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Some Basic Properties and a Two-by-Two Matrix Involving the *k*- Pell Numbers

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Abstract

In this paper we consider the *k*-Pell numbers sequence and present some properties involving the *k*-Pell numbers. The theoretical basis of using generating matrices for deriving the explicit formula for the term of order *n* of the *k*-Pell numbers sequence and also to get the well-known Cassini's identity using linear algebra.

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1. Introduction

The well-known Fibonacci sequence is related to the golden ratio. Such sequence is defined by the recurrence relation, for $n \geq 1, F_{n+1} = F_n + F_{n-1}$, where $F_0 = 0$, $F_1 = 1$. Many papers and research work are dedicated to Fibonacci sequence, such as the work of Hoggatt, in [17] and Vorobiov, in [14],

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among others and more recently we have, for example, the works of Caldwell *et al*. in [4], Marques in [7], and Shattuck, in [12]. Also related with Fibonacci sequence, Falcőn *et al*., in [16], consider some properties for k-Fibonacci numbers obtained from elementary matrix algebra and its identities including generating function and divisibility properties appears in the paper of Bolat *et al.*, in [3]. Other sequence, also important, is the sequence of Pell numbers defined by the recursive recurrence given by $P_n = 2P_{n-1} + P_{n-2}$, $n \geq 2$, with $P_0 = 0$ and $P_1 = 1$. This sequence has been studied and some of its basic properties are known (see, for example, the study of Horadam, in [2]). In [11], we find the matrix method for generating the Pell numbers sequence and comparable matrix generators have been considered by Kalman, in [6], by Bicknell, in [13], for the Fibonacci and Pell sequences. From this sequence, we obtain some types of other sequences namely, Pell-Lucas and Modified Pell sequences and also Dasdemir, in [1], consider some new matrices which are based on these sequences as well as that they have the generating matrices. Also, for the sequence of Jacobsthal number, Koken and Bozkurt, in [9], deduce some properties and the Binet's formula, using matrix method as well as in [10], Yilmaz *et al.* study some more properties related with k- Jacobsthal numbers. Recently, according Jhala *et al.* in [5], Catarino considered in [15] the k-Pell numbers sequence and many properties are proved by easy arguments for the k-Pell numbers. Catarino in [15] obtain the Binet's formula for k-Pell numbers and as a consequence get some properties for k-Pell numbers. Also Catarino [15] gives the generating function for k-Pell sequences and another expression for the general term of the sequence, using the ordinary generating function. In this paper, we continue the study of k-Pell numbers sequence considering several identities involving the k-Pell numbers and using a matrix method for deriving the explicit formula for the term of order n of the k-Pell numbers sequence and also to get the well-known Cassini's identity.

2. The *k*-Pell Number and some basic properties

For any positive real number k, the k-Pell sequence say $(P_{k,n})_{n \in \mathbb{N}}$ is defined recurrently by

$$P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}, \text{ for } n \geq 1, \tag{1}$$

with initial conditions given by, $P_{k,0} = 0$, $P_{k,1} = 1$. Note that using the well-known results involving recursive sequences, the characteristic equation, associated to the recurrence relation (1) is given by $r^2 - 2r - k = 0$ and has two distinct roots $r_1 = 1 + \sqrt{1+k}$ and $r_2 = 1 - \sqrt{1+k}$. Since $\sqrt{1+k} > 1$, then $r_2 < 0$ and so, $r_2 < 0 < r_1$. Also, we obtain that $r_1 + r_2 = 2$, $r_1 - r_2 = 2\sqrt{1+k}$ and $r_1 r_2 = -k$.

Next we present several basic properties of the k-Pell sequence and standard techniques will be used to generate them.

Proposition 1 (Summation formula)

$$P_{k,0} + P_{k,1} + P_{k,2} + \cdots + P_{k,n} = \frac{1}{k+1}\left(-1 + P_{k,n+1} + kP_{k,n}\right).$$

Proof: We have, $P_{k,0} = 0$, $P_{k,1} = \frac{1}{k}\left(P_{k,3} - 2P_{k,2}\right)$, $P_{k,2} = \frac{1}{k}\left(P_{k,4} - 2P_{k,3}\right)$, …, $P_{k,n} = \frac{1}{k}\left(P_{k,n+2} - 2P_{k,n+1}\right)$ and then, $P_{k,0} + P_{k,1} + P_{k,2} + \cdots + P_{k,n} = \frac{1}{k}\left(\left(-2P_{k,2} + P_{k,n+2}\right) - \left(P_{k,3} + P_{k,4} + \cdots + P_{k,n} + P_{k,n+1}\right)\right)$.
Hence, $k(P_{k,0} + P_{k,1} + P_{k,2} + \cdots + P_{k,n}) = \left(-2P_{k,2} + P_{k,n+2}\right) - \left(P_{k,3} + P_{k,4} + \cdots + P_{k,n} + P_{k,n+1}\right)$ and so, $k(P_{k,0} + P_{k,1} + P_{k,2} + \cdots + P_{k,n}) + (P_{k,0} + P_{k,1} + P_{k,2} + \cdots + P_{k,n}) = \left(-2P_{k,2} + P_{k,n+2}\right) + P_{k,1} + P_{k,2} - P_{k,n+1}$. Using (1) and the initial conditions, we obtain that, $(k+1)\left(P_{k,0} + P_{k,1} + P_{k,2} + \cdots + P_{k,n}\right) = -4 + 2P_{k,n+1} + kP_{k,n} + 1 + 2 - P_{k,n+1}$, that is equivalent to $(k+1)\left(P_{k,0} + P_{k,1} + P_{k,2} + \cdots + P_{k,n}\right) = -1 + P_{k,n+1} + kP_{k,n}$, and the result follows immediately.

