \in X;\ \ x \le \alpha\};$$

D_r *a convex subset such that*

$$\alpha \in D_r \subset \{x \in X;\ \ x \ge \alpha\};$$

Let $g_l : D_l \to X$ *be a convex operator such that*

$$\partial g_l(x) \bigcap (-Izom_+(X)) \ne \Phi, \quad \forall x \in D_l \setminus \{\alpha\}.$$

Let $g_r : D_r \to X$ *be a convex operator such that*

$$\partial g_r(x) \bigcap (Izom_+(X)) \ne \Phi, \quad \forall x \in D_r \setminus \{\alpha\}.$$

We also assume that

$$g_l(\alpha) = g_r(\alpha) \ \ \textit{and} \ \ R(g_l) = R(g_r),$$

where $R(g)$ *is the range of* g, *while* $\partial g(x)$ *is the set of all subgradients of* g *at* x. *Let*

$$g : D = D_l \bigcup D_r \to X,$$
$$g(x) = g_l(x)\ \forall x \in D_l, \quad g(x) = g_r(x)\ \forall x \in D_r.$$

Then there exists $F : D \to D$ *such that*

$$g(x) = g(F(x)), \quad \forall\, x \in D$$

F *is strictly decreasing in* D *and it has the following properties:*

(i) α *is the only fixed point of F;*

(ii) *there exists* F^{-1} *and* $F^{-1} = F$ *on* D;

(iii) we have

$$F(x_0) = g_r^{-1}(g_l(x_0)) = \sup\{x \in D_r\ ; g_r(x) \le g_l(x_0)\}\ \forall\, x_0 \in D_l,$$

$$F(x_0) = g_l^{-1}(g_r(x_0)) = \inf\{x \in D_l\ ; g_l(x) \le g_r(x_0)\}\ \forall\, x_0 \in D_r.$$

The proof of this theorem is similar to that from [4].

3. On the analyticity of the solution

Theorem 3.1. *Let* g *be a complex holomophic functions which is the extension of a real analytic function with the properties mention in Theorem* 2.1. *If* g *verifies conditions of Theorem* 10.32 [7], *then the nontrivial solution* f *of the equation*

$$g(z) = g(f(z)), |z - \alpha| < \varepsilon$$

is holomorphic at α *too, so that it is holomorphic and conformal in a complex neighborhood of* α.

The proof is similar to that of Theorem 2.1 [5], point (viii), also using Theorem 10.32 [7]. For some other aspects of the complex case see [3].

4. Applications

Theorem 4.1. *There exists a unique decreasing solution of the equation*

$$x \ln x = f(x) \ln(f(x)), x \in [0,1],$$

such that

(i) $f(0+) = 1,\ f(1-) = 0;$

(ii) $1/e$ *is the unique fixed point of* f;

(iii) $f^{-1} = f$ *in* $[0,1]$;

(iv) f *is analytic in* $]0,1[$, *and* $f'(1/e) = -1,\ f''(1/e) = \frac{2}{3}e;$

(v) *if* $g(x) = x \ln x,\ x \in [0,1],\ g_l = g_{|[0,1/e]},\ g_r = g_{|[1/e,1]}$ *then the following constructive formulae for* f *hold*

$$f(x_0) = \sup\{x \in [1/e,1];\ x \ln x \le x_0 \ln x_0\},\ \forall x_0 \in [0,1/e];$$
$$f(x_0) = \inf\{x \in [0,1/e];\ x \ln x \le x_0 \ln x_0\},\ \forall x \in [1/e,1].$$

The proof is a straightforward application of the Theorem 2.1.

Let H be a Hilbert space, A a selfadjoint operator from H into H, $\mathrm{A}(H)$ the real vector space of all selfadjoint operators acting on H. Let us denote

$$\begin{aligned} X_1 &= X_1(A) = \{T \in \mathrm{A}(H);\ AT = TA\}, \\ X &= \{T \in X_1;\ TU = UT,\ \forall U \in X_1\}. \end{aligned} \tag{1}$$

It is known that X is an order complete Banach lattice with respect to the usual order relation on selfadjoint operators, and a commutative real Banach algebra (see [2]).

