

Approximating Classes of Functions Defined by Operators of Differentiation or Operators of Generalised Translation by Means of Algebraic Polynomials

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Abstract. In this paper, approximation by means of algebraic polynomials of classes of functions defined by a generalised modulus of smoothness of operators of differentiation of these functions is considered. We give structural characteristics of classes of functions defined by the order of best approximation by algebraic polynomials.

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1. INTRODUCTION

In a number of papers (see e.g. [5, 1, 6, 7, 8, 13, 11]) approximation of classes of functions defined by symmetric or asymmetric operators of generalised translation by means of algebraic polynomials is considered.

In our paper we consider the approximation of classes of functions defined by generalised modulus of smoothness of derivatives of these functions. In more general terms, we consider approximation by algebraic polynomials of certain generalised Lipschitz classes of functions.

By $L_p[a, b]$ we denote the set of functions f such that for $1 \leq p < \infty$ f is a measurable function on the segment $[a, b]$ and

$$\|f\|_p = \left(\int_a^b |f(x)| dx \right)^{1/p} < \infty,$$

and for $p = \infty$ the function f is continuous on the segment $[a, b]$ and

$$\|f\|_\infty = \max_{a \leq x \leq b} |f(x)|.$$

In case that $[a, b] = [-1, 1]$ we simply write L_p instead of $L_p[-1, 1]$.

Denote by $L_{p,\alpha,\beta}$ the set of functions f such that $f(x)(1-x)^\alpha(1+x)^\beta \in L_p$, and put

$$\|f\|_{p,\alpha,\beta} = \|f(x)(1-x)^\alpha(1+x)^\beta\|_p.$$

By $E_n(f)_{p,\alpha,\beta}$ we denote the best approximation of the function $f \in L_{p,\alpha,\beta}$ by means of algebraic polynomials of degree not greater than $n-1$, in $L_{p,\alpha,\beta}$ metrics, i.e.

$$E_n(f)_{p,\alpha,\beta} = \inf_{P_n} \|f - P_n\|_{p,\alpha,\beta},$$

where P_n is an algebraic polynomial of degree not greater than $n-1$.

For a summable function f we define the asymmetric operator of generalised translation $\tilde{T}_t(f, x)$ by

$$\begin{aligned} \tilde{T}_t(f, x) &= \frac{1}{\pi(1-x^2)} \\ &\times \int_{-1}^1 \left(1 - R^2 - 2(1-z^2)\sin^2 t + 4(1-x^2)(1-z^2)^2\sin^2 t \right) f(R) \frac{dz}{\sqrt{1-z^2}}, \end{aligned}$$

where $R = x \cos t + z\sqrt{1-x^2} \sin t$.

For a function $f \in L_{p,\alpha,\beta}$ we define by means of this operator of generalised translation the generalised modulus of smoothness by

$$\tilde{\omega}(f, \delta)_{p,\alpha,\beta} = \sup_{|t| \leq \delta} \left\| \tilde{T}_t(f, x) - f(x) \right\|_{p,\alpha,\beta}.$$

We say that φ is a function of modulus of continuity type if

1. φ is continuous and non-negative function on the interval $(-1, 1]$,
2. $\varphi(t_1) \leq C_{\varphi,1}\varphi(t_2)$ ($0 < t_1 \leq t_2 \leq 1$),
3. $\varphi(2t) \leq C_{\varphi,2}\varphi(t)$ ($0 < t \leq \frac{1}{2}$).

We say that a function $f(x)$ has the derivative of order r inside of the interval $(-1, 1)$ if the function $f(x)$ has the absolutely continuous derivative of order $r-1$ in every segment $[a, b] \subset (-1, 1)$. From the last condition it follows that almost everywhere on the segment $[a, b]$ there exists the finite derivative of order r , which is a summable function on that interval.

Denote by $D_{x,\nu,\mu}$ the following operator of differentiation

$$D_{x,\nu,\mu} = (1 - x^2) \frac{d^2}{dx^2} + (\mu - \nu - (\nu + \mu + 2)x) \frac{d}{dx},$$

and put

$$\begin{aligned} D_{x,\nu,\mu}^1 f(x) &= D_{x,\nu,\mu} f(x), \\ D_{x,\nu,\mu}^r f(x) &= D_{x,\nu,\mu} (D_{x,\nu,\mu}^{r-1} f(x)) \quad (r = 1, 2, \dots). \end{aligned}$$

We say that $f \in AD(p, \alpha, \beta)$ if $f \in L_{p,\alpha,\beta}$, the function f has the derivative $\frac{d}{dx}f(x)$ absolutely continuous on every segment $[a, b] \subset (-1, 1)$ and $D_{x,\nu,\mu}f(x) \in L_{p,\alpha,\beta}$.

