Int. Journal of Math. Analysis, Vol. 7, 2013, no. 26, 1285 - 1300
HIKARI Ltd, www.m-hikari.com

On Defining Complex Uncertainty Data Points by Type-2 Fuzzy Number: Two Specials Cases

Rozaimi Zakaria

Department of Mathematics, Faculty of Science and Technology
Universiti Malaysia Terengganu, Malaysia
rozaimi_z@yahoo.com

Abd. Fatah Wahab

Department of Mathematics, Faculty of Science and Technology
Universiti Malaysia Terengganu, Malaysia
fatah@umt.edu.my

Abstract

In this paper, we will discuss about the defining complex uncertainty data points based on the definition of type-2 fuzzy number (T2FN) concepts. This defining process of complex uncertainty data points have two cases that are the complex uncertainty data points which defined when the complex uncertainty happened at their membership function values, and the complex uncertainty data points are defined when the complex uncertainty happened at their footprint. After the complex uncertainty data points had been defined which known as type-2 fuzzy data points (T2FDPs) for both cases, we applied the fuzzification (alpha-cut operation), type-reduction and defuzzification processes to show the data forms processes from T2FDPs to crisp T2FDPs solution which in singular data forms. These steps for obtaining the crisp T2FDPs solution are being modeled through non-uniform rational B-spline (NURBS) curve function as the addition for more understanding.

Keywords: Type-2 fuzzy number, type-2 fuzzy data points, alpha-cut operation, type-reduction, defuzzification

1 Introduction

In dealing the uncertain matter, various types of theories and concepts are required in order to define the uncertainty based on the expert research fields where the uncertainty problems are arisen. The arising uncertainty problems based on the imperfections of human thoughts in measuring and valuating and the influenced of environmental circumstance, which lead the researchers have to face the difficulty in making the decision. These uncertainty problems are also existed in data collective sets, which known as the uncertainty data. Usually, these uncertainty data set are rejected for analysis due to their inaccuracy and only used the perfect data. Therefore, the analysis of perfect data became lack of results because the part of the data set are being rejected.

However, this problem can be solved by using the specific theory and concept such as type-1 fuzzy set theory (fuzzy set theory) which was introduced by Zadeh [21]. This theory became the fundamental theory in defining the uncertainty problem like the type-1 fuzzy number(T1FN) concept. This T1FN concept used to define the uncertainty problem involving measuring and handling the uncertainty real number form.

Although the T1FN concept was used to define the uncertainty data, but there is a problem which T1FN unable to define the uncertainty. The problem arises when the uncertainty data become more uncertainty, which known as complex uncertainty data. Therefore, we need a high level definition of T1FN to define them. Then, the definition of T2FN concepts was introduced based on the type-2 fuzzy set theory (T2FST) where T2FST was discussed by Zadeh in his trilogy papers [22].

Therefore, in this paper we will discuss about the way complex uncertainty data points can be defined through T2FN concepts, which involving two cases. The two cases are either the type-2 membership function values are complex uncertainty where the footprints are in single values or the footprints are complex uncertainty which the type-2 membership function values are certain. Then, these two cases of defining the complex uncertainty data points will be fuzzified, reduced and defuzzified in order to obtain crisp T2FDPs solution. In addition, we demonstrate the crisp T2FDPs solution in a curve illustrated including the fuzzification, type-reduction and defuzzification processes by using NURBS curve function. The modeling of both cases through curves representation can give more comprehended on the T2FDPs form.

This paper constructed as follows: Section 2 gives the basic definition of T2FST, T2FN and type-2 fuzzy relations(T2FR) which are being used to define the complex uncertainty data points where in Section 3. Section 4 discusses about T2FDPs in different membership function values(T2FDPs of membership function) by assume this kind of T2FDPs as the case 1. Meanwhile, for case 2, this case is the T2FDPs, which has the footprint uncertainty but the same membership function values(T2FDPs of footprint) is in Section 5. For both cases, we give the numerical example together with their illustration, which modeled by NURBS curve function where in Section 6. Then, in Section 7, the discussion and conclusion of both cases are explained here.

2 Preliminaries

In this Section 2, we will discuss about the definition of T2SFT, T2FN and T2FR which these definitions will lead us to define the complex uncertainty data points.

Definition 2.1. A type-2 fuzzy set(T2FS), denoted $\ddot{A}$, is characterized by a type-2 membership function $\mu_{\ddot{A}}(x,u)$, where $x \in X$ and $u \in U_x \subseteq [0,1]$ that is,

$$\ddot{A} = \left\{ \left((x,u), \mu_{\ddot{A}}(x,u) \right) \mid \forall x \in X, \forall u \in U_x \subseteq [0,1] \right\}$$

in which, $0 \leq \mu_{\ddot{A}}(x,u) \leq 1$ [15].

Definition 2.2. A T2FN is broadly defined as a T2FS that has a numerical domain. An interval T2FS is defined using the following four constraints, where $\ddot{A}_\alpha = \left\{ [a^\alpha, b^\alpha], [c^\alpha, d^\alpha] \right\}$, $\forall \alpha \in [0,1]$, $\forall\ a^b, b^\alpha, c^\alpha, d^\alpha \in$ R (Fig. 2.1) [1,23,24]:

1. $a^\alpha \leq b^\alpha \leq c^\alpha \leq d^\alpha$
2. $[a^\alpha, d^\alpha]$ and $[b^\alpha, c^\alpha]$ generate a function that is convex and $[a^\alpha, d^\alpha]$ generate a function is normal.
3. $\forall \alpha_1, \alpha_2 \in [0,1] : (\alpha_2 > \alpha_1) \Rightarrow \left(\left[a^{\alpha_1}, c^{\alpha_1} \right] \supset \left[a^{\alpha_2}, c^{\alpha_2} \right], \right.$ $\left. \left[b^{\alpha_1}, d^{\alpha_1} \right] \supset \left[b^{\alpha_2}, d^{\alpha_2} \right] \right)$, for $c^{\alpha_2} \geq b^{\alpha_2}$.
4. If the maximum of the membership function generated by $[b^\alpha, c^\alpha]$ is the level α_m, that is, $[b^{\alpha_m}, c^{\alpha_m}]$, then $\left[b^{\alpha_m}, c^{\alpha_m} \right] \subset \left[a^{\alpha=1}, d^{\alpha=1} \right]$.

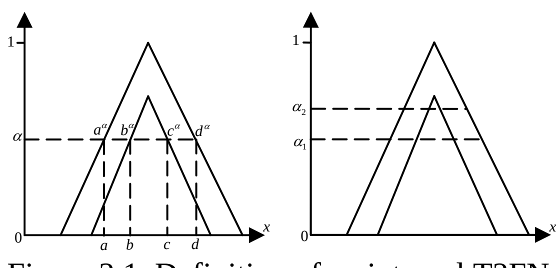


Figure 2.1. Definition of an interval T2FN.

Definition 3. A T2FR is a type-2 fuzzy set which defined on the Cartesian product of crisp sets $X_1, X_2, ..., X_n$, where the tuples $(x_1, x_2, ..., x_n)$ have varying degree of membership which are type-1 fuzzy sets(T1FSs) [11].

