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CR-Submanifolds of a Lorentzian Para-Sasakian Manifold Endowed with a Semi-symmetric Non-metric Connection

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Abstract

We define a semi-symmetric non-metric connection in a Lorentzian para-Sasakian manifold and study *CR*-submanifolds of a Lorentzian para-Sasakian manifold endowed with a semi-symmetric non-metric connection. Moreover, we also obtain integrability conditions of the distributions on *CR*-submanifolds.

Keywords: Lorentzian para-Sasakian manifold, CR-submanifolds, semi-symmetric non-metric connection, parallel horizontal distribution.

1. Introduction

The notion of *CR*-submanifolds of a Kaehler manifold was introduced by A. Bejancu in [1]. Later, *CR*-submanifolds of Sasakian manifold were studied by M. Kobayashi in [9]. K. Matsumotu introduced the idea of Lorentzian para-

Sasakian structure and studied its several properties in [6]. B. Prasad [4] and S. Prasad, R.H. Ojha [11] studied submanifolds of a Lorentzian para-Sasakian manifolds. U. C. De and Anup Kumar Sengupta studied *CR*-submanifolds of a Lorentzian para-Sasakian manifold in [12]. In this paper, we study *CR*-submanifolds of a Lorentzian para-Sasakian manifold endowed with a semi-symmetric non-metric connection. On the otherhand, A. Friedmann and J. A. Schouten introduced the idea of a semi-symmetric linear connection in ([3], [5]). A linear connection ∇ is said to be semi-symmetric connection if its torsion tensor T is of the form

$$T(X,Y)=\eta(Y)X-\eta(X)Y,$$

where η is a 1-form. In [7], K. Yano studied some properties of semi-symmetric metric connection. In [10], N.S. Agashe and M.R. Chaffle studied some properties of semi-symmetric non-metric connection. In [8], the first author and C. Ozgur defined a semi-symmetric non-metric connection and studied some properties of hypersurfaces of almost $r-$paracontact Riemannian manifold with semi-symmetric non-metric connection. In this paper, we study *CR*-submanifolds of a Lorentzian para-Sasakian manifold endowed with a semi-symmetric non-metric connection.

2. Preliminaries

Let $\overline{M}$ be an $n-$dimensional almost contact metric manifold with almost contact metric structure (ϕ,η,ξ,g). Then we have by definition

$$\phi^2X = X+\eta(X)\xi,\quad \eta(\xi)=-1, \tag{2.1}$$

$$g(\phi X,\phi Y)=g(X,Y)+\eta(X)\eta(Y), \tag{2.2}$$

$$g(X,\xi)=\eta(X), \tag{2.3}$$

$$g(\phi X,Y)=g(X,\phi Y)=\psi(X,Y) \tag{2.4}$$

for all vector fields X,Y tangent to $\overline{M}$. Such a structure (ϕ,η,ξ,g) is termed as Lorentzian para-contact structure [6].
Also in a Lorentzian para-contact structure the following relations hold:

$$\phi\xi=0,\quad \eta(\phi X)=0,\quad \text{rank}\,(\phi)=n-1.$$

A Lorentzian para-contact manifold $\overline{M}$ is called Lorentzian para-Sasakian (LP-Sasakian manifold if [6])

$$(\overline{\overline{\nabla}}_X\phi)(Y)=g(X,Y)+\eta(Y)X+2\eta(X)\eta(Y)\xi, \tag{2.5}$$

$$\overline{\overline{\nabla}}_X\xi=\phi X \tag{2.6}$$

for all vector fields X,Y tangent to $\overline{M}$, where $\overline{\overline{\nabla}}$ is the Riemannian connection with respect to g. We define a semi-symmetric non-metric connection $\overline{\nabla}$ in LP-Sasakian manifold by

(2.7) $\overline{\nabla}_X Y = \overline{\overline{\nabla}}_X Y + \eta(Y)X$

such that $(\overline{\nabla}_X g)(Y,Z) = -\eta(Y)g(X,Z) - \eta(Z)g(X,Y)$ for any $X,Y \in TM,$ where $\overline{\overline{\nabla}}$ is induced connection with respect to g on $M.$

