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Simultaneous Approximation by Complex Polynomials

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Abstract

In this paper we prove the following theorem for simultaneous approximation in L_p spaces for p<1:Let $m \in N$. For any $f \in A^m(\mathbb{D})$ and $n \geq m$, there exists a polynomial $P_n$of degree $\leq n$, such that for all j=0,…,mwehave$\|f^{(j)} - Pn^{(j)}\|_p \leq c(p)n^{j-m}E_{n-m}(f^{(m)})_p$where $c(p) > 0$ depends on p butit isindependent of n and f. Here $E_n(f^{(m)})$ =inf{$\| f^{(m)} - P \|_p$, P is polynomial of degree $\leq$ n}

Keywords: complex approximation, shape preserving approximation, interpolate-ing polynomial*s*

1. Introduction

As a fundamental definitions In our paper we have, $\mathcal{D}$= {$z \in \mathbb{C}$; |z|<1}be the open unit disk and let$E_n(f) = \inf\{\| f - P \|_p$,be the degree of best approximation of f by P, where Pis polynomial of degree $\leq$ n} .y best approximation to f from y if and only if$\| f - y \| = \inf \| f - w \|, w \in Y$.the following two results are known:

Theorem 1.1[6]: Let $m \in \mathbb{N}$.For any $f \in A^m(\mathbb{D})$and $n \geq m$, there exists a polynomial $P_n$of degree $\leq n$, such that for all j = 0, . . . , m we have$|f^{(j)}(z) - p_n^{(j)}(z)| \leq A_n^{j-m}\omega_1\left(f^{(m)};\frac{1}{n}\right), \forall z \in \partial D$, whereA is independent of nandz. Here $\omega_1(g;\delta) = \sup\{|f(u) - f(v)|; u, v \in \overline{\mathbb{D}}, |u - v| \leq \delta\}$.

Theorem 1.2[1]: Let us suppose that m, q, r $\in \mathbb{N}, f \in A^m(\mathbb{D})$ and consider the distinct points $|z_l| = 1$, l= 1, . . . , q. Then, for any $n \in \mathbb{N}, n \geq qm + r$,there exists a polynomial $p_n$of degree $\leq n$, such that for all j = 0, . . . , mwe have$|f^{(j)}(z) - p_n^{(j)}(z)| \leq c_n^{j-m}\omega_r^*\left(f^m;\frac{1}{n}\right), \forall z \in \partial\mathbb{D}$,and $p_n^{(j)}(z_l) = f^{(j)}(z_l), l = 1, \dots, q$,Where C is independent of n and z.Here$\omega_r^*(g;\delta) := \sup_{z\in\overline{\mathbb{D}}}\{E_{r-1}(g;\overline{\mathbb{D}} \cap B(z;\delta))\}$,$B(z;\delta) = \{\xi \in \mathbb{C}; |\xi - z| \leq \delta\}, E_m(g;M) := \inf\{\|g - p\|_M$; P complex polynomial of degree $\leq m\}$.Theorem 1.1was proved by (N.N.Vorob'ev) in [6] for the so-called domains of type1.1in the complex plane (including the unit disk) and Theorem1.2 was proved by (V.V. Andrievskii, I. Pritsker and R. Varga)in [1] for general continuain the complex plane (including the unit disk).In our paper , we will re-state them here for the particular case of unit disk only. Unfortunately, the constants appearing in these estimates, are claimed in the corresponding papers, as independent of n and zonly, without to be mentioned the independence of ftoo. Because of complicated technical details, it seems to be very difficult to deduce from the proofs of Theorems 1.1and 1.2that possibly these constants are independent of f too. For this reason, in the case of unit disk, by our main result Theorem 3.1, in the following section we improve these theorems to the spaces L_p,for $p < 1$ with new simple proof which clearly shows that the constant is independent of n, z and f too.

2. The Auxiliary results

To prove our theorem we need the following definitions and results for $f \in A^m(\mathbb{D}), \sigma_{n,h}(f)(z) = \frac{1}{h+1}\sum_{k=n-h}^{n} T_k(f)(z)$, where $T_k(f)(z) = \sum_{j=0}^{k}\frac{f^{(j)}(0)}{j!}z^j$

Lemma 2.1: $T'_k(f)(z) = T_{k-1}(f')(z)$.

