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Riesz Representation Theorem on Generalized n-Inner Product Spaces

Pudji Astuti

Faculty of Mathematics and Natural Sciences
Institut Teknologi Bandung
Jln. Ganesha 10 Bandung 40132, Indonesia
pudji@math.itb.ac.id

Abstract

In this paper we develop a generalization of Riesz representation theorem on generalized n-inner product spaces. We show that any alternating n-linear functional on a finitely generated standard generalized n-inner product space can be represented by a finite number of n-tuples of vectors. The proof of the theorem utilizes an inner product on a quotient tensor product space induced by the generalized n-inner product.

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1 Introduction

This paper deals with a generalization of Riesz representation theorem on generalized n-inner product spaces. The concept of generalized n-inner product was introduced by Trenčevsky and Malčeski [8] as a generalization of an n-inner product by Misiak [5]. Following this introduction, a number of results concerning the structure and properties of generalized n-inner product spaces can be found in literature (see [1], [2] and [6]). Gozali etal. [3] exposed a question whether Riesz type representation theorem holds on standard generalized

n-inner product spaces; that is whether any bounded n-linear functional on a separable Hilbert space can be represented by n unique vectors. We show that the question has negative answer and prove an enhancement of Riesz representation theorem on standard generalized n-inner product spaces.

In this paper we proceed as follows. In section 2 we review the concept of generalized n-inner product and n-linear functional, and also present two examples which lead to the main result. The main result of this paper is Theorem 2.6 where the proof will be presented in section 4. Section 3 will discuss auxiliary results concerning induced inner product on a quotient tensor product space which will be used later.

2 Definitions, notation and main theorem

The notation $\mathbb{R}$ stands for the real number field and n is a positive integer. The concept of a generalized n-inner product was introduced by Trenčevsky and Malčeski [8] as a generalization of an n-inner product by Misiak [5] which, to some extent, is more proper notion as a generalization of an inner product.

Definition 2.1. *[8] Let V be a real vector space having dimension at least n. A real valued function with domain V^{2n}, the Cartesian product of $2n$ times of V,*

$$\langle -,\ldots,-|-,\ldots,-\rangle : V^n \times V^n \longrightarrow \mathbb{R}$$

is called generalized n-inner product on V if it satisfies the following conditions.

i) (Quasi Positive Definite)
$\langle \mathbf{a}_1,\ldots,\mathbf{a}_n|\mathbf{a}_1,\ldots,\mathbf{a}_n\rangle \geq 0$ *for all* $\mathbf{a}_1,\ldots,\mathbf{a}_n \in V$ *and* $\langle \mathbf{a}_1,\ldots,\mathbf{a}_n|\mathbf{a}_1,\ldots,\mathbf{a}_n\rangle = 0$ *if and only if the set* $\{\mathbf{a}_1,\ldots,\mathbf{a}_n\} \subseteq V$ *is linearly dependent.*

ii) (Quasi Symmetry)
$\langle \mathbf{a}_1,\ldots,\mathbf{a}_n|\mathbf{b}_1,\ldots,\mathbf{b}_n\rangle = \langle \mathbf{b}_1,\ldots,\mathbf{b}_n|\mathbf{a}_1,\ldots,\mathbf{a}_n\rangle$
for all $\mathbf{a}_1,\ldots,\mathbf{a}_n,\mathbf{b}_1,\ldots,\mathbf{b}_n \in V$.

iii) (Linear at the first variable)
$\langle \lambda\mathbf{a}_1,\ldots,\mathbf{a}_n|\mathbf{b}_1,\ldots,\mathbf{b}_n\rangle = \lambda\langle \mathbf{a}_1,\ldots,\mathbf{a}_n|\mathbf{b}_1,\ldots,\mathbf{b}_n\rangle$
$\langle \mathbf{a}_1+\mathbf{c},\ldots,\mathbf{a}_n|\mathbf{b}_1,\ldots,\mathbf{b}_n\rangle = \langle \mathbf{a}_1,\ldots,\mathbf{a}_n|\mathbf{b}_1,\ldots,\mathbf{b}_n\rangle + \langle \mathbf{c},\mathbf{a}_2,\ldots,\mathbf{a}_n|\mathbf{b}_1,\ldots,\mathbf{b}_n\rangle$
for all $\lambda \in \mathbb{R}$ *and* $\mathbf{a}_1,\ldots,\mathbf{a}_n,\mathbf{b}_1,\ldots,\mathbf{b}_n,\mathbf{c} \in V$.