By $P_n^{(\nu,\mu)}(x)$ ($n = 0, 1, \dots$) we denote the Jacobi's polynomials, i.e. algebraic polynomials of order n , orthogonal to each other with weight $(1-x)^\nu(1+x)^\mu$ on the segment $[-1, 1]$ and normed by the condition $P_n^{(\nu,\mu)}(1) = 1$ ($n = 0, 1, \dots$).

Let $\nu \geq \mu \geq -\frac{1}{2}$. The following symmetric operators of generalised translation (see e.g. [5, 1, 6, 7, 8]) will have an auxiliary role below:

1. for $\nu = \mu = -\frac{1}{2}$

$$S_t(f, x, \nu, \mu) = \frac{1}{2}(f(Q_{x,t,1,1}) - f(Q_{x,-t,1,1}));$$

2. for $\nu = \mu > -\frac{1}{2}$

$$S_t(f, x, \nu, \mu) = \frac{1}{\gamma(\nu)} \int_{-1}^1 f(Q_{x,t,z,1}) (1 - z^2)^{\nu-\frac{1}{2}} dz;$$

3. for $\nu > \mu = -\frac{1}{2}$

$$S_t(f, x, \nu, \mu) = \frac{1}{\gamma(\nu)} \int_{-1}^1 f(Q_{x,t,1,z}) (1 - z^2)^{\nu-\frac{1}{2}} dz;$$

4. for $\nu > \mu > -\frac{1}{2}$

$$\begin{aligned} S_t(f, x, \nu, \mu) &= \frac{1}{\gamma(\nu, \mu)} \int_0^1 \int_{-1}^1 f(Q_{x,t,z,u}) (1 - z^2)^{\nu-\mu-1} z^{2\mu+1} (1 - u^2)^{\mu-\frac{1}{2}} du dz, \end{aligned}$$

where

$$Q_{x,t,z,u} = x \cos t + zu\sqrt{1-x^2} \sin t - (1-u^2)(1-x) \sin^2 \frac{t}{2},$$

$$\gamma(\nu) = \int_{-1}^1 (1 - z^2)^{\nu-\frac{1}{2}} dz,$$

$$\gamma(\nu, \mu) = \int_0^1 \int_{-1}^1 (1 - z^2)^{\nu-\mu-1} z^{2\mu+1} (1 - u^2)^{\mu-\frac{1}{2}} du dz.$$

2. AUXILIARY STATEMENTS

We need the following lemmas in order to prove our results.

Lemma 2.1. *Let $P_n(x)$ be an algebraic polynomial of order not greater than $n - 1$, $1 \leq p \leq \infty$, $\rho \geq 0$, $\sigma \geq 0$;*

$$\begin{aligned} \alpha &> -\frac{1}{p}, \quad \beta > -\frac{1}{p} \quad \text{for } 1 \leq p < \infty, \\ \alpha &\geq 0, \quad \beta \geq 0 \quad \text{for } p = \infty. \end{aligned}$$

The following inequalities hold true

$$\begin{aligned} \|P'_n(x)\|_{p, \alpha + \frac{1}{2}, \beta + \frac{1}{2}} &\leq C_1 n \|P_n\|_{p, \alpha, \beta}, \\ \|P_n\|_{p, \alpha, \beta} &\leq C_2 n^{2 \max\{\rho, \sigma\}} \|P_n\|_{p, \alpha + \rho, \beta + \sigma}, \end{aligned}$$

where constants C_1 and C_2 do not depend on n .

Lemma 2.1 is proved in [3].

Lemma 2.2. *Let be given numbers p , α , β and γ such that $1 \leq p \leq \infty$, $\gamma = \min\{\alpha, \beta\}$;*

$$\begin{aligned} \gamma &> 1 - \frac{1}{2p} \quad \text{for } 1 \leq p < \infty, \\ \gamma &\geq 1 \quad \text{for } p = \infty. \end{aligned}$$

Let ε be an arbitrary number from the interval $0 < \varepsilon < \frac{1}{2}$ and let

$$\gamma_1 = \begin{cases} \alpha - \beta & \text{if } \alpha > \beta \\ 0 & \text{if } \alpha \leq \beta, \end{cases} \quad \gamma_2 = \begin{cases} 0 & \text{if } \alpha > \beta \\ \beta - \alpha & \text{if } \alpha \leq \beta; \end{cases}$$

for $1 < p \leq \infty$

$$\gamma_3 = \begin{cases} \gamma - \frac{3}{2} + \frac{1}{2p} + \varepsilon & \text{if } \gamma \geq \frac{3}{2} - \frac{1}{2p} \\ 0 & \text{if } \gamma < \frac{3}{2} - \frac{1}{2p}, \end{cases}$$

for $p = 1$

$$\gamma_3 = \begin{cases} \gamma - 1 & \text{if } \gamma \geq 1 \\ 0 & \text{if } \gamma < 1. \end{cases}$$