Theorem 4.2. *Consider the following subsets of the space* X :

$$D_l = \{U \in X;\, \sigma(U) \subset [0, e^{-1}[\} \cup \{e^{-1}I\},$$
$$D_r = \{U \in X;\, \sigma(U) \subset]e^{-1}, 1]\} \cup \{e^{-1}I\},$$

where $\sigma(U)$ *is the spectrum of* U *and* I *is the identity operator on* H. *Let* $D = D_l \cup D_r$. *Then there exists a unique decreasing operator* $F : D \to D$ *such that*

$$U \ln U = F(U) \ln F(U),\ \forall U \in D;$$

(i) $e^{-1}I$ *is the unique fixed point of* F;

(ii) F *is invertible, being its own inverse on* D;

(iii) F *is "constructed" following the formulae*

$$F(U_0) = \sup\{U \in D_r;\, U \ln U \le U_0 \ln U_0\},\ U_0 \in D_l,$$
$$F(U_0) = \inf\{U \in D_l;\, U \ln U \le U_0 \ln U_0\},\ U_0 \in D_r.$$

Proof. We verify the conditions from the statement of Theorem 2.3. Using the fact that the algebra X is commutative, and also the convexity of the function $g(x) = x \ln x$ on $[0,1]$, one obtains:

$$[(1-t)U_1 + tU_2] \cdot \ln((1-t)U_1 + tU_2) =$$
$$\iint_{\sigma(U_1)\times\sigma(U_2)} ((1-t)x_1 + tx_2) \cdot \ln((1-t)x_1 + tx_2)\, dE_{U_1} dE_{U_2} \le$$
$$(1-t) \iint_{\sigma(U_1)\times\sigma(U_2)} x_1 \ln x_1 dE_{U_1} dE_{U_2} + t \iint_{\sigma(U_1)\times\sigma(U_2)} x_2 \ln x_2 dE_{U_1} dE_{U_2} =$$
$$(1-t)U_1 \ln U_1 + tU_2 \ln U_2,\ U_1, U_2 \in \{U \in X;\, \sigma(U) \subset [0,1]\},\ t \in [0,1].$$

In particular, the restrictions of the function $G(U) = U \ln U$ to the convex subsets D_l, D_r are convex. We have used the fact that the two spectral measures attached to U_1, U_2 are positive measures on the corresponding spectrums. For the restriction of G to D_l we have:

$$G_l'(U) = \ln U + I < 0,\ \sigma(U) \subset]0, e^{-1}[$$

because of the relations

$$0 \le U < e^{-1} I = \exp(-I), \forall U \in D_l \setminus \{e^{-1} I\}$$

Since the algebra X is commutative, we infer that $G_s'(U) < 0, U \in D_s \setminus \{e^{-1} I\}$ Similarly, we have

$$G_r'(U) > 0, U \in D_r \setminus \{e^{-1} I.\}$$

Finally, we prove that the ranges $R(G_l)$ and $R(G_r)$ are equal. Let $U_1 \in D_l$. Integrating the scalar equation

$$g_l(x) = g_r(f(x)), x \in \left[0, e^{-1}\right]$$

with respect to the spectral measure dE_{U_1} leads to

$$G_l(U_1) = \int_{\sigma(U_1)} g_l(x) dE_{U_1} = \int_{\sigma(U_1)} g_r(f(x)) dE_{U_1} =$$
$$G_r(F(U_1)) = G_r(U_2) \in R(G_r),$$

since f applies the interval $[0, 1/e]$ onto the interval $[1/e, 1]$, following the decreasing sense. Hence $U_2 = F(U_1) \in D_r, \forall U_1 \in D_l$. In the same way, one proves that $F(D_r) \subset D_l$. Using also the relations $f^{-1} = f, F^{-1} = F$, one proves that actually $F(D_l) = D_r, F(D_r) = D_l$. Now all requirements of theorem 2.3 are accomplished. The conclusion follows by an application of the latter theorem. Now the proof is complete. □

5. Approximating the solution

Theorem 5.1. *In a small interval centered at* e^{-1}, *we have*

$$f(x) \approx \frac{x}{2 \ln x + 3},$$

Proof. Let f be the solution from Theorem 4.1. Our next goal is to give an approximation around $\alpha = e^{-1}$ of this solution. We have:

$$x\ln x = f(x)\ln(f(x)) \Leftrightarrow \frac{f(x)}{x} = \frac{\ln x}{\ln(f(x))} \Leftrightarrow$$

$$1+\frac{f(x)-x}{x} = 1+\frac{\ln\left(1+\dfrac{x-f(x)}{f(x)}\right)}{\ln f(x)} \Leftrightarrow$$

$$\frac{f(x)-x}{x} \approx \frac{1}{\ln(f(x))}\left[\frac{x-f(x)}{f(x)} - \frac{(x-f(x))^2}{2f^2(x)}\right] \Leftrightarrow$$

$$\frac{f(x)\ln(f(x))}{x} = \ln x \approx -1 \pm \frac{x-f(x)}{2f(x)}.$$

From the last relation, using the known fact that e^{-1} must be a fixed point of f, $f'\left(e^{-1}\right) = -1$, we conclude that

$$f(x) \approx \frac{x}{2\ln x + 3},$$

where x is in a small interval centered at e^{-1}. □

Corollary 5.1. *With the above notations, if $U \in D \subset X$ has the spectrum contained in a small interval centered at e^{-1}, then we have*

$$F(U) \approx U \cdot (2\ln U + 3I)^{-1}.$$

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