iv) (Quasi Skew-Symmetry)
$\langle \mathbf{a}_1,\ldots,\mathbf{a}_n|\mathbf{b}_1,\ldots,\mathbf{b}_n\rangle = -\langle \mathbf{a}_{\sigma(1)},\ldots,\mathbf{a}_{\sigma(n)}|\mathbf{b}_1,\ldots,\mathbf{b}_n\rangle$ *for all* $\mathbf{a}_1,\ldots,\mathbf{a}_n$,

$\mathbf{b}_1,\ldots,\mathbf{b}_n \in V$ *and* σ *odd permutation on* $\{1,\ldots,n\}$.

v. *If* $\langle \mathbf{a}_1, \mathbf{b}_1, \dots, \mathbf{b}_{(i-1)}, \mathbf{b}_{(i+1)}, \dots, \mathbf{a}_n | \mathbf{b}_1, \dots, \mathbf{b}_n \rangle = 0$ *for all* $i \in \{1, 2, \dots, n\}$ *then for any* $\mathbf{a}_2, \dots, \mathbf{a}_n \in V$ $\langle \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n | \mathbf{b}_1, \dots, \mathbf{b}_n \rangle = 0$.

A vector space V *coupled with a generalized* n*-inner product on it is called a generalized* n*-inner product space or* n*-prehilbert space.*

□

A standard generalized n-inner product space is an inner product space coupled with the standard generalized n-inner product induced by the inner product [8]. Let V be an inner product space where its inner product is denoted by $\langle -, - \rangle$. The standard generalized n-inner product on V induced by the inner product $\langle -, - \rangle$, is the function

$$\langle -, \dots, - | -, \dots, - \rangle_S : V^n \times V^n \longrightarrow \mathbb{R}$$

$$\langle \mathbf{a}_1, \dots, \mathbf{a}_n | \mathbf{b}_1, \dots, \mathbf{b}_n \rangle_S = \begin{vmatrix} \langle \mathbf{a}_1, \mathbf{b}_1 \rangle & \langle \mathbf{a}_1, \mathbf{b}_2 \rangle & \dots & \langle \mathbf{a}_1, \mathbf{b}_n \rangle \\ \langle \mathbf{a}_2, \mathbf{b}_1 \rangle & \langle \mathbf{a}_2, \mathbf{b}_2 \rangle & \dots & \langle \mathbf{a}_2, \mathbf{b}_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{a}_n, \mathbf{b}_1 \rangle & \langle \mathbf{a}_n, \mathbf{b}_2 \rangle & \dots & \langle \mathbf{a}_n, \mathbf{b}_n \rangle \end{vmatrix}$$

for all $\mathbf{a}_1, \dots, \mathbf{a}_n$, $\mathbf{b}_1, \dots, \mathbf{b}_n \in V$. The proof of $\langle -, \dots, - | -, \dots, - \rangle_S$ being a generalized n-inner product can be found in [8]. In this paper we deal with standard generalized n-inner product spaces.

One interesting research topic connected to generalized n-inner product spaces is multilinear function.

Definition 2.2. *[7] Let* V, W *be two real vector spaces. A real function*

$$f : V^n \longrightarrow W$$

is called multilinear or n*-linear function if it is linear in each variables. That is, if*

$$f(\mathbf{u}_1, \dots, \mathbf{u}_{k-1}, \alpha \mathbf{v}_1 + \beta \mathbf{v}_2, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n) =$$

$$\alpha f(\mathbf{u}_1, \dots, \mathbf{u}_{k-1}, \mathbf{v}_1, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n) + \beta f(\mathbf{u}_1, \dots, \mathbf{u}_{k-1}, \mathbf{v}_2, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n)$$

for all $k \in \{1, \dots, n\}$, $\alpha, \beta \in \mathbb{R}$, $\mathbf{v}_1, \mathbf{v}_2 \in$ *and* $\mathbf{u}_i \in V$ *with* $i \neq k$. *A multilinear function* f *with codomain* $W = \mathbb{R}$ *the real number field is called multilinear or* n*-linear functional.*