Definition 4. Let $X, Y, U_x, V_y \subseteq R$ and

$$\ddot{\overleftrightarrow{A}} = \left\{\left((x,u), \mu_{\ddot{\overleftrightarrow{A}}}(x,u)\right) \mid \forall x \in X, \forall u \in U_x \subseteq [0,1]\right\} \text{ and}$$

$$\ddot{\overleftrightarrow{B}} = \left\{\left((y,v), \mu_{\ddot{\overleftrightarrow{B}}}(y,v)\right) \mid \forall y \in Y, \forall v \in V_y \subseteq [0,1]\right\}$$

are two T2FSs. Then, $\ddot{\overleftrightarrow{R}} = \left\{\left(\left((x,u),(y,v)\right), \quad \mu_{\ddot{\overleftrightarrow{R}}}\left(\mu_{\ddot{\overleftrightarrow{A}}}(x,u), \mu_{\ddot{\overleftrightarrow{B}}}(y,v)\right)\right) \mid \left(\forall x \in X, \forall u \in U_x\right) \times \left(\forall y \in Y, \forall u \in V_y\right) \subseteq [0,1]\right\}$ is a T2FR on $\ddot{\overleftrightarrow{A}}$ and $\ddot{\overleftrightarrow{B}}$ if $\mu_{\ddot{\overleftrightarrow{R}}}\left(\mu_{\ddot{\overleftrightarrow{A}}}(x,u), \mu_{\ddot{\overleftrightarrow{B}}}(y,v)\right) \leq \mu_{\ddot{\overleftrightarrow{A}}}(x,u)$,

$$\forall\left((x,u),(y,v)\right) \in \left(\forall x \in X, \forall u \in U_x\right) \times \left(\forall y \in Y, \forall v \in V_y\right)$$ [23,24].

3 Type-2 Fuzzy Data Points

This section discusses about defining the complex uncertainty data points which finally known as T2FDPs after been defined. In this section also, we will discuss about the fuzzification(alpha-cut operation), type-reduction and defuzzification processes for obtaining the crisp T2FDPs solution as the final result which is in singular data forms.

Based on the T2FN and T2FR definitions, the T2FDPs definition can be given as follows.

Definition 3.1. Let $P = \{x \mid x \text{ type-2 fuzzy point}\}$ and $\ddot{\overleftrightarrow{P}} = \{P_i \mid P_i \text{ data point}\}$ which is set of type-2 fuzzy data point with $P_i \in P \subset X$, where X is a universal set and $\mu_P(P_i) : P \to [0,1]$ is the membership function which defined as $\mu_P(P_i) = 1$ and formulated by $\ddot{\overleftrightarrow{P}} = \{(P_i, \mu_p(P_i)) \mid P_i \in \mathrm{R}\}$. Therefore,

$$\mu_P(P_i) = \begin{cases} 0 & \text{if } P_i \notin X \\ c \in (0,1) & \text{if } P_i \,\ddot{\in}\, X \\ 1 & \text{if } P_i \in X \end{cases} \tag{3.1}$$

with $\mu_P(P_i) = \left\langle \mu_P(\ddot{\overleftrightarrow{P}}_i^{\leftarrow}), \mu_P(P_i), \mu_P(\ddot{\overleftrightarrow{P}}_i^{\rightarrow}) \right\rangle$ which $\mu_P(\ddot{\overleftrightarrow{P}}_i^{\leftarrow})$ and $\mu_P(\ddot{\overleftrightarrow{P}}_i^{\rightarrow})$ are left and right footprint of membership values with $\mu_P(\ddot{\overleftrightarrow{P}}_i^{\leftarrow}) = \left\langle \mu_P(\overleftrightarrow{P}_i^{\overleftarrow{\leftarrow}}), \mu_P(\overleftrightarrow{P}_i^{\leftarrow}), \mu_P(\overleftrightarrow{P}_i^{\overrightarrow{\leftarrow}}) \right\rangle$ where, $\mu_P(\overleftrightarrow{P}_i^{\overleftarrow{\leftarrow}})$, $\mu_P(\overleftrightarrow{P}_i^{\leftarrow})$ and $\mu_P(\overleftrightarrow{P}_i^{\overrightarrow{\leftarrow}})$ are left-left, left, right-left membership grade values and $\mu_P(\overleftrightarrow{P}_i^{\overrightarrow{\rightarrow}})$, $\mu_P(\overleftrightarrow{P}_i^{\rightarrow})$ and $\mu_P(\overleftrightarrow{P}_i^{\overleftarrow{\rightarrow}})$ are right-right, right, left-right membership grade values, which can be written as

$$\overset{\leftrightarrow}{P} = \left\{ \overset{\leftrightarrow}{P}_i : i = 0,1,2,\ldots,n \right\} \tag{3.2}$$

for every i, $\overset{\leftrightarrow}{P}_i = \left\langle \overset{\leftrightarrow}{P}_i^{\leftarrow}, P_i, \overset{\leftrightarrow}{P}_i^{\rightarrow} \right\rangle$ with $\overset{\leftrightarrow}{P}_i^{\leftarrow} = \left\langle \overset{\leftarrow}{\overset{\leftrightarrow}{P}}{}_i^{\leftarrow}, \overset{\leftrightarrow}{P}_i^{\leftarrow}, \overset{\rightarrow}{\overset{\leftrightarrow}{P}}{}_i^{\leftarrow} \right\rangle$ where $\overset{\leftarrow}{\overset{\leftrightarrow}{P}}{}_i^{\leftarrow}$, $\overset{\leftrightarrow}{P}_i^{\leftarrow}$ and $\overset{\rightarrow}{\overset{\leftrightarrow}{P}}{}_i^{\leftarrow}$ are left-left, left and right-left T2FDPs and $\overset{\leftrightarrow}{P}_i^{\rightarrow} = \left\langle \overset{\leftarrow}{\overset{\leftrightarrow}{P}}{}_i^{\rightarrow}, \overset{\leftrightarrow}{P}_i^{\rightarrow}, \overset{\rightarrow}{\overset{\leftrightarrow}{P}}{}_i^{\rightarrow} \right\rangle$ where $\overset{\leftarrow}{\overset{\leftrightarrow}{P}}{}_i^{\rightarrow}$, $\overset{\leftrightarrow}{P}_i^{\rightarrow}$ and $\overset{\rightarrow}{\overset{\leftrightarrow}{P}}{}_i^{\rightarrow}$ are left-right, right and right-right T2FDPs respectively [23,24]. This can be illustrated as in Fig. 3.1.

The illustration of T2FDP was shown in Fig. 3.1 which T1FDP becomes the primary membership function bounded by upper bound, $\left[\overset{\leftarrow}{\overset{\leftrightarrow}{P}}{}^{\leftarrow}, P, \overset{\rightarrow}{\overset{\leftrightarrow}{P}}{}^{\rightarrow} \right]$ and lower bound, $\left[\overset{\rightarrow}{\overset{\leftrightarrow}{P}}{}^{\leftarrow}, P, \overset{\leftarrow}{\overset{\leftrightarrow}{P}}{}^{\rightarrow} \right]$ respectively. The process of defining T2FDP can be shown through Fig. 3.2.