■

Proposition 2

$$\begin{cases} P_{k,1}r_1 = r_1 - kP_{k,0} \\ P_{k,n}r_1 = r_1^n - kP_{k,n-1} \end{cases}, \ n \geq 2. \tag{2}$$

Proof: We prove the proposition by using induction on *n*. Note that, for $n = 1$, the first identity is true using the initial conditions. Suppose now that is true for n and so we obtain $r_1^{n+1} - kP_{k,n} = r_1 r_1^n - kP_{k,n} = r_1\left(P_{k,n}r_1 + kP_{k,n-1}\right) - kP_{k,n} = P_{k,n}r_1^2 + kP_{k,n-1}r_1 - kP_{k,n} = \left(1 + \sqrt{1+k}\right)\left(2P_{k,n} + kP_{k,n-1}\right) = r_1 P_{k,n+1}$, as required.

■

Analogously the following result is shown.

Proposition 3

$$\begin{cases} P_{k,1}r_2 = r_2 - kP_{k,0} \\ P_{k,n}r_2 = r_2^n - kP_{k,n-1} \end{cases}, \ n \geq 2. \tag{3}$$

■

The Binet's formula of the *k*-Pell numbers sequence was considered by Catarino in [15]. Next we present again this formula, but we get it using the identities (2) and (3).

Proposition 4 (Binet's formula)
The nth k-Pell number is given by

$$P_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}, \tag{4}$$

where r_1 *and* r_2 *are the roots of the characteristic equation associated to the recurrence relation* (1) *and* $r_1 > r_2$.

Proof: Easily, if we subtract (3) to (2), we get, for $n = 1$, that $P_{k,1}r_2 - P_{k,1}r_1 = r_2 - kP_{k,0} - r_1 + kP_{k,0}$. Then $r_1 - r_2 = P_{k,1}(r_1 - r_2)$, and we get the Binet's formula for this case. Now, for $n \geq 2$, once more we obtain that $r_1^n - r_2^n = P_{k,n}\, r_1 - P_{k,n}r_2$, and the result follows.

■

Proposition 5
If $m \geq 0$ *and* $n > 0$, *then* $P_{k,n+m} = kP_{k,n-1}P_{k,m} + P_{k,n}P_{k,m+1}$.

Proof: We prove by induction on m. For $m = 0$, we get $P_{k,n} = kP_{k,n-1}P_{k,0} + P_{k,n}P_{k,1}$, that is valid using the initial conditions of the *k*-Pell sequence. Also for $m = 1$, and once more using the initial conditions, we obtain, in this case, that $P_{k,n+1} = kP_{k,n-1}P_{k,1} + P_{k,n}P_{k,2} = kP_{k,n-1} + 2P_{k,n}$, a valid identity. Suppose now that $q > 1$ and that the property is true for all k, $1 \leq k \leq q$ and for all $n > 0$. Using the induction hypothesis

$$P_{k,n+(q-1)} = kP_{k,n-1}P_{k,q-1} + P_{k,n}P_{k,q} \qquad (5)$$

$$P_{k,n+q} = kP_{k,n-1}P_{k,q} + P_{k,n}P_{k,q+1} \qquad (6)$$

Now multiplying (5) by k and (6) by 2 and adding the identities, we have that $kP_{k,n+(q-1)} + 2P_{k,n+q} = k\big(kP_{k,n-1}P_{k,q-1} + P_{k,n}P_{k,q}\big) + 2\big(kP_{k,n-1}P_{k,q} + P_{k,n}P_{k,q+1}\big)$, and using (1) three times, we conclude that$P_{k,n+(q+1)} = kP_{k,n-1}P_{k,q+1} + P_{k,n}P_{k,q+2}$. Therefore the identity is also valid for $q + 1$ whenever $n > 0$ and our aim follows.

■

Using a particular case of the last property we have the following result:

Proposition 6
For all $n \geq 1$, $\left(P_{k,n+1}^2 - k^2 P_{k,n-1}^2\right) = 2P_{k,2n}$.

Proof: Consider $m = n$ in the last property and so $P_{k,2n} = kP_{k,n-1}P_{k,n} + P_{k,n}P_{k,n+1}$. Now replacing $P_{k,n}$ by the expression given by (1), we obtain that $P_{k,2n} = kP_{k,n-1}\left(\frac{P_{k,n+1} - kP_{k,n-1}}{2}\right) + \left(\frac{P_{k,n+1} - kP_{k,n-1}}{2}\right)P_{k,n+1} = \frac{P_{k,n+1}^2 - k^2 P_{k,n-1}^2}{2}$, as we required.