□

Let V be an n-prehilbert space with the generalized n-inner product denoted by $\langle -, \ldots, - | -, \ldots, - \rangle$ and $\mathbf{a}_1, \ldots, \mathbf{a}_n \in V$. For these n vectors, one can define a functional

$$f_{(\mathbf{a}_1,\ldots,\mathbf{a}_n)} : V^n \longrightarrow R$$

by mapping for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in V^n$ to

$$f_{(\mathbf{a}_1,\ldots,\mathbf{a}_n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \langle \mathbf{x}_1, \ldots, \mathbf{x}_n | \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle$$

The property that $\langle -, \ldots, - | -, \ldots, - \rangle$ being multilinear induces multilinear property of the functional $f_{(\mathbf{a}_1,\ldots,\mathbf{a}_n)}$. The converse of this result, exposed by [3] as Riesz type representation theorem, is an interesting problem to be addressed. Unfortunately the expectation that any bounded n-linear functional on a real separable Hilbert space can be represented by n unique vectors is not true as shown in the following example.

Example 2.3. *Let V be a standard generalized 2-inner product space with the inner product denoted by $\langle -, - \rangle$. The standard generalized 2-inner product on V; is defined as*

$$\langle \mathbf{u}_1, \mathbf{u}_2 | \mathbf{v}_1, \mathbf{v}_2 \rangle_S = \begin{vmatrix} \langle \mathbf{u}_1, \mathbf{v}_1 \rangle & \langle \mathbf{u}_1, \mathbf{v}_2 \rangle \\ \langle \mathbf{u}_2, \mathbf{v}_1 \rangle & \langle \mathbf{u}_2, \mathbf{v}_2 \rangle \end{vmatrix}$$

for all $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in V$. For this space, the inner product $\langle -, - \rangle$ is 2-linear functional but it does not satisfy condition that there exist $\mathbf{b}_1, \mathbf{b}_2 \in V$ such that

$$\langle \mathbf{x}_1, \mathbf{x}_2 \rangle = \langle \mathbf{x}_1, \mathbf{x}_2 | \mathbf{b}_1, \mathbf{b}_2 \rangle_S \quad \textit{for all} \quad \mathbf{x}_1, \mathbf{x}_2 \in V. \tag{1}$$

Suppose there do exist $\mathbf{b}_1, \mathbf{b}_2 \in V$ such that (1) holds. Then we have for $\mathbf{v} \in V$ $\mathbf{v} \neq 0$, $\langle \mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{v} | \mathbf{b}_1, \mathbf{b}_2 \rangle_S = 0$, which is a contradiction to the positive definite property of the inner product. □

Definition 2.4. *[7] A multilinear function $f : V^n \longrightarrow W$ is called alternating if it satisfies*

$$f(\mathbf{u}_1, \ldots, \mathbf{u}_i, \ldots, \mathbf{u}_j, \ldots, \mathbf{u}_n) = 0$$

for all $\mathbf{u}_1, \ldots, \mathbf{u}_n \in V$ with $\mathbf{u}_i = \mathbf{u}_j$ for some $i < j$.

□

Further investigation concerning Riesz type representation problem observes that the n-linear functional on an n-prehilbert space generated by n vectors is alternating. This gives an idea to confine the investigation of Riesz type representation problem in the subspace of bounded alternating n-linear functionals. However, similar to the above observation, a negative answer to this inquiry is obtained as shown in the following example.

Example 2.5. *Let $\mathbb{R}^4$ be the Euclidean space coupled with the standard generalized 2-inner product, for every $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^4$*

$$\langle \mathbf{u}_1, \mathbf{u}_2 | \mathbf{v}_1, \mathbf{v}_2 \rangle = \begin{vmatrix} \langle \mathbf{u}_1, \mathbf{v}_1 \rangle & \langle \mathbf{u}_1, \mathbf{v}_2 \rangle \\ \langle \mathbf{u}_2, \mathbf{v}_1 \rangle & \langle \mathbf{u}_2, \mathbf{v}_2 \rangle \end{vmatrix}$$