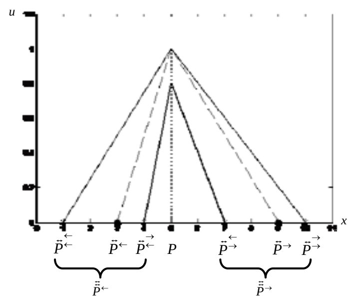


Figure 3.1. T2FDP around 5.

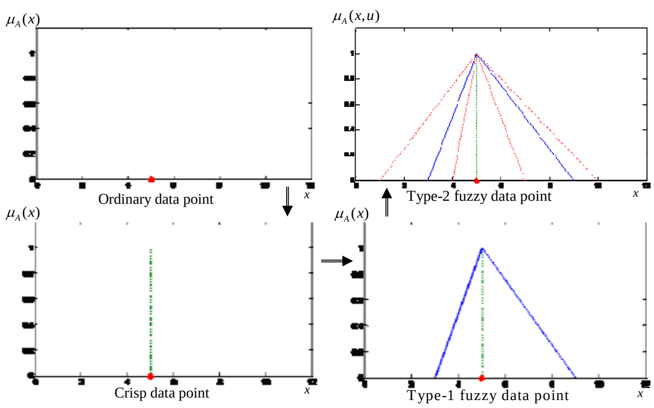


Figure 3.2. Process of defining T2FDP.

Definition 3.2. Based on Def. 3.1, let $\overleftrightarrow{P}$ be the set of T2FDPs with $\overleftrightarrow{P}_i \in \overleftrightarrow{P}$ where $i = 0,1,...,n-1$. Then $\overleftrightarrow{P}^{\alpha}$ is the α-cut operation of T2FDPs which is given in equation as follows.

$$
\begin{aligned}
\overset{\leftrightarrow}{\overleftrightarrow{P}}{}_i^{\alpha} &= \left\langle \overset{\leftrightarrow}{\overleftrightarrow{P}}{}_i^{\alpha\leftarrow}, P_i, \overset{\leftrightarrow}{\overleftrightarrow{P}}{}_i^{\alpha\rightarrow} \right\rangle \\
&= \left\langle \left\langle \overset{\leftarrow}{\overleftrightarrow{P}}{}_i^{\alpha\leftarrow}; \overleftrightarrow{P}_i^{\alpha\leftarrow}; \overset{\rightarrow}{\overleftrightarrow{P}}{}_i^{\alpha\leftarrow} \right\rangle, P_i, \left\langle \overset{\leftarrow}{\overleftrightarrow{P}}{}_i^{\alpha\rightarrow}; \overleftrightarrow{P}_i^{\alpha\rightarrow}; \overset{\rightarrow}{\overleftrightarrow{P}}{}_i^{\alpha\rightarrow} \right\rangle \right\rangle \\
&= \left\langle \left[\left(P_i - \left\langle \overset{\leftarrow}{\overleftrightarrow{P}}{}_i^{\leftarrow}; \overleftrightarrow{P}_i^{\leftarrow}; \overset{\rightarrow}{\overleftrightarrow{P}}{}_i^{\leftarrow} \right\rangle \right) \alpha + \left\langle \overset{\leftarrow}{\overleftrightarrow{P}}{}_i^{\leftarrow}; \overleftrightarrow{P}_i^{\leftarrow}; \overset{\rightarrow}{\overleftrightarrow{P}}{}_i^{\leftarrow} \right\rangle \right], P_i, \right. \\
&\qquad \left. \left[-\left(\left\langle \overset{\leftarrow}{\overleftrightarrow{P}}{}_i^{\rightarrow}; \overleftrightarrow{P}_i^{\rightarrow}; \overset{\rightarrow}{\overleftrightarrow{P}}{}_i^{\rightarrow} \right\rangle - P_i \right) \alpha + \left\langle \overset{\leftarrow}{\overleftrightarrow{P}}{}_i^{\rightarrow}; \overleftrightarrow{P}_i^{\rightarrow}; \overset{\rightarrow}{\overleftrightarrow{P}}{}_i^{\rightarrow} \right\rangle \right] \right\rangle
\end{aligned}
\qquad (3.3)
$$

This definition can be illustrated through Fig. 3.3.

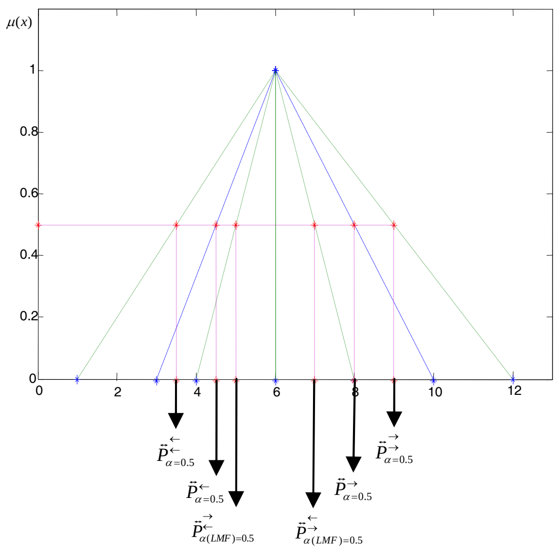


Figure 3.3. The alpha-cut operation towards T2FDP.

After the fuzzifictaion processes had been applied towards T2FDPs, then the next process in order to obtain crisp T2FDP solution is the type-reduction process [2-8,10,12-15,18,19]. This type-reduction process is the process which reduces the T2FDPs to become T1FDPs for allowing the defuzzification process of type-1 can be applied to obtain the crisp T2FDPs solution. Therefore, the type-reduction and defuzzification definitions can be given by Def. 3.3 and Def 3.4 as follows.

Definition 3.3. Let $\overleftrightarrow{P}_i$ be a set $(n+1)$ T2FDPs, then the type-reduction method of α-T2FDPs(after fuzzification), $\overline{\overleftrightarrow{P}}_i^{\alpha}$ is defined by

$$\overline{\overleftrightarrow{P}}^{\alpha}=\left\{\overline{\overleftrightarrow{P}}_i^{\alpha}=\left\langle \overline{\overleftrightarrow{P}}_i^{\alpha\leftarrow},P_i,\overline{\overleftrightarrow{P}}_i^{\alpha\rightarrow}\right\rangle ;\ i=0,1,\ldots,n\right\} \tag{3.4}$$

where $\overline{\overleftrightarrow{P}}_i^{\alpha\leftarrow}$ is left type-reduction of α-cut T2FDPs, $\overline{\overleftrightarrow{P}}_i^{\alpha\leftarrow}=\frac{1}{3}\sum_{i=0,\ldots,n}\left\langle \overleftarrow{\overleftrightarrow{P}}_i^{\alpha\leftarrow}+\overleftrightarrow{P}_i^{\alpha\leftarrow}+\overrightarrow{\overleftrightarrow{P}}_i^{\alpha\leftarrow}\right\rangle$, P_i is the crisp point and $\overline{\overleftrightarrow{P}}_i^{\alpha\rightarrow}$ is right type-reduction of α-cut T2FDPs, $\overline{\overleftrightarrow{P}}_i^{\alpha\rightarrow}=\frac{1}{3}\sum_{i=0,\ldots,n}\left\langle \overleftarrow{\overleftrightarrow{P}}_i^{\alpha\rightarrow}+\overleftrightarrow{P}_i^{\alpha\rightarrow}+\overrightarrow{\overleftrightarrow{P}}_i^{\alpha\rightarrow}\right\rangle$ [23,24].