Then the following inequality holds true

$$\begin{aligned} \left\| \tilde{T}_t(f, x) \right\|_{p, \alpha, \beta} &\leq C \left(\|f\|_{p, \alpha, \beta} + t^{2(\gamma_1 + \gamma_2)} \|f\|_{p, \alpha - \gamma_1, \beta - \gamma_2} \right. \\ &\quad \left. + t^{2\gamma_3} \|f\|_{p, \alpha - \gamma_3, \beta - \gamma_3} + t^{2(\gamma_1 + \gamma_2 + \gamma_3)} \|f\|_{p, \alpha - \gamma_1 - \gamma_3, \beta - \gamma_2 - \gamma_3} \right), \end{aligned}$$

where constant C does not depend on f and t .

Lemma 2.2 is proved in [13].

Lemma 2.3. *Let be given positive integers q and m and let $f \in L_{1,2,2}$. The function*

$$Q(x) = \frac{1}{\gamma_m} \int_0^\pi \tilde{T}_t(f, x) \left(\frac{\sin \frac{mt}{2}}{\sin \frac{t}{2}} \right)^{2q+4} \sin^3 t \, dt,$$

where

$$\gamma_m = \int_0^\pi \left(\frac{\sin \frac{mt}{2}}{\sin \frac{t}{2}} \right)^{2q+4} \sin^3 t \, dt,$$

is an algebraic polynomial of degree not greater than $(q + 2)(m - 1)$.

Lemma 2.3 is also proved in [13]

Lemma 2.4. *Let $f \in L_{p,\alpha,\beta}$ and let be given numbers p, α, β, ρ and σ such that $1 \leq p \leq \infty, \rho \geq 0, \sigma \geq 0$;*

$$\begin{aligned} \alpha > -\frac{1}{p}, \quad \beta > -\frac{1}{p} \quad \text{for } 1 \leq p < \infty, \\ \alpha \geq 0, \quad \beta \geq 0 \quad \text{for } p = \infty. \end{aligned}$$

Let φ be a function of modulus of continuity type such that

$$(1) \quad \sum_{j=n+1}^\infty j^{2\lambda_0-1} \varphi\left(\frac{1}{j}\right) \leq C_{\varphi,3} n^{2\lambda_0} \varphi\left(\frac{1}{n}\right),$$

where $\lambda_0 = \max\{\rho, \sigma\}$ and constant $C_{\varphi,3}$ does not depend on n . If there exists a sequence of algebraic polynomials $P_n(x)$ of degree not greater than $n - 1$ ($n = 0, 1, \dots$) such that

$$\|f - P_n\|_{p,\alpha+\rho,\beta+\sigma} \leq C_1 \varphi\left(\frac{1}{n}\right),$$

then there exists a sequence of algebraic polynomials $R_n(x)$ of degree not greater than $n - 1$ ($n = 0, 1, \dots$) such that

$$\|f - R_n\|_{p,\alpha,\beta} \leq C_2 n^{2\lambda_0} \varphi\left(\frac{1}{n}\right),$$

where constants C_1 and C_2 do not depend on f and n . Also we have

$$R_{2^N}(x) = P_{2^N}(x).$$

Proof. We consider the sequence of algebraic polynomials $Q_n(x)$ of degree not greater than $2^n - 1$ given by

$$Q_k(x) = P_{2^k}(x) - P_{2^{k-1}}(x) \quad (k = 1, 2, \dots)$$

and $Q_0(x) = P_1(x)$. From the conditions of the lemma it follows that

$$\begin{aligned} \|Q_k\|_{p,\alpha+\rho,\beta+\sigma} &\leq \|P_{2^k} - f\|_{p,\alpha+\rho,\beta+\sigma} + \|f - P_{2^{k-1}}\|_{p,\alpha+\rho,\beta+\sigma} \\ &\leq C_3 \left(\varphi\left(\frac{1}{2^k}\right) + \varphi\left(\frac{1}{2^{k-1}}\right) \right). \end{aligned}$$

Considering the properties of the function φ we get

$$\|Q_k\|_{p,\alpha+\rho,\beta+\sigma} \leq C_4 \varphi\left(\frac{1}{2^k}\right).$$

Applying Lemma 2.1 and that evaluate we obtain

$$\|Q_k\|_{p,\alpha,\beta} \leq C_5 2^{2k\lambda_0} \varphi\left(\frac{1}{2^k}\right).$$