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\} \subseteq \mathbb{R}^4$ be the standard basis of $\mathbb{R}^4$. Consider the 2-linear functional defined on $\mathbb{R}^4$ as follows; for every $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^4$

$$f(\mathbf{x}_1, , \mathbf{x}_2) = \langle \mathbf{x}_1, \mathbf{x}_2 | \mathbf{e}_1, \mathbf{e}_2 \rangle + \langle \mathbf{x}_1, \mathbf{x}_2 | \mathbf{e}_3, \mathbf{e}_4 \rangle$$

The 2-linear functional f is alternating. Suppose f can be represented by two vectors in $\mathbb{R}^4$, that is there exist $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^4$ satisfying

$$f(\mathbf{x}_1, \mathbf{x}_2) = \langle \mathbf{x}_1, \mathbf{x}_2 | \mathbf{b}_1, \mathbf{b}_2 \rangle \quad \text{for all} \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^4. \tag{2}$$

Let $\mathbf{b}_1 = (b_{11}, b_{12}, b_{13}, b_{14})^t, \mathbf{b}_2 = (b_{21}, b_{22}, b_{23}, b_{24})^t$. Then we have the following equations

$$\begin{array}{lllll} f(\mathbf{e}_1, \mathbf{e}_2) & = & 1 & = & b_{11}b_{22} - b_{21}b_{12} \\ f(\mathbf{e}_1, \mathbf{e}_3) & = & 0 & = & b_{11}b_{23} - b_{21}b_{13} \\ f(\mathbf{e}_1, \mathbf{e}_4) & = & 0 & = & b_{11}b_{24} - b_{21}b_{14} \\ f(\mathbf{e}_3, \mathbf{e}_4) & = & 1 & = & b_{13}b_{24} - b_{23}b_{14} \end{array}$$

The first three equations imply that

$$\begin{array}{c} (b_{11}, b_{21}) \neq (0, 0) \\ (b_{13}, b_{23}) = \alpha(b_{11}, b_{21}) \\ (b_{14}, b_{24}) = \beta(b_{11}, b_{21}) \end{array}$$

for some $\alpha, \beta \in \mathbb{R}$. As a result we obtain $\{(b_{13}, b_{23}), (b_{14}, b_{24})\}$ is linearly dependent. In contrast the last equation implies that $\{(b_{13}, b_{23}), (b_{14}, b_{24})\}$ is linearly independent, which is a contradiction statement to the previous result. Thus f cannot be represented by two vectors in $\mathbb{R}^4$. □

Even though, to a certain extent, the above Example 2.3 and Example 2.5 are disappointed, the expression of the 2-linear functional f as sum of two generalized 2-inner products inspires to investigate further: can any alternating n-linear functional be represented as finite sum of generalized n-inner products? The following theorem is the main result of this paper and it can be recognized as an enhancement of Riesz representation theorem on standard n-prehilbert spaces.

Theorem 2.6. *Let V be a finitely dimensional standard generalized n-inner product space with the n-inner product $\langle -, \ldots, - | -, \ldots, - \rangle_S$ is induced by an inner product $\langle -, - \rangle$ on V and $\mathrm{Dim}_{\mathbb{R}}(V) = m \geq n$. Let $f : V^n \longrightarrow \mathbb{R}$ be an* *alternating n-linear functional. Then there exists a finite number of elements $(\mathbf{a}_{i1},\ldots,\mathbf{a}_{in})\in V^n$, $i=1,\ldots,k$ such that*

$$f(\mathbf{x}_1,\ldots,\mathbf{x}_n)=\sum_{i=1}^{k}\langle \mathbf{x}_1,\ldots,\mathbf{x}_n|\mathbf{a}_{i1},\ldots,\mathbf{a}_{in}\rangle_S \quad \text{for all} \quad \mathbf{x}_1,\ldots,\mathbf{x}_n\in V.$$

□

The proof of the theorem will be explained in section 4 and it utilizes properties of the induced inner product on a quotient tensor product space.

3 Induced inner product

Definition 3.1. *[7] Let V be a real vector space. The n-fold tensor product of V is a real vector space T such that there exists a multilinear function, called the tensor map, $t: V^n \longrightarrow T$ having the property that any multilinear function*

$f: V^n \longrightarrow W$, where W is a real vector space, can be decomposed by t as $f=g\circ t$ for a unique linear map $g: T\longrightarrow W$.