Definition 3.4. Let α-TR is the type-reduction method after α-cut process had been applied for every T2FDPs, $\overline{\overleftrightarrow{P}}_i^{\alpha}$. Then $\overline{\overline{\overleftrightarrow{P}}}_i^{\alpha}$ named as defuzzification T2FDPs for $\overline{\overleftrightarrow{P}}_i^{\alpha}$ if for every $\overline{\overleftrightarrow{P}}_i^{\alpha}\in\overline{\overleftrightarrow{P}}^{\alpha}$,

$$\overline{\overline{\overleftrightarrow{P}}}^{\alpha}=\left\{\overline{\overline{\overleftrightarrow{P}}}_i^{\alpha}\right\}\qquad \text{for}\qquad i=0,1,\ldots,n \tag{3.5}$$

where for every $\overline{\overline{\overleftrightarrow{P}}}_i^{\alpha}=\frac{1}{3}\sum_{i=0}<\overline{\overleftrightarrow{P}}_i^{\alpha\leftarrow},P_i,\overline{\overleftrightarrow{P}}_i^{\alpha\rightarrow}>$ [23,24]. The process in defuzzifying T2FDPs can be illustrated at Fig. 3.4.

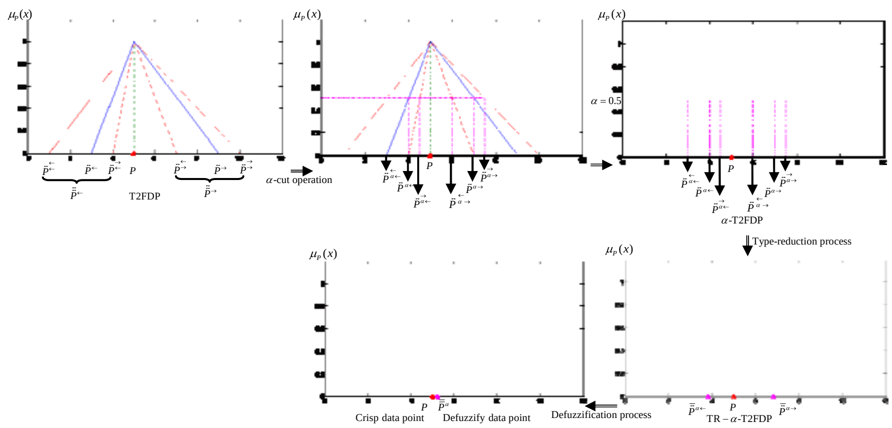


Figure 3.4. Defuzzification process of T2FDP.

4 Type-2 Fuzzy Data Points: Case 1

In this section, we will discuss about the definition of T2FDPs which has the uncertain membership function. The uncertain membership function is when all the upper, crisp(type-1) and lower footprints are equal. Therefore, this following definition is given based on Def. 3.1.

Definition 4.1. Based on Def. 3.1, the T2FDPs of membership function, ${}^{MF}\overleftrightarrow{\overleftrightarrow{P}}$, can be written as

$$ {}^{MF}\overleftrightarrow{\overleftrightarrow{P}} = \left\{ {}^{MF}\overleftrightarrow{\overleftrightarrow{P}}_i : i = 0,1,\ldots,n \right\} \tag{4.1}$$

where for every *i*, ${}^{MF}\overleftrightarrow{\overleftrightarrow{P}}_i = \left\langle \overleftrightarrow{\overleftrightarrow{P}}_i^{\leftarrow}, P_i, \overleftrightarrow{\overleftrightarrow{P}}_i^{\rightarrow} \right\rangle$ with $\overleftrightarrow{\overleftrightarrow{P}}_i^{\leftarrow} = \left\langle \overleftarrow{\overleftrightarrow{\overleftrightarrow{P}}_i^{\leftarrow}}, \overleftrightarrow{\overleftrightarrow{P}}_i^{\leftarrow}, \overrightarrow{\overleftrightarrow{\overleftrightarrow{P}}_i^{\leftarrow}} \right\rangle$ where $\overleftarrow{\overleftrightarrow{\overleftrightarrow{P}}_i^{\leftarrow}}$, $\overleftrightarrow{\overleftrightarrow{P}}_i^{\leftarrow}$ and $\overrightarrow{\overleftrightarrow{\overleftrightarrow{P}}_i^{\leftarrow}}$ are left-left, left and right-left T2FDPs which $\overleftarrow{\overleftrightarrow{\overleftrightarrow{P}}_i^{\leftarrow}} = \overleftrightarrow{\overleftrightarrow{P}}_i^{\leftarrow} = \overrightarrow{\overleftrightarrow{\overleftrightarrow{P}}_i^{\leftarrow}}$ and ${}^{MF}\overleftrightarrow{\overleftrightarrow{P}}_i^{\rightarrow} = \left\langle \overleftarrow{\overleftrightarrow{\overleftrightarrow{P}}_i^{\rightarrow}}, \overleftrightarrow{\overleftrightarrow{P}}_i^{\rightarrow}, \overrightarrow{\overleftrightarrow{\overleftrightarrow{P}}_i^{\rightarrow}} \right\rangle$ where $\overleftarrow{\overleftrightarrow{\overleftrightarrow{P}}_i^{\rightarrow}}$, $\overleftrightarrow{\overleftrightarrow{P}}_i^{\rightarrow}$ and $\overrightarrow{\overleftrightarrow{\overleftrightarrow{P}}_i^{\rightarrow}}$ are left-right, right and right-right T2FDPs which $\overleftarrow{\overleftrightarrow{\overleftrightarrow{P}}_i^{\rightarrow}} = \overleftrightarrow{\overleftrightarrow{P}}_i^{\rightarrow} = \overrightarrow{\overleftrightarrow{\overleftrightarrow{P}}_i^{\rightarrow}}$. This definition can be illustrated by Fig 4.1 as follows.

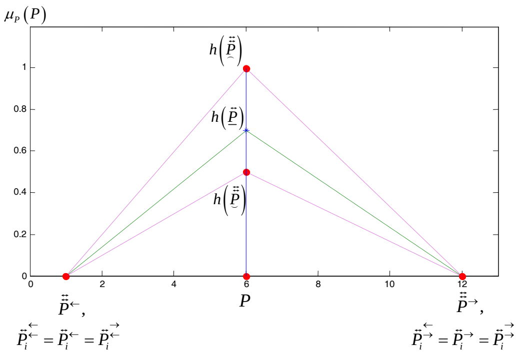


Figure 4.1. The T2FDP with the equal footprint.