Inserting (2.7) in (2.5) and (2.6), we have

(2.8) (a) $(\overline{\nabla}_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi - \eta(Y)\phi X,$

(2.8) (b) $\overline{\nabla}_X \xi = \phi X - X.$

We denote by g the metric tensor of $\overline{M}$ as well as that induced on $M.$ Let $\overline{\nabla}$ be the semi-symmetric non-metric connection on $\overline{M}$ and ∇ be the induced connection on M with respect to unit normal $N.$ Gauss equation and Weingarten formula for *CR*-submanifolds of LP-Sasakian manifold with semi-symmetric non-metric connection are given respectively by

(2.11) $\overline{\nabla}_X Y = \nabla_X Y + h(X,Y),$

(2.12) $\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N$

for $X,Y \in TM,$ $N \in T^{\perp}M,$ h (resp. A_N) is the second fundamental form (resp. tensor) of M in $\overline{M}$ and $\nabla^{\perp}$ denotes the operator of the normal connection. Moreover, we have [1]

(2.13) $g(h(X,Y),N) = g(A_N X,Y).$

Any vector X tangent to M is given as

(2.14) $X = PX + QX,$

where PX and QX belong to the distribution D and $D^{\perp}$ respectively.

For any vector field N normal to $M,$ we put

(2.15) $\phi N = BN + CN,$

where BN (resp. CN) denotes the tangential (resp. normal) component of $\phi N.$

3. Integrabilities of horizontal distribution D and vertical distribution $D^{\perp}$

Lemma 3.1. Let M be a *CR*-submanifold of an LP-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection. Then

(3.1)
$$P\nabla_X \phi PY - PA_{\phi QY}X = \phi P\nabla_X Y + g(X,Y)P\xi + \eta(Y)PX - \eta(Y)\phi PX + 2\eta(X)\eta(Y)P\xi,$$

(3.2)
$$Q\nabla_X \phi PY - QA_{\phi QY}X = g(X,Y)Q\xi + \eta(Y)QX - \eta(Y)\phi QX + 2\eta(X)\eta(Y)Q\xi + Bh(X,Y),$$

(3.3) $h(X,\phi PY) + \nabla_X^{\perp}\phi QY = \phi Q\nabla_X Y + Ch(X,Y)$

for $X, Y \in TM$.

Proof. By virtue of (2.8) (a), (2.11), (2.12), (2.14) and (2.15), we can easily get

$$P\nabla_X \phi PY - Q\nabla_X \phi PY + h(X, \phi PY) - PA_{\phi QY} X - QA_{\phi QY} X + \nabla^{\perp}_X \phi QY$$
$$= g(X,Y)P\xi + g(X,Y)Q\xi + \eta(Y)PX + \eta(Y)QX - \eta(Y)\phi PX$$
$$-\eta(Y)\phi QX + 2\eta(X)\eta(Y)P\xi + 2\eta(X)\eta(Y)Q\xi + \phi P\nabla_X Y$$
$$+\phi Q\nabla_X Y + Bh(X,Y) + Ch(X,Y).$$

Equations (3.1)-(3.3) follow by equating horizontal, vertical and normal components.

Lemma 3.2. Let M be a *CR*-submanifold of an LP-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection. Then

$$\phi P[X,Y] = A_{\phi Y} Z - A_{\phi Z} Y + \eta(Y)Z - \eta(Z)Y - \eta(Y)\phi Z + \eta(Z)\phi Y$$

for all $Y, Z \in D^{\perp}$.

Proof. By virtue of (2.8) (a), (2.11) and (2.12) we have

$$-A_{\phi Z} Y + \nabla^{\perp}_Y \phi Z = g(Y,Z)\xi + \eta(Z)Y - \eta(Z)\phi Y + 2\eta(Y)\eta(Z)\xi$$
$$+\phi(\nabla_Y Z + h(Y,Z))$$

for any $Y, Z \in D^{\perp}$.