□

It is well known that the n-fold tensor product of V is unique up to isomorphism. One way to construct the n-fold tensor product of V is as shown in [7]. The tensor product space can be identified by the quotient space $\mathcal{F}/\mathcal{S}$ where $\mathcal{F}$ is the real vector space with the Cartesian product V^n as its basis and $\mathcal{S}$ is its subspace generated by the set of all elements $\mathcal{F}$ of the form

$$\alpha(\mathbf{u}_1,\ldots,\mathbf{u}_{k-1},\mathbf{v}_1,\mathbf{u}_{k+1},\ldots,\mathbf{u}_n)+\beta(\mathbf{u}_1,\ldots,\mathbf{u}_{k-1},\mathbf{v}_2,\mathbf{u}_{k+1},\ldots,\mathbf{u}_n)$$

$$-(\mathbf{u}_1,\ldots,\mathbf{u}_{k-1},\alpha\mathbf{v}_1+\beta\mathbf{v}_2,\mathbf{u}_{k+1},\ldots,\mathbf{u}_n)$$

for some $\alpha,\beta\in\mathbb{R}$ and $\mathbf{v}_1,\mathbf{v}_2,\mathbf{u}_i\in V$ with $i\neq k$. In this case, the tensor map t is defined as

$$t(\mathbf{u}_1,\ldots,\mathbf{u}_n)=(\mathbf{u}_1,\ldots,\mathbf{u}_n)+\mathcal{S}$$

for all $\mathbf{u}_1,\ldots,\mathbf{u}_n\in V$. From now on, the n-fold tensor product of V will be denoted by $V^{\otimes n}$ and the coset $(\mathbf{u}_1,\ldots,\mathbf{u}_n)+\mathcal{S}$, for any $(\mathbf{u}_1,\ldots,\mathbf{u}_n)\in V^n$, called decomposable tensor, will be denoted by

$$\mathbf{u}_1\otimes\mathbf{u}_2\cdots\otimes\mathbf{u}_n=(\mathbf{u}_1,\ldots,\mathbf{u}_n)+\mathcal{S}$$

As a consequence, any element of $V^{\otimes n}$ is a finite sum of decomposable tensors $\sum_i \mathbf{u}_{i1}\otimes\cdots\otimes\mathbf{u}_{in}$.

Let V be an m-dimensional n-prehilbert space, $m \geq n$, with the generalized n-inner product, denoted by $\langle -, \ldots, -|-, \ldots, -\rangle_S$, is induced by an inner product on V $\langle -, -\rangle$. One can construct a bilinear form on the n-fold tensor product space $V^{\otimes n}$ induced by the generalized n-inner product as the following. For any $\sum_i \mathbf{u}_{i1} \otimes \ldots \mathbf{u}_{in}$, $\sum_j \mathbf{v}_{j1} \otimes \ldots \mathbf{v}_{jn} \in V^{\otimes n}$ define

$$\langle \textstyle\sum_i \mathbf{u}_{i1} \otimes \ldots \mathbf{u}_{in}, \sum_j \mathbf{v}_{j1} \otimes \ldots \mathbf{v}_{jn}\rangle_t = \sum_i \sum_j \langle \mathbf{u}_{i1}, \ldots, \mathbf{u}_{in} | \mathbf{v}_{j1}, \ldots, \mathbf{v}_{jn}\rangle_S$$

It can be shown that the above definition is a well defined real valued function on $V^{\otimes n} \times V^{\otimes n}$. Moreover, as a results of properties of the generalized n-inner product, we obtain that the functional

$$\langle -, -\rangle_t : V^{\otimes n} \times V^{\otimes n} \longrightarrow \mathbb{R}$$

is a symmetric bilinear form on $V^{\otimes n}$. However, it is degenerate. We have that for any $\mathbf{v} \in V$, $\langle \mathbf{v} \otimes \cdots \otimes \mathbf{v}, \mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_2\rangle_t = 0$ for all $\mathbf{u}_1, \ldots, \mathbf{u}_n \in V$. Whereas for $\mathbf{v} \neq 0$ we have $\mathbf{v} \otimes \cdots \otimes \mathbf{v} \neq 0$.

