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Some Identities on the q-Genocchi Polynomials with Weak Weight α and Bernstein Polynomials

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Abstract

In this paper, by using fermionic p-adic q-integral on $\mathbb{Z}_p$, we give some interesting relationship between Bernstein polynomials and q-Genocchi polynomials with weak weight α.

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1 Introduction

Let p be a fixed odd prime number. Throughout this paper, we always make use of the following notations: $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The p-adic absolute value is defined by $|x|_p = \frac{1}{p^r}$, where $x = p^r \frac{s}{t}$($r \in \mathbb{Q}$ and $s, t \in \mathbb{Z}$ with $(s,t) = (p,s) = (p,t) = 1$). In this paper we assume that $q \in \mathbb{C}_p$ with $|q-1|_p < 1$ as an indeterminate. The q-number is defined by

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q}, \text{ see [1, 2, 3, 4, 5, 6, 7].}$$

Note that $\lim_{q \to 1}[x]_q = x$. For

$$f \in UD(\mathbb{Z}_p) = \{f | f : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(f)=\int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x)=\lim_{N\to\infty}\frac{[2]_q}{1+q^{p^N}}\sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ see [2, 3] }. \quad (1.1)$$

For $\alpha\in\mathbb{Z}$ and $q\in\mathbb{C}_p$ with $|1-q|_p\le 1$, the q-Genocchi polynomials $\widetilde{G}_{n,q}^{(\alpha)}$ with weak weight α are defined by

$$\widetilde{G}_{n,q}^{(\alpha)}(x)=n\int_{\mathbb{Z}_p}(x+y)^{n-1}d\mu_{-q^\alpha}(y). \quad (1.2)$$

In the special case, $x=0$, $\widetilde{G}_{n,q}^{(\alpha)}(0)=\widetilde{G}_{n,q}^{(\alpha)}$ are called the n-th q-Genocchi numbers with weak weight α(see [1]). In this paper we investigate some relations between Bernstein polynomials and q-Genocchi numbers with weak weight α. From these relations, we derive some interesting identities on the q-Genocchi numbers with weak weight α.

2 q-Genocchi polynomials with weak weight α and Bernstein polynomials

The following elementary properties of q-Genocchi polynomials $\widetilde{G}_{n,q}^{(\alpha)}(x)$ are readily derived from (1.1) and (1.2) . We, therefore, choose to omit the details involved. More studies and results in this subject we may see references [1], [7].

Proposition 2.1 *([7]) For any positive integer n and $\alpha\in\mathbb{Z}$, we have*

$$\widetilde{G}_{n,q}^{(\alpha)}(x)=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}\widetilde{G}_{k,q}^{(\alpha)}=\left(x+\widetilde{G}_q^{(\alpha)}\right)^n.$$

By (2.5), we have

$$\widetilde{\mathbf{G}}_{q^{-1}}^{(\alpha)}(1-t,-x)=\sum_{n=0}^{\infty}(-1)^{n-1}\widetilde{G}_{n,q^{-1}}^{(\alpha)}(1-x)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\widetilde{G}_{n,q}^{(\alpha)}(x)\frac{t^n}{n!}.$$

Thus we have the following theorem.

Theorem 2.2 *For any positive integer n, we have*

$$\widetilde{G}_{n,q}^{(\alpha)}(x)=(-1)^{n-1}\widetilde{G}_{n,q^{-1}}^{(\alpha)}(1-x)$$

For q-Genocchi numbers with weak weight α, we have the following theorem.

Proposition 2.3 *([7]) For $n \in \mathbb{Z}_+$, we have*

$$\widetilde{G}_{0,q}^{(\alpha)} = 0, \quad q^{\alpha}\widetilde{G}_{n,q}^{(\alpha)}(1) + \widetilde{G}_{n,q}^{(\alpha)} = \begin{cases} [2]_{q^\alpha}, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$$

with the usual convention about replacing $\left(\widetilde{G}_q^{(\alpha)}\right)^n$ by $\widetilde{G}_{n,q}^{(\alpha)}$ in the binomial expansion.

Proposition 2.4 *([7]) For $n \in \mathbb{Z}_+$, we have*

$$\widetilde{G}_{0,q}^{(\alpha)} = 0, \quad q^{\alpha}(\widetilde{G}_{q}^{(\alpha)} + 1)^n + \widetilde{G}_{n,q}^{(\alpha)} = \begin{cases} [2]_{q^\alpha}, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

with the usual convention of replacing $(\widetilde{G}_q^{(\alpha)})^n$ by $\widetilde{G}_{n,q}^{(\alpha)}$.