Fig. 4.1 shows that the illustration of Def. 4.1, which shows that the left and right footprints are equal to a single value with different of height values. The heights of upper membership function(UMF), $h\left(\underset{\frown}{\overleftrightarrow{\overleftrightarrow{P}}}\right)$, type-1 membership function(T1MF), $h\left(\underline{\overleftrightarrow{P}}\right)$ and lower membership function(LMF), $h\left(\underset{\smile}{\overleftrightarrow{\overleftrightarrow{P}}}\right)$ can be formulized as $\mu_P(P) = \left\langle h\left(\underset{\frown}{\overleftrightarrow{\overleftrightarrow{P}}}\right), h\left(\underline{\overleftrightarrow{P}}\right), h\left(\underset{\smile}{\overleftrightarrow{\overleftrightarrow{P}}}\right) \right\rangle \in (0,1]$ where, $0 < h\left(\underset{\frown}{\overleftrightarrow{\overleftrightarrow{P}}}\right) < h\left(\underline{\overleftrightarrow{P}}\right) < h\left(\underset{\smile}{\overleftrightarrow{\overleftrightarrow{P}}}\right) = 1$.

After the T2FDPs of membership function has been defined, then the fuzzification processes of these T2FDPs are next, which will be given as follows.

Definition 4.2. Based on Def. 3.1, let $\overleftrightarrow{\overleftrightarrow{P}}$ be the set of T2FDPs with $\overleftrightarrow{\overleftrightarrow{P}}_i \in \overleftrightarrow{\overleftrightarrow{P}}$ where $i = 0,1,\ldots,n-1$. Then, ${}^{MF}\overleftrightarrow{\overleftrightarrow{P}}_\alpha$ is the α-cut operation of T2FDPs of membership function which is given as equation as follows.

$$\begin{aligned}
{}^{MF}\overleftrightarrow{\ddot{P}}_{i_\alpha} &= \left\langle \overleftrightarrow{\ddot{P}}^{\leftarrow}_{i_\alpha}, P_i, \overleftrightarrow{\ddot{P}}^{\rightarrow}_{i_\alpha} \right\rangle \\
&= \left\langle \left\langle \overleftarrow{\vec{\ddot{P}}}^{\leftarrow}_{i_\alpha}; \vec{\ddot{P}}^{\leftarrow}_{i_\alpha}; \overrightarrow{\vec{\ddot{P}}}^{\leftarrow}_{i_\alpha} \right\rangle, P_i, \left\langle \overleftarrow{\vec{\ddot{P}}}^{\rightarrow}_{i_\alpha}; \vec{\ddot{P}}^{\rightarrow}_{i_\alpha}; \overrightarrow{\vec{\ddot{P}}}^{\rightarrow}_{i_\alpha} \right\rangle \right\rangle \\
&= \left\langle \left[\left(P_i - \left\langle \overleftarrow{\vec{\ddot{P}}}^{\leftarrow}_{i} = \vec{\ddot{P}}^{\leftarrow}_{i} = \overrightarrow{\vec{\ddot{P}}}^{\leftarrow}_{i} \right\rangle \right) \left(\alpha; \frac{\alpha}{\alpha_{T1MF}}; \frac{\alpha}{\alpha_{LMF}} \right) + \left\langle \overleftarrow{\vec{\ddot{P}}}^{\leftarrow}_{i} = \vec{\ddot{P}}^{\leftarrow}_{i} = \overrightarrow{\vec{\ddot{P}}}^{\leftarrow}_{i} \right\rangle \right], P_i, \right. \\
&\quad \left. \left[-\left(\left\langle \overleftarrow{\vec{\ddot{P}}}^{\rightarrow}_{i} = \vec{\ddot{P}}^{\rightarrow}_{i} = \overrightarrow{\vec{\ddot{P}}}^{\rightarrow}_{i} \right\rangle - P_i \right) \left(\frac{\alpha}{\alpha_{LMF}}; \frac{\alpha}{\alpha_{T1MF}}; \alpha \right) + \left\langle \overleftarrow{\vec{\ddot{P}}}^{\rightarrow}_{i} = \vec{\ddot{P}}^{\rightarrow}_{i} = \overrightarrow{\vec{\ddot{P}}}^{\rightarrow}_{i} \right\rangle \right] \right\rangle
\end{aligned} \tag{4.2}$$

This definition can be illustrated through Fig. 4.2.

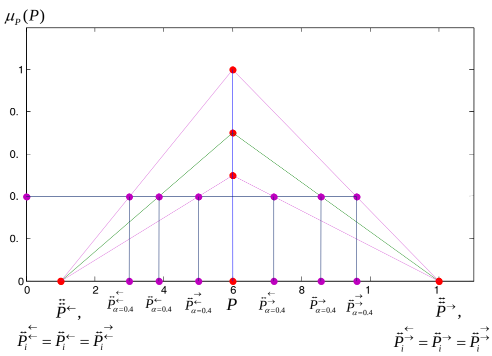


Figure 4.2. The fuzzification(alpha-cut operation) process against T2FDPs of membership function, ${}^{MF}\overleftrightarrow{\ddot{P}}_{\alpha}$.

After the fuzzificatiopn process of ${}^{MF}\overleftrightarrow{\ddot{P}}$ has been applied, then the type-reduction and defuzzification processes are being applied in order to obtain crisp T2FDPs solution as the final result which these T2FDPs become singular data points. Therefore, the definitions of type-reduction and defuzzification processes can be given as following definitions.

Definition 4.3. Let ${}^{MF}\overleftrightarrow{\ddot{P}}_{i}$ be a set $(n+1)$ T2FDPs of membership function, then the type-reduction method of α -T2FDPs membership function(after fuzzification), ${}^{MF}\overline{\overline{\ddot{P}}}_{i_\alpha}$ is defined by

$${}^{MF}\overline{\overline{\ddot{P}}}_{\alpha} = \left\{ {}^{MF}\overline{\overline{\ddot{P}}}_{i_\alpha} = \left\langle \overline{\overline{\ddot{P}}}^{\leftarrow}_{i_\alpha}, P_i, \overline{\overline{\ddot{P}}}^{\rightarrow}_{i_\alpha} \right\rangle ; \ i = 0,1,\ldots,n \right\} \tag{4.3}$$

where $\overline{\overline{\ddot{P}}}^{\leftarrow}_{i_\alpha}$ is left type-reduction of α -cut T2FDPs,

$\overleftrightarrow{\overline{\overline{P}}}_{i_\alpha}^{\leftarrow} = \frac{1}{3}\sum_{i=0,\dots,n}\left\langle \overleftrightarrow{P}_{i_\alpha}^{\overleftarrow{\leftarrow}} + \overleftrightarrow{P}_{i_\alpha}^{\leftarrow} + \overleftrightarrow{P}_{i_\alpha}^{\overrightarrow{\leftarrow}} \right\rangle$, P_i is the crisp point and $\overleftrightarrow{\overline{\overline{P}}}_{i_\alpha}^{\rightarrow}$ is right type-reduction of α -cut T2FDPs, $\overleftrightarrow{\overline{\overline{P}}}_{i_\alpha}^{\rightarrow} = \frac{1}{3}\sum_{i=0,\dots,n}\left\langle \overleftrightarrow{P}_{i_\alpha}^{\overleftarrow{\rightarrow}} + \overleftrightarrow{P}_{i_\alpha}^{\rightarrow} + \overleftrightarrow{P}_{i_\alpha}^{\overrightarrow{\rightarrow}} \right\rangle$.