According to Greub [4] the above symmetric bilinear form on $V^{\otimes n}$ can induce a non-degenerate symmetric bilinear form on the quotient space $V^{\otimes n}/D$ where D is the subspace of $V^{\otimes n}$ generated by all degenerate elements. That is

$$D = \{X \in V^{\otimes n} \mid \langle X, Y\rangle_t = 0 \text{ for all } Y \in V^{\otimes n}\}.$$

The induced bilinear between cosets $X + D, Y + D \in V^{\otimes n}/D$ is defined as

$$\langle X + D, Y + D\rangle_q = \langle X, Y\rangle_t.$$

Now we shall show that the bilinear $\langle -, -\rangle_q$ is an inner product on $V^{\otimes n}/D$. It is enough by showing that $V^{\otimes n}/D$ has an orthonormal basis with respect to $\langle -, -\rangle_q$. Let $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m\}$ be an orthonormal ordered basis of V. According to Roman [7] we obtain

$$B^{\otimes n} = \{\mathbf{b}_{i_1} \otimes \cdots \otimes \mathbf{b}_{i_n} \mid \mathbf{b}_{i_1}, \ldots, \mathbf{b}_{i_n} \in B\}$$

is a basis of $V^{\otimes n}$. Let us partition $B^{\otimes n}$ becomes three subsets:

$$B_1^{\otimes n} = \{\mathbf{b}_{i_1} \otimes \cdots \otimes \mathbf{b}_{i_n} \in B^{\otimes n} \mid \mathbf{b}_{i_1}, \ldots, \mathbf{b}_{i_n} \in B, i_j < i_k \text{ for } j < k\}$$
$$B_2^{\otimes n} = \{\mathbf{b}_{i_1} \otimes \cdots \otimes \mathbf{b}_{i_n} \in B^{\otimes n} \mid \mathbf{b}_{i_\ell} \in B, i_j \neq i_k \text{ for } j \neq k, \text{ and } \exists j < k \ni i_j > i_k\}$$
$$B_3^{\otimes n} = \{\mathbf{b}_{i_1} \otimes \cdots \otimes \mathbf{b}_{i_n} \in B^{\otimes n} \mid \mathbf{b}_{i_1}, \ldots, \mathbf{b}_{i_n} \in B, i_j = i_k \text{ for some } j \neq k\}.$$

1. Because of the orthonormality of B, the subset $B_1^{\otimes n}$ is orthonormal with respect to the bilinear form $\langle -, -\rangle_t$.

2. For every $\mathbf{b}_{i_1} \otimes \cdots \otimes \mathbf{b}_{i_n} \in B_2^{\otimes n}$ there exists σ a permutation on $\{i_1, i_2, \ldots, i_n\}$ such that $\mathbf{b}_{\sigma(i_1)} \otimes \cdots \otimes \mathbf{b}_{\sigma(i_n)} \in B_1^{\otimes n}$. Let $\bar{B}_2^{\otimes n}$ be the set of elements

$$\mathbf{b}_{i_1} \otimes \cdots \otimes \mathbf{b}_{i_n} - (-1)^{sgn(\sigma)} \mathbf{b}_{\sigma(i_1)} \otimes \cdots \otimes \mathbf{b}_{\sigma(i_n)}$$

for all $\mathbf{b}_{i_1}\otimes\cdots\otimes\mathbf{b}_{i_n}\in B_2^{\otimes n}$ where σ is the permutation on $\{i_1,i_2,\ldots,i_n\}$ such that $\mathbf{b}_{\sigma(i_1)}\otimes\cdots\otimes\mathbf{b}_{\sigma(i_n)}\in B_1^{\otimes n}$. We obtain that the set $B_1^{\otimes n}\cup\bar{B}_2^{\otimes n}\cup B_3^{\otimes n}$ forms a basis of $V^{\otimes n}$. Moreover $\bar{B}_2^{\otimes n}$ is a subset of D.

3. Since any element in $B_3^{\otimes n}$ is generated by less then n elements of B, we obtain $B_3^{\otimes n}\subseteq D$.

Up to now we obtain that $B_1^{\otimes n}\cup\bar{B}_2^{\otimes n}\cup B_3^{\otimes n}$ is a basis of $V^{\otimes n}$ and the independent subset $\bar{B}_2^{\otimes n}\cup B_3^{\otimes n}$ is inside D. If we can show that $\bar{B}_2^{\otimes n}\cup B_3^{\otimes n}$ is a basis of D and because $B_1^{\otimes n}$ is orthonormal with respect to $\langle -,-\rangle_t$, we obtain that the set

$$\{\mathbf{b}_{i_1}\otimes\cdots\otimes\mathbf{b}_{i_n}+D \mid \mathbf{b}_{i_1}\otimes\cdots\otimes\mathbf{b}_{i_n}\in B_1^{\otimes n}\}$$

is an orthonormal basis of $V^{\otimes n}/D$ with respect to $\langle -,-\rangle_q$.