By Proposition 2.1 and Proposition 2.3, we obtain

$$\begin{aligned}\widetilde{G}_{n,q}^{(\alpha)}(2) &= \sum_{l=0}^{n}\binom{n}{l}2^{n-l}\widetilde{G}_{l,q}^{(\alpha)} = \left(\widetilde{G}_q^{(\alpha)} + 1 + 1\right)^n \\ &= \frac{n[2]_{q^\alpha}}{q^\alpha} - \frac{1}{q^\alpha}\sum_{l=1}^{n}\binom{n}{l}\widetilde{G}_{l,q}^{(\alpha)}(1) \\ &= \frac{n[2]_{q^\alpha}}{q^\alpha} + \frac{1}{q^{2\alpha}}\widetilde{G}_{n,q}^{(\alpha)} \text{ if } n > 1.\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 2.5 *For $n \in \mathbb{N}$ with $n > 1$, we have*

$$q^{2\alpha}\widetilde{G}_{n,q}^{(\alpha)}(2) = n[2]_{q^\alpha}q^\alpha + \widetilde{G}_{n,q}^{(\alpha)}.$$

By Theorem 2.5, we have the following corollary.

Corollary 2.6 *For $n \in \mathbb{N}$ with $n > 1$, we have*

$$\widetilde{G}_{n,q^{-1}}^{(\alpha)}(2) = n[2]_{q^\alpha} + q^{2\alpha}\widetilde{G}_{n,q^{-1}}^{(\alpha)}.$$

As well known definition, Bernstein polynomials of degree n are given by

$$B_{k,n}(x) = \binom{n}{k}x^k(1-x)^{n-k}, \text{ where } x \in [0,1], n, k \in \mathbb{Z}_+, \text{ see } [5,6]. \quad (2.1)$$

By (2.1), we get the symmetry of Bernstein polynomials as follows:

$$B_{k,n}(x) = B_{n-k,n}(1-x).$$

By Theorem 2.2 and Corollary 2.6, we have

$$\begin{aligned}\int_{\mathbb{Z}_p}(1-x)^n d\mu_{-q^\alpha}(x) &= (-1)^n\int_{\mathbb{Z}_p}(x-1)^n d\mu_{-q^\alpha}(x)\\ &= (-1)^n\frac{\widetilde{G}^{(\alpha)}_{n+1,q}(-1)}{n+1}\\ &= (-1)^n\frac{(-1)^n\widetilde{G}^{(\alpha)}_{n+1,q^{-1}}(2)}{n+1}\\ &= [2]_{q^\alpha}+q^{2\alpha}\frac{\widetilde{G}^{(\alpha)}_{n+1,q^{-1}}}{n+1}.\end{aligned}$$

Let us take the fermionic p-adic q-integral on $\mathbb{Z}_p$ for the Bernstein polynomials of degree n as follows:

$$\begin{aligned}\int_{\mathbb{Z}_p}B_{k,n}(x)d\mu_{-q^\alpha}(x) &= \binom{n}{k}\int_{\mathbb{Z}_p}x^k(1-x)^{n-k}d\mu_{-q^\alpha}(x)\\ &= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\frac{\widetilde{G}^{(\alpha)}_{l+k+1,q}}{l+k+1}.\end{aligned}\tag{2.2}$$

Let $n,k\in\mathbb{Z}_+$ with $n>k+1$. Then we get

$$\begin{aligned}&\int_{\mathbb{Z}_p}B_{k,n}(x)d\mu_{-q^\alpha}(x) = \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\int_{\mathbb{Z}_p}(1-x)^{n-l}d\mu_{-q^\alpha}(x)\\ &= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left([2]_{q^\alpha}+q^{2\alpha}\frac{\widetilde{G}^{(\alpha)}_{n-l+1,q^{-1}}}{n-l+1}\right).\end{aligned}\tag{2.3}$$

Therefore, by (2.2) and (2.3), we obtain the following theorem.

Theorem 2.7 *Let $n,k\in\mathbb{Z}_+$ with $n>k+1$. Then we have*

$$\int_{\mathbb{Z}_p}B_{k,n}(x)d\mu_{-q^\alpha}(x) = \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\left([2]_{q^\alpha}+q^{2\alpha}\frac{\widetilde{G}^{(\alpha)}_{n-l+1,q^{-1}}}{n-l+1}\right).$$

Moreover,

$$\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\frac{\widetilde{G}^{(\alpha)}_{l+k+1,q}}{l+k+1} = \sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\left([2]_{q^\alpha}+q^{2\alpha}\frac{\widetilde{G}^{(\alpha)}_{n-l+1,q^{-1}}}{n-l+1}\right).$$

Let $n_1,n_2,k\in\mathbb{Z}_+$ with $n_1+n_2>2k+1$. Then we get

$$\begin{aligned}&\int_{\mathbb{Z}_p}B_{k,n_1}(x)B_{k,n_2}(x)d\mu_{-q^\alpha}(x)\\ &= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\left([2]_{q^\alpha}+q^{2\alpha}\frac{\widetilde{G}^{(\alpha)}_{n_1+n_2-l+1,q^{-1},}}{n_1+n_2-l+1}\right).\end{aligned}\tag{2.4}$$

Therefore, by (2.4), we have the following theorem.