Definition 4.4. Let α -TR is the type-reduction method after α -cut process which had been applied for every T2FDPs of T2FPDs, ${}^{MF}\overleftrightarrow{\overline{\overline{P}}}_i^{\alpha}$. Then ${}^{MF}\overline{\overline{P}}_{i_\alpha}$ named as defuzzification T2FDPs of membership function for ${}^{MF}\overleftrightarrow{\overline{P}}_{i_\alpha}$ if for every ${}^{MF}\overleftrightarrow{\overline{\overline{P}}}_{i_\alpha} \in {}^{MF}\overleftrightarrow{\overline{\overline{P}}}_{\alpha}$,

$$ {}^{MF}\overline{\overline{P}}_{\alpha} = \left\{ {}^{MF}\overline{\overline{P}}_{i_\alpha} \right\} \quad \text{for} \quad i = 0,1,\dots,n \tag{4.4} $$

where for every ${}^{MF}\overline{\overline{P}}_{i_\alpha} = \frac{1}{3}\sum_{i=0} < \overleftrightarrow{\overline{\overline{P}}}_{i_\alpha}^{\leftarrow}, P_i, \overleftrightarrow{\overline{\overline{P}}}_{i_\alpha}^{\rightarrow} >$.

5 Type-2 Fuzzy Data Points: Case 2

This Section 5 will discuss about the T2FDPs of footprint uncertainty definition were also known as perfectly normal T2FDPs which mentioned in [23]. Therefore, T2FDPs of footprint uncertainty definition can be given as the perfectly normal T2FDPs which involving the fuzzification, type-reduction and defuzzification process's definitions. A a result, the definition of T2FDPs of footprint uncertainty can be referred to [23].

6 Two Type-2 Fuzzy Data Points Modeling

After the two types of T2FDPs had been discussed in Section 4 and Section 5 respectively, then we demonstrate them by modeling these following numerical examples as in Table 6.1 and Table 6.2. Table 6.1 give the numerical example of calculating the T2FDPs of membership function, which involving the fuzzification, type-reduction and defuzzification processes. For Table 6.2 is same as the Table 6.1 where Table 6.2 give the numerical example of T2FDPs of the footprint(perfectly normal T2FDPs).

Then, both types of T2FDPs are being modeled through curves by using NURBS curves [9,16,17,20] which are shown in Table 6.3 as the comparison between those two types of T2FDPs. This Table 6.3 give us more understanding on the both types of T2FPDs forms.

Table 6.1. The fuzzification, type-reduction and defuzzification processes of T2FDPs of membership function.

T2FDPs of Memebrship Function	${}^{MF}\overleftrightarrow{\overleftrightarrow{P}}_i$						
	$\overleftrightarrow{P}_i^{\overset{\leftarrow}{\leftarrow}}$	$\overleftrightarrow{P}_i^{\leftarrow}$	$\overleftrightarrow{P}_i^{\overset{\rightarrow}{\leftarrow}}$	P_i	$\overleftrightarrow{P}_i^{\overset{\leftarrow}{\rightarrow}}$	$\overleftrightarrow{P}_i^{\rightarrow}$	$\overleftrightarrow{P}_i^{\overset{\rightarrow}{\rightarrow}}$
$i=0$	(-11, 0)			(-5, 0)	(6,0)		
$i=1$	(15, 26)			(15, 20)	(15,14)		
$i=2$	(15, -15)			(10, -20)	(5,-25)		
$i=3$	(32, 10)			(40, 10)	(48,10)		
α -Cut	${}^{MF}\overleftrightarrow{\overleftrightarrow{P}}_{i_\alpha}$						
	$\overleftrightarrow{P}_{i_{\alpha=0.5}}^{\overset{\leftarrow}{\leftarrow}}$	$\overleftrightarrow{P}_{i_{\alpha=0.5/0.8}}^{\leftarrow}$	$\overleftrightarrow{P}_{i_{\alpha=0.5/0.6}}^{\overset{\rightarrow}{\leftarrow}}$	P_i	$\overleftrightarrow{P}_{i_{\alpha=0.5/0.6}}^{\overset{\leftarrow}{\rightarrow}}$	$\overleftrightarrow{P}_{i_{\alpha=0.5/0.8}}^{\rightarrow}$	$\overleftrightarrow{P}_{i_{\alpha=0.5}}^{\overset{\rightarrow}{\rightarrow}}$
$i=0$	(-8, 0)	(-7.25, 0)	(-6, 0)	(-5, 0)	(-3.1667, 0)	(-0.875, 0)	(0.5, 0)
$i=1$	(15, 23)	(15, 22.25)	(15, 21)	(15, 20)	(15, 19)	(15, 17.75)	(15, 17)
$i=2$	(12.5, -17.5)	(11.875,-18.125)	(10.833, -19.1667)	(10, -20)	(9.1667, -20.8333)	(8.125, -21.875)	(7.5, -22.5)
$i=3$	(36, 10)	(37, 10)	(38.6667, 10)	(40, 10)	(41.3333, 10)	(43, 10)	(44, 10)
Type-Reduction	${}^{MF}\overline{\overline{P}}_{i_\alpha}$						
	$\overline{\overline{P}}_{i_\alpha}^{\leftarrow}$			P_i	$\overline{\overline{P}}_{i_\alpha}^{\rightarrow}$		
$i=0$	(-7.8333, 0)			(-5, 0)	(0.5, 0)		
$i=1$	(15, 23.1667)			(15, 20)	(15, 17)		
$i=2$	(12.5, -17.5)			(10, -20)	(7.5, -22.5)		
$i=3$	(36, 10)			(40, 10)	(44, 10)		
Type-2 Defuzzification				${}^{MF}\overline{\overline{\overline{P}}}_{i_\alpha}$			
				P_i	${}^{MF}\overline{\overline{\overline{P}}}_{i_\alpha}$		
$i=0$				(-5, 0)	(-4.4213, 0)		
$i=1$				(15, 20)	(15, 20)		
$i=2$				(10, -20)	(10, -20)		
$i=3$				(40, 10)	(40, 10)		

Table 6.2. The fuzzification, type-reduction and defuzzification processes of T2FDPs of footprint [23].