Proof: $T_k(f)(z) = \sum_{j=0}^{k}\frac{f^{(j)}(0)}{j!}z^j$,k= 0, …, m$T'_k(f)(z) = \sum_{j=0}^{k} j\frac{f^{(j)}(0)}{j(j-1)!}z^{j-1} = \sum_{j=0}^{k} j\frac{(f^{(j-1)}(0))'}{j(j-1)!}z^{j-1} = \sum_{j=0}^{k-1}\frac{(f^{(j)})'(0)}{j!}z^j = T_{k-1}(f')(z)$■

Lemma 2.2: $\sigma_{2n,n-m}^{(k)}(f)(z) = \sigma_{2n-k,n-m}\left(f^{(k)}\right)(z)$

Proof: $\sigma_{n,h}(f)(z) = \frac{1}{h+1}\sum_{k=n-h}^{n} T_k(f)(z)$ where $n = 2n, h = n - m$,k=0,…,m, $\sigma_{2n,n-m}(f)(z) = \frac{1}{n-m+1}\sum_{k=2n-n+m}^{2n} T_k(f)(z)$ by lemma 2.1 we get

$$\sigma'_{2n,n-m}(f)(z) = \frac{1}{n-m+1}\sum_{k=2n-n+m}^{2n} T_{k-1}(f')(z) = \sigma_{2n-1,n-m}(f')(z)$$

$$\sigma''_{2n,n-m}(f)(z) = \frac{1}{n-m+1}\sum_{k=2n-n+m}^{2n} T_{k-2}(f'')(z) = \sigma_{2n-2,n-m}(f'')(z)$$

$$\sigma'''_{2n,n-m}(f)(z) = \frac{1}{n-m+1}\sum_{k=2n-n+m}^{2n} T_{k-3}(f''')(z) = \sigma_{2n-3,n-m}(f''')(z)$$

$\vdots$

$$\sigma^{k}_{2n,n-m}(f)(z) = \frac{1}{n-m+1}\sum_{k=2n-n+m}^{2n} T_{k-k}(f^{k})(z) = \sigma_{2n-k,n-m}(f^{k})(z) \blacksquare$$

Lemma 2.3 [5]: $\|f - \sigma_{n,m}(f)\|_\infty \leq A\sum_{j=0}^{n}\frac{E_{n-m+j}(f)_\infty}{m+j+1}$ where: $E_n(f)_\infty = \inf\{\|f - p\|_\infty, p \text{ is polynomail of degree} \leq n\}$,A is an absolute constant independent of f, n and m.

Lemma 2.4[3]: $\sup_{x\in[a,b]}|p_n(x)|^p \leq \frac{c_p}{b-a}\|p_n\|^p_{Lp[a,b]}$,$0 < p < \infty$

Proposition 2.5: $\| f - \sigma_{n,h}(f) \|_p \leq c(p)\sum_{j=0}^{n}\frac{E_{n-m+j}(f)_p}{m+j+1}$
where $c(p)$ is an absolute constant independent of f, n and m.

Proof: It is sufficient to show
$E_{n-m+j}(f)_\infty \leq c(p)E_{n-m+j}(f)_p$ Assume
$E_n(f)_p = \| f - Q \|_P$, $0 < P < \infty$, *degree* $Q \leq n$ *For* $n \in N$ *define* l *by* $2^l = n$
And P_{2^i} be a polynomial of best approximation of f. Then we may write $f - p_n = \sum_{j=1}^{\infty} p_{2^j_n} - p_{2^{j-1}_n}$. Thus S.N.Bernstein inequality [2] yields $E_n(f)_\infty = \| \sum_{j=0}^{n} p^j_{2n} - p^{j-1}_{2n} \|_\infty$ Then by lemma 2.4 we obtain

$$E_n(f)_\infty \leq c(p) \| \sum_{j=0}^{n} p^j_{2n} - p^{j-1}_{2n} \|_p \leq c(p) \| f - Q \|_P \quad \ldots\ldots\ldots (1)$$

$= c(p)E_n(f)_p$Hence$\| f-\sigma_{n,m}(f)\|_p \le \| f-\sigma_{n,m}(f)\|_\infty \le A\sum_{j=0}^{n}\frac{E_{n-m+j}(f)_\infty}{m+j+1}$

Then from(1)we get $\| f-\sigma_{n,m}(f)\|_p \le c(p)\sum_{j=0}^{n}\frac{E_{n-m+j}(f)_p}{m+j+1}$ ■

Lemma 2.6: Ifn $< n+1$,then $E_{n+1}(f)_p \le E_n(f)_p$

Proof: $E_{n+1}(f)_p = \inf_{p_{n+1}\in\prod_{n+1}} \| f-p_{n+1}\|_p$, where $\prod_n \le \prod_{n+1} \le \inf_{p_n\in\prod_n} \| f-p_n\|_p = E_n(f)_p$ ■

Lemma 2.7:$\| f-\sigma_{2n-k,n-m}(f)\|_p \le c(p)E_{n+m-k}(f)_p$
where $c(p)$is an absolute constant independent of f, n and m.