Theorem 2.8 *For $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k + 1$, we have*

$$\int_{\mathbb{Z}_p} B_{k,n_1}(x) B_{k,n_2}(x) d\mu_{-q^\alpha}(x)$$
$$= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\left([2]_{q^\alpha} + q^{2\alpha}\frac{\widetilde{G}^{(\alpha)}_{n_1+n_2-l+1,q^{-1},}}{n_1+n_2-l+1}\right).$$

From the binomial theorem, we can derive the following equation.

$$\int_{\mathbb{Z}_p} B_{k,n_1}(x) B_{k,n_2}(x) d\mu_{-q^\alpha}(x)$$
$$= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{n_1+n_2-2k}(-1)^l\binom{n_1+n_2-2k}{l}\frac{\widetilde{G}^{(\alpha)}_{2k+l+1,q}}{2k+l+1}. \tag{2.5}$$

Thus, by (2.5) and Theorem 2.8, we obtain the following corollary.

Corollary 2.9 *Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k + 1$. Then we have*

$$\sum_{l=0}^{n_1+n_2-2k}(-1)^l\binom{n_1+n_2-2k}{l}\frac{\widetilde{G}^{(\alpha)}_{2k+l+1,q}}{2k+l+1}$$
$$= \sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\left([2]_{q^\alpha} + q^{2\alpha}\frac{\widetilde{G}^{(\alpha)}_{n_1+n_2-l+1,q^{-1}}}{n_1+n_2-l+1}\right).$$

For $x \in \mathbb{Z}_p$ and $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk + 1$. Then we take the fermionic p-adic q-integral on $\mathbb{Z}_p$ for the Bernstein polynomials of degree n as follows:

$$\int_{\mathbb{Z}_p} \underbrace{B_{k,n_1}(x)\cdots B_{k,n_s}(x)}_{s-\text{times}} d\mu_{-q^\alpha}(x)$$
$$= \binom{n_1}{k}\cdots\binom{n_s}{k}\int_{\mathbb{Z}_p} x^{sk}(1-x)^{n_1+\cdots+n_s-sk} d\mu_{-q^\alpha}(x)$$
$$= \binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\left([2]_{q^\alpha} + q^{2\alpha}\frac{\widetilde{G}^{(\alpha)}_{n_1+\cdots+n_s-l+1,q^{-1}}}{n_1+\cdots+n_s-l+1}\right).$$

Therefore we have the following theorem.

Theorem 2.10 *For $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk + 1$. Then we get*

$$\int_{\mathbb{Z}_p} \underbrace{B_{k,n_1}(x) \cdots B_{k,n_s}(x)}_{s-times} d\mu_{-q^\alpha}(x)$$
$$= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \left([2]_{q^\alpha} + q^{2\alpha} \frac{\widetilde{G}^{(\alpha)}_{n_1+\cdots+n_s-l+1,q^{-1}}}{n_1 + \cdots + n_s - l + 1} \right).$$

By the definition of Bernstein polynomials and the binomial theorem, we easily get

$$\int_{\mathbb{Z}_p} \underbrace{B_{k,n_1}(x) \cdots B_{k,n_s}(x)}_{s-\text{times}} d\mu_{-q^\alpha}(x)$$
$$= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{n_1+\cdots+n_s-sk} (-1)^l \binom{n_1 + \cdots + n_s - sk}{l} \frac{\widetilde{G}^{(\alpha)}_{sk+l+1,q}}{sk + l + 1}. \tag{2.6}$$

Therefore, by Theorem 2.10 and (2.6), we have the following corollary.

Corollary 2.11 *For $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk + 1$. Then we have*

$$\sum_{l=0}^{n_1+\cdots+n_s-sk} (-1)^l \binom{n_1 + \cdots + n_s - sk}{l} \frac{\widetilde{G}^{(\alpha)}_{sk+l+1,q}}{sk + l + 1}$$
$$= \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \left([2]_{q^\alpha} + q^{2\alpha} \frac{\widetilde{G}^{(\alpha)}_{n_1+\cdots+n_s-l+1,q^{-1}}}{n_1 + \cdots + n_s - l + 1} \right).$$

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