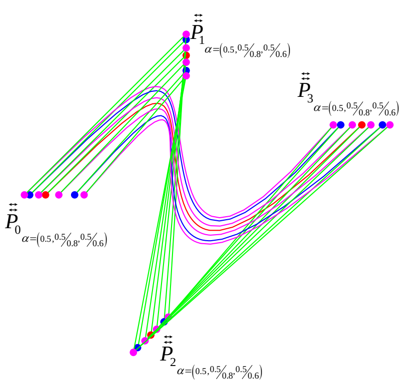
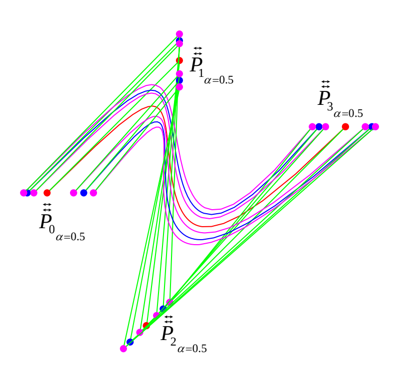
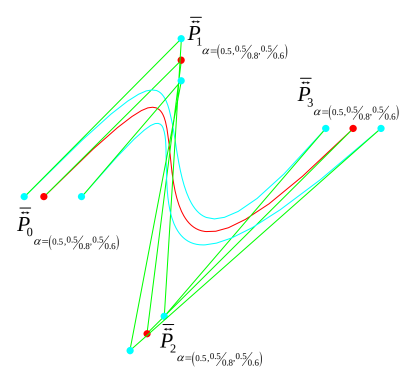
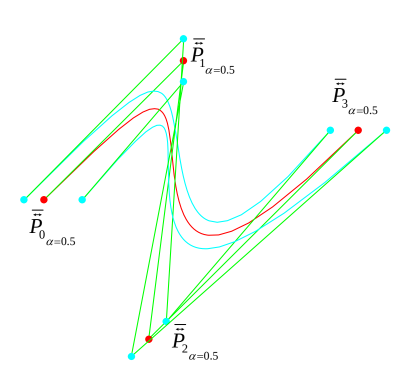
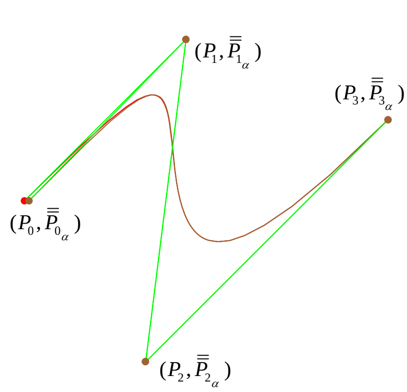
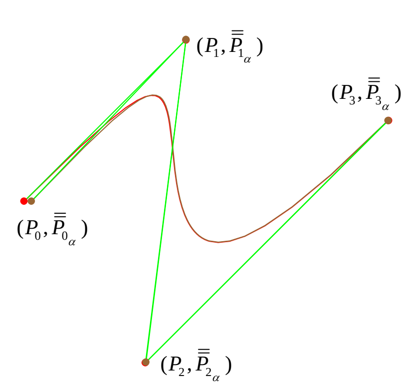
T2FDPs of Footprint	$\ddot{\tilde{P}}_i$						
	$\overleftarrow{\ddot{P}}_i^{\leftarrow}$	$\ddot{P}_i^{\leftarrow}$	$\overrightarrow{\ddot{P}}_i^{\leftarrow}$	P_i	$\overleftarrow{\ddot{P}}_i^{\rightarrow}$	$\ddot{P}_i^{\rightarrow}$	$\overrightarrow{\ddot{P}}_i^{\rightarrow}$
$i=0$	(-12, 0)	(-11, 0)	(-9, 0)	(-5, 0)	(3, 0)	(6,0)	(9,0)
$i=1$	(15, 28)	(15, 26)	(15, 25)	(15, 20)	(15, 16)	(15,14)	(15,12)
$i=2$	(17, -13)	(15, -15)	(13, -17)	(10, -20)	(8, -22)	(5,-25)	(3,-27)
$i=3$	(30, 10)	(32, 10)	(34, 10)	(40, 10)	(46, 10)	(48,10)	(49,10)

α -Cut, $\alpha = 0.5$	$\ddot{\tilde{P}}_{i_\alpha}$						
	$\overleftarrow{\ddot{P}}_{i_\alpha}^{\leftarrow}$	$\ddot{P}_{i_\alpha}^{\leftarrow}$	$\overrightarrow{\ddot{P}}_{i_\alpha}^{\leftarrow}$	P_i	$\overleftarrow{\ddot{P}}_{i_\alpha}^{\rightarrow}$	$\ddot{P}_{i_\alpha}^{\rightarrow}$	$\overrightarrow{\ddot{P}}_{i_\alpha}^{\rightarrow}$
$i=0$	(-8.5, 0)	(-8, 0)	(-7, 0)	(-5, 0)	(-1, 0)	(0.5, 0)	(2, 0)
$i=1$	(15, 24)	(15, 23)	(15, 22.5)	(15, 20)	(15, 18)	(15, 17)	(15, 16)
$i=2$	(13.5, -16.5)	(12.5, -17.5)	(11.5, -18.5)	(10, -20)	(9, -21)	(7.5, -22.5)	(6.5, -23.5)
$i=3$	(35, 10)	(36, 10)	(37, 10)	(40, 10)	(43, 10)	(44, 10)	(44.5, 10)

Type-Reduction	$\bar{\ddot{\tilde{P}}}_{i_\alpha}$		
	$\bar{\ddot{P}}_{i_\alpha}^{\leftarrow}$	P_i	$\bar{\ddot{P}}_{i_\alpha}^{\rightarrow}$
$i=0$	(-7.8333, 0)	(-5, 0)	(0.5, 0)
$i=1$	(15, 23.1667)	(15, 20)	(15, 7)
$i=2$	(12.5, -17.5)	(10, -20)	(7.6667, -22.3333)
$i=3$	(36, 10)	(40, 10)	(48.8333, 10)

Type-2 Defuzzification	$\bar{\bar{\tilde{P}}}_{i_\alpha}$	
	P_i	$\bar{\bar{P}}_{i_\alpha}$
$i=0$	(-5, 0)	(-4.1111, 0)
$i=1$	(15, 20)	(15, 20.0556)
$i=2$	(10, -20)	(10.0556, -19.9444)
$i=3$	(40, 10)	(39.9444, 10)

Table 6.3. The fuzzification, type-reduction and defuzzification processes of both cases of T2FDPs via NURBS curve.