Now we shall show that $\bar{B}_2^{\otimes n}\cup B_3^{\otimes n}$ is a basis of D. Let $X\in V^{\otimes n}$ with $X\notin \mathrm{Span}\{\bar{B}_2^{\otimes n}\cup B_3^{\otimes n}\}$. Write $X=X_1+X_2$ where $X_1\in\mathrm{Span}\{B_1^{\otimes n}\}$ and $X_2\in\mathrm{Span}\{\bar{B}_2^{\otimes n}\cup B_3^{\otimes n}\}$. Then $X_1\neq 0$ and

$$X_1=\sum_{\mathbf{b}_{i_1}\otimes\cdots\otimes\mathbf{b}_{i_n}\in B_1^{\otimes n}}\alpha_{i_1\ldots i_n}\mathbf{b}_{i_1}\otimes\cdots\otimes\mathbf{b}_{i_n}$$

with some $\alpha_{i_1\ldots i_n}\neq 0$. Let $\mathbf{b}_{i_1}\otimes\cdots\otimes\mathbf{b}_{i_n}\in B_1^{\otimes n}$ with $\alpha_{i_1\ldots i_n}\neq 0$. Then, because of the orthonormality of $B_1^{\otimes n}$ with respect to $\langle -,-\rangle_t$,

$$\langle X,\mathbf{b}_{i_1}\otimes\cdots\otimes\mathbf{b}_{i_n}\rangle_t=\langle X_1,\mathbf{b}_{i_1}\otimes\cdots\otimes\mathbf{b}_{i_n}\rangle_t=\alpha_{i_1\ldots i_n}\neq 0$$

which results in $X\notin D$. Thus, we have $\bar{B}_2^{\otimes n}\cup B_3^{\otimes n}$ is a generator of D which results in $\bar{B}_2^{\otimes n}\cup B_3^{\otimes n}$ is a basis of D.

4 The proof of theorem 2.6

Let V be an m-dimensional standard generalized n-inner product space, $m\geq n$, with the n-inner product $\langle -,\ldots,-|-,\ldots,-\rangle_S$ induced by an inner product $\langle -,-\rangle$ on V. Let $f;V^n\longrightarrow\mathbb{R}$ be an alternating n-linear functional. Referring to the above explanation let $g:V^{\otimes n}\longrightarrow\mathbb{R}$ be the unique linear functional such that $f=g\circ t$ where $t:V^n\longrightarrow V^{\otimes n}$ is the tensor map satisfying

$$t(\mathbf{u}_1,\ldots,\mathbf{u}_n)=\mathbf{u}_1\otimes\cdots\otimes\mathbf{u}_n$$

for all $\mathbf{u}_1,\ldots,\mathbf{u}_n\in V$. We shall show that the degenerate subspace D is contained in the kernel of g so that we can lift g to the induced inner product space $V^{\otimes n}/D$. It is enough by showing the basis $\bar{B}_2^{\otimes n}\cup B_3^{\otimes n}\subseteq\mathrm{Ker}(g)$.

Let $\mathbf{b}_{i_1} \otimes \cdots \otimes \mathbf{b}_{i_n} - (-1)^{sgn(\sigma)} \mathbf{b}_{\sigma(i_1)} \otimes \cdots \otimes \mathbf{b}_{\sigma(i_n)} \in \bar{B}_2^{\otimes n}$. Then