Using (3.3), we get

$$\phi P\nabla_Y Z = -A_{\phi Z} Y - g(Y,Z)\xi - \eta(Z)Y + \eta(Z)\phi Y - 2\eta(Y)\eta(Z)\xi$$
$$-Bh(Y,Z).$$

Interchanging Y and Z, we find

$$\phi P\nabla_Z Y = -A_{\phi Y} Z - g(Z,Y)\xi - \eta(Y)Z + \eta(Y)\phi Z - 2\eta(Z)\eta(Y)\xi$$
$$-Bh(Z,Y).$$

On subtracting above two equations, we obtain

$$\phi P[Y,Z] = A_{\phi Y} Z - A_{\phi Z} Y + \eta(Y)Z - \eta(Z)Y + \eta(Z)\phi Y - \eta(Y)\phi Z$$

for $Y, Z \in D^{\perp}$. Thus we have

Theorem 3.3. Let M be a *CR*-submanifold of an LP-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection. Then the distribution $D^{\perp}$ is integrable if and only if

$$A_{\phi Z} Y - A_{\phi Y} Z = \eta(Y)Z - \eta(Z)Y + \eta(Z)\phi Y - \eta(Y)\phi Z$$

for all $Y, Z \in D^{\perp}$.

4. Parallel horizontal distributions of *CR*-submanifolds

Definition. The horizontal distribution D is said to be parallel with respect to the connection ∇ on M if $\nabla_X Y \in D$ for all vector fields $X, Y \in D$.

Proposition 4.1. Let M be a $\xi-$vertical *CR*-submanifold of a Lorentzian para-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection. Then the distribution $D^{\perp}$ is parallel with respect to the connection ∇ on M, if and only if, $A_N X \in D^{\perp}$ for each $X \in D^{\perp}$ and $N \in TM^{\perp}$.

Proof: Let $Y, X \in D^{\perp}$. Then using (2.11) and (2.12), we have

$$-A_{\phi Y} X + \nabla_X^{\perp} \phi Y = \phi \nabla_X Y + \phi h(Y, X) + \eta(Y) X + g(Y, X)\xi$$
$$-\eta(Y)\phi X + 2\eta(X)\eta(Y)\xi.$$

Taking inner product with $Z \in D,$ we get

$$-g(A_{\phi Y} X, Z) = g(\nabla_X Y, \phi Z).$$

Therefore, $\nabla_X Y = 0$ if and only if $A_{\phi Y} X \in D^{\perp}$ for all $X \in D^{\perp}$. From which our assertion follows.

Definition. A *CR*-submanifold M of an LP-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection is said to be totally geodesic if $h(X,Y) = 0.$ for $X \in D$ and $Y \in D^{\perp}$.

It follows immediately that a *CR*-submanifold is mixed totally geodesic if and only if $A_N X \in D$ for each $X \in D$ and $N \in T^{\perp}M$.

Let M be a mixed totally geodesic $\xi-$vertical *CR*-submanifold of an LP-Sasakian manifold $\overline{M}$ admitting a semi-symmetric non-metric connection.

From (2.8) (a), we have

$$(\overline{\nabla}_X \phi)N = 0$$

for $X \in D$ and $Y \in \phi D^{\perp}$.

Since $\overline{\nabla}_X \phi N = (\overline{\nabla}_X \phi)N + \phi(\overline{\nabla}_X N)$ so that $\overline{\nabla}_X \phi N = \phi(\overline{\nabla}_X N)$.

Using (2.11) and (2.12) in above equation, we get

$$\nabla_X (\phi N) = -\phi A_N X + \phi \nabla_X^{\perp} N$$

as $\phi A_N X \in D,$ so that $\nabla_X \phi N \in D$ if and only if $\phi \nabla_X^{\perp} N = 0.$

Thus we have the following theorem.

Theorem 4.2. Let M be a mixed totally $\xi-$vertical *CR*-submanifold of an LP-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection. Then the normal section $N \in \phi D^{\perp}$ is $D-$parallel if and only if $\nabla_X (\phi N) \in D$ for $X \in D.$

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