T2FDPs	T2FDPs of Membership Function	T2FDPs of Footprint
	${}^{MF}\ddot{\tilde{P}}_{i=0,\ldots,3}$	$\ddot{\tilde{P}}_{i=0,\ldots,3}$
Fuzzification, $\alpha = 0.5$ $\overline{\overline{{}^{\alpha}RBsC}}_{k,n}(t) = \frac{\sum_{i=1}^{n} w_i N_i^k(t)\ddot{\tilde{P}}_{i_\alpha}}{\sum_{r=1}^{n} w_r N_r^k(t)}$		
Type-reduction $\overline{{}^{\alpha}RBsC}_{k,n}(t) = \frac{\sum_{i=1}^{n} w_i N_i^k(t)\overline{\tilde{P}}_{i_\alpha}}{\sum_{r=1}^{n} w_r N_r^k(t)}$		
Defuzzification $\overline{\overline{{}^{\alpha}RBsC}}_{k,n}(t) = \frac{\sum_{i=1}^{n} w_i N_i^k(t)\overline{\overline{P}}_{i_\alpha}}{\sum_{r=1}^{n} w_r N_r^k(t)}$		

7 Discussion and Conclusion

The defining T2FDPs for both cases which are T2FDPs of membership function and footprint were given earlier in Section 4 and Section 5 respectively. The

T2FDPs of membership function explained that the left and right-hands sides of T2FDPs are equal to a singular value at the left and right-hands sides of T2FDPs. However, although the left and right-hands sides of T2FDPs are in singular value, the membership function values of each triangular forming the T2FDPs has different values. These different values of membership function were shown in Table 6.1 and illustrated in Table 6.3.

For T2FDPs which has left and right footprint uncertainty, the membership function values are equal were called perfectly normal T2FDPs which has been discussed on the previous research paper [23]. In this case of T2FDPs, the illustration of the phenomenon on getting the crisp T2FDPs solution from T2FDPs are shown in Table 6.3.

Based on Table 6.3, the both T2FDPs forms are illustrated as the comparison among them for each processes, including the fuzzification, type-reduction and defuzzification processes by the NURBs curve function. This illustration gives us the more understanding in define the complex uncertainty data points based on the human presumption at the confident level (membership function value) or the confident interval (footprint of uncertainty). This thought can be extended about defined the complex uncertainty data points by T2FST through many cases, which are may exists.

Acknowledgement

The authors would like to thank Research Management and Innovation Centre (RMIC) of Universiti Malaysia Terengganu and Ministry of Higher Education (MOHE) Malaysia for funding(FRGS, vot59244) and providing the facilities to carry out this research.

References

[1] J.R. Aguero, A. Vargas. (2007). Calculating Functions of Interval Type-2 Fuzzy Numbers for Fault Current Analysis. *IEEE Transactions on Fuzzy Systems*, *15*(1), 31-40.

[2] L. Chengdong, Y. Jianqiang, Z. Dongbin. (2008). A Novel Type-Reduction Method for Interval Type-2 Fuzzy Logic Systems, *Fuzzy Systems and Knowledge Discovery, 2008. FSKD '08. Fifth International Conference on* (Vol. 1, pp. 157-161).

[3] H. Cheul, R. Frank Chung-Hoon. (2007). Uncertain Fuzzy Clustering: Interval Type-2 Fuzzy Approach to C-Means. *IEEE Transactions on Fuzzy Systems*, *15*(1), 107-120.

[4] Y. Chi-Yuan, W.H.R. Jeng, L. Shie-Jue. (2011). An Enhanced Type-Reduction Algorithm for Type-2 Fuzzy Sets. *IEEE Transactions on Fuzzy Systems*, *19*(2), 227-240.

[5] S. Coupland. (2007). Type-2 Fuzzy Sets: Geometric Defuzzification and Type-Reduction, *IEEE Symposium on Foundations of Computational Intelligence, 2007. FOCI 2007* (pp. 622-629).

[6] S. Coupland, R. John. (2006). An Investigation into Alternative Methods for the Defuzzification of an Interval Type-2 Fuzzy Set, *IEEE International Conference on Fuzzy Systems 2006* (pp. 1425-1432).

[7] U. Dongrui, N. Maowen. (2011). Comparison and practical implementation of type-reduction algorithms for type-2 fuzzy sets and systems, *IEEE International Conference on Fuzzy Systems (FUZZ), 2011* (pp. 2131-2138).

[8] W. Dongrui. (2012). An overview of alternative type-reduction approaches for reducing the computational cost of interval type-2 fuzzy logic controllers, *IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), 2012* (pp. 1-8).

[9] G. Farin. (1999). *NURBS for Curve and Surface Design: from Projective Geometry to Practical Use* (2nd ed.): AK Peters, Ltd.

[10] S. Greenfield, F. Chiclana, R. John. (2009). Type-reduction of the discretised interval type-2 fuzzy set, *IEEE International Conference on Fuzzy Systems, 2009. FUZZ-IEEE* (pp. 738-743).

[11] R. John, S. Lake. (2001). Type-2 fuzzy sets for modelling nursing intuition, *IFSA World Congress and 20th NAFIPS International Conference, 2001. Joint 9th* (Vol. 4, pp. 1920-1925 vol.1924).

[12] N.N. Karnik, J.M. Mendel. (1998). Type-2 fuzzy logic systems: type-reduction, *IEEE International Conference on Systems, Man, and Cybernetics, 1998.* (Vol. 2, pp. 2046-2051 vol.2042).

[13] O. Linda, M. Manic. (2012). Monotone Centroid Flow Algorithm for Type-Reduction of General Type-2 Fuzzy Sets. *IEEE Transactions on Fuzzy Systems, 20*(5), 805 - 819.

[14] N. Maowen, T. Woei Wan. (2008). Towards an efficient type-reduction method for interval type-2 fuzzy logic systems, *IEEE International Conference on Fuzzy Systems, 2008. FUZZ-IEEE 2008. (IEEE World Congress on Computational Intelligence).* (pp. 1425-1432).

[15] J.M. Mendel. (2001). *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions*. Upper Saddle River, NJ: Prentice Hall PTR.

[16] D.F. Rogers. (2001). *An Introduction to NURBS: With Historical Perspective*. USA: Academic Press.

[17] D. Salomon. (2006). *Curves and Surfaces for Computer Graphics*. USA: Springer.

[18] W. Shen, M. Mahfouf. (2012). A new computationally efficient mamdani interval type- 2 fuzzy modelling framework, *IEEE International Conference on Fuzzy Systems(FUZZ-IEEE), 2012* (pp. 1-8).

[19] W. Shen, M. Mahfouf. (2012). Multi-objective optimisation for fuzzy modelling using interval type-2 fuzzy sets, *IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), 2012* (pp. 1-8).

[20] F. Yamaguchi. (1988). *Curves and Surfaces in Computer Aided Geometric Design*. Germany: Springer-Verlag.

[21] L. Zadeh. (1965). Fuzzy Sets. *Information and Control, 8*, 338-353.

[22] L.A. Zadeh. (1975). The concept of a linguistic variable and its application to approximate reasoning-Part I-II-III *Information Science*, *8, 8, 9*, 199-249, 301-357, 143-180.

[23] R. Zakaria, A.F. Wahab, R.U. Gobithaasan. (2013). Perfectly Normal Type-2 Fuzzy Interpolation B-spline Curve. *Applied Mathematical Sciences, 7*(21), 1043-1055.

[24] R. Zakaria, A.F. Wahab, R.U. Gobithaasan. (2013). Normal Type-2 Fuzzy Rational B-spline Curve. *International Journal of Mathematical Analysis, 7*(16), 789-806.

Received: January 25, 2013