$$\begin{aligned}
& g(\mathbf{b}_{i_1} \otimes \cdots \otimes \mathbf{b}_{i_n} - (-1)^{sgn(\sigma)} \mathbf{b}_{\sigma(i_1)} \otimes \cdots \otimes \mathbf{b}_{\sigma(i_n)}) \\
&= g(\mathbf{b}_{i_1} \otimes \cdots \otimes \mathbf{b}_{i_n}) - (-1)^{sgn(\sigma)} g(\mathbf{b}_{\sigma(i_1)} \otimes \cdots \otimes \mathbf{b}_{\sigma(i_n)}) \\
&= g(t(\mathbf{b}_{i_1}, \ldots, \mathbf{b}_{i_n})) - (-1)^{sgn(\sigma)} g(t(\mathbf{b}_{\sigma(i_1)}, \ldots, \mathbf{b}_{\sigma(i_n)}) \\
&= f(\mathbf{b}_{i_1}, \ldots, \mathbf{b}_{i_n})) - (-1)^{sgn(\sigma)} f(\mathbf{b}_{\sigma(i_1)}, \ldots, \mathbf{b}_{\sigma(i_n)}) \\
&= 0
\end{aligned}$$

since f is alternating. Let $\mathbf{b}_{i_1} \otimes \cdots \otimes \mathbf{b}_{i_n} \in B_3^{\otimes n}$. Then $\mathbf{b}_{i_j} = \mathbf{b}_{i_k}$ for some $j \neq k$. Hence, since f is alternating, we obtain

$$g(\mathbf{b}_{i_1} \otimes \cdots \otimes \mathbf{b}_{i_n}) = g(t(\mathbf{b}_{i_1}, \ldots, \mathbf{b}_{i_n}) = f(\mathbf{b}_{i_1}, \ldots, \mathbf{b}_{i_n}) = 0.$$

Thus $D \subseteq \mathrm{Ker}(g)$. According to [7] there exists a unique linear functional

$$\bar{g} : V^{\otimes n}/D \longrightarrow \mathbb{R}$$

such that $g = \bar{g} \circ \pi_D$ where

$$\pi_D : \ V^{\otimes n} \ \longrightarrow \ V^{\otimes n}/D$$

is the canonical projection defined as $\pi_D(X) = X + D$ for all $X \in V^{\otimes n}$. Now we have $\bar{g}$ a linear functional on $V^{\otimes n}/D$, an inner product space with the inner product $\langle -, - \rangle_q$, Riesz representation theorem [7] guarantees the existence of a unique $A + D \in V^{\otimes n}/D$ such that

$$\bar{g}(X + D) = \langle X + D, A + D \rangle_q \quad \text{for any} \quad X + D \in V^{\otimes n}/D.$$

Consider that $A \in V^{\otimes n}$ is of the form as a finite sum of decomposable tensors. Let $(\mathbf{a}_{i1}, \ldots, \mathbf{a}_{in}) \in V^n$ for $i = 1, 2, \ldots, k$ such that $A = \sum_{i=1}^k \mathbf{a}_{i1} \otimes \cdots \otimes \mathbf{a}_{in}$. Then we obtain for any $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in V$

$$\begin{aligned}
f(\mathbf{x}_1, \ldots, \mathbf{x}_n) &= (g \circ t)(\mathbf{x}_1, \ldots, \mathbf{x}_n) \\
&= g(t(\mathbf{x}_1, \ldots, \mathbf{x}_n)) \\
&= g(\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_n) \\
&= (\bar{g} \circ \pi_D)(\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_n) \\
&= \bar{g}(\pi_D(\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_n)) \\
&= \bar{g}(\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_n + D) \\
&= \langle \mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_n + D, A + D \rangle_q \\
&= \langle \mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_n, A \rangle_t \\
&= \langle \mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_n, \textstyle\sum_{i=1}^k \mathbf{a}_{i1} \otimes \cdots \otimes \mathbf{a}_{in} \rangle_t \\
&= \textstyle\sum_{i=1}^k \langle \mathbf{x}_1, \ldots, \mathbf{x}_n | \mathbf{a}_{i1}, \ldots, \mathbf{a}_{in} \rangle_S.
\end{aligned}$$

□

5 Concluding remark

In this paper we have developed an adaptation of Riesz representation theorem for alternating multilinear functionals on standard n-prehilbert spaces. We obtained that any alternating multilinear functional on a finitely generated inner product space can be represented by a finite number of n-tuples of vectors. The uniqueness of the representation n-tuples of vectors is not guaranteed since any n-tuple of vectors which forms a degenerate decomposable tensor can be added to the representation n-tuples. However it would be interesting to study whether the minimum number of representation n-tuples of vectors is connected to the rank matrix of the associated tensor product. Extension of the result to any n-prehilbert spaces would also be an interesting problem to be investigated.

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