Proof: UsingProposition 2.5 to obtain

$$\| f-\sigma_{2n-k,n-m}(f)\|_p \le c(p)\sum_{j=0}^{2n-k}\frac{E_{n+m-k+j}(f)_p}{n-m+j+1}$$

$$\le c(p)E_{n+m-k}(f)_p\sum_{j=0}^{2n-k}\frac{1}{n-m+j+1} \le c(p)E_{n+m-k}(f)_p\frac{2n-k+1}{n-m+1}$$

byn $\ge$ msothat$\| f-\sigma_{2n-k,n-m}(f)\|_p \le c(p)E_{n+m-k}(f)_p\frac{2n+1}{n-m+1} \le c(p)(2m+1)E_{n+m-k}(f)_p$ ■

Lemma 2.8[4]: $E_n(f)_\infty \le c(p)n^{-m}E_{n-m}(f^{(m)})_\infty$
where $c(p)$is a constant depending only on p.

Proposition 2.9: $E_n(f)_p \le c(p)n^{-m}E_{n-m}(f^{(m)})_p$
where $c(p)$is a constant depending only on p.

Proof: Using lemma2.8 we have $E_n(f)_p \le c(p)E_n(f)_\infty \le c(p)n^{-m}E_{n-m}(f^{(m)})_\infty$
Using the same lines used for the proof of proposition 2.5we can get $E_n(f)_p \le c(p)E_n(f)_\infty \le c(p)n^{-m}E_{n-m}(f^{(m)})_\infty \le c(p)n^{-m}E_{n-m}(f^{(m)})_p$ ■

3. The main result

Theorem 3.1:$\| f^{(k)}-P_n^{(k)}\|_p \le c(p)n^{-m+k}E_{n-m}(f^{(m)})_p$

where $c(p)$ is a constant depending only on p.

Proof: $\| f^{(k)} - P_n^{(k)} \|_p \leq \| f^{(k)} - \sigma_{2n,n-m}^{(k)}(f) \|_p + \| \sigma_{2n,n-m}^{(k)}(f) - p_n^{(k)} \|_p$
Usinglemma2.2weget $\| f^{(k)} - P_n^{(k)} \|_p = \| f^{(k)} - \sigma_{2n-k,n-m}(f^{(k)}) \|_p + \| \sigma_{2n,n-m}^{(k)}(f) - p_n^{(k)} \|_p$ Usinglemma2.7weobtain
$\| f^{(k)} - P_n^{(k)} \|_p \leq c(p)E_{n+m-k}(f^{(k)})_p + \| (\sigma_{2n,n-m}(f) - p_n)^{(k)} \|_p$
by Bernstein's inequality we have $\| f^{(k)} - P_n^{(k)} \|_p \leq c(p)E_{n+m-k}(f^{(k)})_p + (2n)^k \| \sigma_{2n,n-m}(f) - p_n \|_p \leq c(p)E_{n+m-k}(f^{(k)})_p + C(p)n^k[\| \sigma_{2n,n-m}(f) - f \|_p + \| f - P_n \|_p \leq c(p)E_{n+m-k}(f^{(k)})_p + C(p)n^k[E_{n+m}(f) + E_n(f)]_p \leq c(p)E_{n+m-k}(f^{(k)})_p + C(p)n^k E_n(f)_p$ byProposition2.9weget $\| f^{(k)} - P_n^{(k)} \|_p \leq C(p)E_{n+m-k}(f^{(k)})_p + C(p)n^k n^{-m} E_{n-m}(f^{(m)})_p$ by Proposition 2. 9we obtain ,for $n + m - k$ $\| f^{(k)} - P_n^{(k)} \|_p \leq C(p)(n + m - k)^{-m+k} E_n(f^{(m)})_p + C(p)n^{-m+k} E_{n-m}(f^{(m)})_p \leq C(p)n^{-m+k} E_{n-m}(f^{(m)})_p$ ■

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