■

3. Generating matrix for the *k*-Pell sequences

One of the most usual and recurrent recent methods for the study of the recurrence sequences is to define the so-called generating matrix. Next we shall study this problem for the *k*-Pell sequences. Theorems may then be cited from linear algebra so as to give short proofs. Write $|A|$ for the determinant of a matrix A. Then it is well known that $|AB| = |A|\,|B|$, and in general,$|A^n| = |A|^n$. The *k*-Pell sequence is one (among others) of the special cases of a sequence which is defined recursively as a linear combination of the preceding p terms

$$a_{n+p} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{p-1} a_{n+p-1}, \tag{7}$$

where $c_0,\ c_1, \ldots,\ c_{p-1}$ are real constants. In [6], Kalman derives a number of closed-form formulas for the generalized sequence by companion matrix method. Also in [8] the method matrix is used for the case of the generalized order-*k* Pell numbers. Using the matrix method, consider a matrix T of order $p \times p$ such that the last line is consisting of the constants $c_0,\ c_1, \ldots,\ c_{p-1}$ and the entries $t_{i,i+1} = 1$, for $i = 1, \ldots p-1$ and the remaining entries zero. Also define a matrix $A_n = \left(a_n, \ldots, a_{n+p-1}\right)^T$ associated with (7). It is easy to show that $TA_n = A_{n+1}$ and $A_n = T^n A_0$, where $A_0 = \left(a_0, \ldots, a_{p-1}\right)^T$ and results of linear algebra will be used in what is follows in order to give another expression for the general term of the *k*-Pell sequence. Also we shall use the eigenvalues and the respective eigenvectors in this process. If we recall the recurrence (1), then according (7), we have that $p = 2$, $c_0 = k$ and $c_1 = 2$. Hence the matrix associated is given by

$$T = \begin{pmatrix} 0 & 1 \\ c_0 & c_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ k & 2 \end{pmatrix},$$

with $|T| = -k$. Considering all powers of T, we have the following result:

Proposition 7

$$T^n = \begin{pmatrix} kP_{k,n-1} & P_{k,n} \\ kP_{k,n} & P_{k,n+1} \end{pmatrix}, \tag{8}$$

for all $n \geq 1$.

Proof: We shall use induction on n. For $n = 1$, $T = \begin{pmatrix} kP_{k,0} & P_{k,1} \\ kP_{k,1} & P_{k,2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ k & 2 \end{pmatrix}$, that is true using the initial conditions of sequence. Suppose now that (8) is valid for n and then we get that

$$\begin{aligned} T^{n+1} = TT^n &= \begin{pmatrix} 0 & 1 \\ k & 2 \end{pmatrix} \begin{pmatrix} kP_{k,n-1} & P_{k,n} \\ kP_{k,n} & P_{k,n+1} \end{pmatrix} \\ &= \begin{pmatrix} kP_{k,n} & P_{k,n+1} \\ k^2 P_{k,n-1} + 2kP_{k,n} & kP_{k,n} + 2P_{k,n+1} \end{pmatrix} \\ &= \begin{pmatrix} kP_{k,n} & P_{k,n+1} \\ k\left(kP_{k,n-1} + 2P_{k,n}\right) & kP_{k,n} + 2P_{k,n+1} \end{pmatrix} \\ &= \begin{pmatrix} kP_{k,n} & P_{k,n+1} \\ kP_{k,n+1} & P_{k,n+2} \end{pmatrix}, \end{aligned}$$

as required. ■

Using the properties involving the determinant of the matrices T and T^n, we can obtain the Cassini's identity and the following result follows immediately:

Proposition 8 (Cassini's identity)

$$P_{k,n-1}P_{k,n+1} - P_{k,n}^2 = (-1)^n k^{n-1}$$

■

Calculating the eigenvalues of T, we obtain two distinct eigenvalues that coincides with $r_1$and r_2. The eigenvectors associated with r_1 and r_2 are respectively,$\begin{pmatrix} x \\ r_1 x \end{pmatrix}$, $\begin{pmatrix} x \\ r_2 x \end{pmatrix}$, with x non zero. In particular, for $x = 1$, we get the eigenvectors $v_1 = \begin{pmatrix} 1 \\ r_1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ r_2 \end{pmatrix}$. Writing $A_0 = \begin{pmatrix} P_{k,0} \\ P_{k,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha_1 v_1 + \alpha_2 v_2$, we obtain that $\alpha_1 = \frac{1}{r_1 - r_2}$ and $\alpha_2 = -\frac{1}{r_1 - r_2}$. Finally, applying$T^n$, we get $A_n = T^n A_0 = \begin{pmatrix} P_{k,n} \\ P_{k,n+1} \end{pmatrix}$ and then we achieve the Binet's formula for the *k*-Pell sequence, obtained in a different way that Catarino did in [15].

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