There from

$$\sum_{k=0}^{\infty} \|Q_k\|_{p,\alpha,\beta} \leq C_5 \sum_{k=0}^{\infty} 2^{2k\lambda_0} \varphi\left(\frac{1}{2^k}\right).$$

Note that considering the properties of the function φ we have

$$\begin{aligned} \sum_{j=2^k}^{2^{k+1}-1} j^{2\lambda_0-1} \varphi\left(\frac{1}{j}\right) &\geq C_{\varphi,1}^{-1} C_{\varphi,2}^{-1} \varphi\left(\frac{1}{2^k}\right) \sum_{j=2^k}^{2^{k+1}-1} j^{2\lambda_0-1} \\ &\geq C_6 \varphi\left(\frac{1}{2^k}\right) 2^k 2^{k(\lambda_0-1)} = C_6 2^{2k\lambda_0} \varphi\left(\frac{1}{2^k}\right). \end{aligned}$$

So, we get

$$\sum_{k=0}^{\infty} \|Q_k\|_{p,\alpha,\beta} \leq C_7 \sum_{k=0}^{\infty} \sum_{j=2^k}^{2^{k+1}-1} j^{2\lambda_0-1} \varphi\left(\frac{1}{j}\right) = C_7 \sum_{k=0}^{\infty} k^{2\lambda_0-1} \varphi\left(\frac{1}{k}\right).$$

Thus, inequality (1) yields

$$\sum_{k=0}^{\infty} \|Q_k\|_{p,\alpha,\beta} < \infty.$$

Hence, considering the conditions of the lemma it follows that the series $\sum_{k=0}^{\infty} Q_k(x)$ converge to $f(x)$ in terms of $L_p[a, b]$ for every segment $[a, b] \subset (-1, 1)$.

Now we consider the expression

$$I = \|f - P_{2^N}\|_{p,\alpha,\beta}.$$

From what we said above it follows that

$$\begin{aligned} I &\leq \sum_{k=N+1}^{\infty} \|Q_k\|_{p,\alpha,\beta} \leq C_5 \sum_{k=N+1}^{\infty} 2^{2k\lambda_0} \varphi\left(\frac{1}{2^k}\right) \\ &\leq C_8 \sum_{k=N+1}^{\infty} \sum_{j=2^k}^{2^{k+1}-1} j^{2\lambda_0-1} \varphi\left(\frac{1}{j}\right) = C_8 \sum_{k=2^{N+1}}^{\infty} k^{2\lambda_0-1} \varphi\left(\frac{1}{k}\right). \end{aligned}$$

Considering the inequality (1) and the properties of the function φ we obtain that

$$I \leq C_9 2^{2(N+1)\lambda_0} \varphi\left(\frac{1}{2^{N+1}}\right) \leq C_{10} 2^{2N\lambda_0} \varphi\left(\frac{1}{2^N}\right),$$

where constant C_{10} does not depend on f and N .

Put

$$R_n(x) = P_{2^N}(x) \quad (2^{N-1} < n \leq 2^N),$$

we get

$$\|f - R_n\|_{p,\alpha,\beta} \leq C_{10} 2^{2N\lambda_0} \varphi\left(\frac{1}{2^N}\right) \leq C_{11} n^{2\lambda_0} \varphi\left(\frac{1}{n}\right).$$

Lemma 2.4 is proved. □

Lemma 2.5. *Let be given numbers p, α, β, ν , and μ such that $1 \leq p \leq \infty$, $\nu \geq \mu \geq -\frac{1}{2}$;*

1. *if $\nu = \mu = -\frac{1}{2}$, then $\alpha = \beta = -\frac{1}{2p}$;*
2. *if $\nu = \mu > -\frac{1}{2}$, then $\alpha = \beta$, and*

$$\begin{aligned} -\frac{1}{2} < \alpha \leq \nu & \quad \text{for } p = 1, \\ -\frac{1}{2p} < \alpha < \nu + \frac{1}{2} - \frac{1}{2p} & \quad \text{for } 1 < p < \infty, \\ 0 \leq \alpha < \nu + \frac{1}{2} & \quad \text{for } p = \infty; \end{aligned}$$

3. *if $\nu > \mu = -\frac{1}{2}$, then $\beta = -\frac{1}{2p}$, and*

$$\begin{aligned} -\frac{1}{2} < \alpha \leq \nu & \quad \text{for } p = 1, \\ -\frac{1}{2p} < \alpha < \nu + \frac{1}{2} - \frac{1}{2p} & \quad \text{for } 1 < p < \infty, \\ 0 \leq \alpha < \nu + \frac{1}{2} & \quad \text{for } p = \infty; \end{aligned}$$

4. *if $\nu > \mu > -\frac{1}{2}$, then $\nu - \mu > \alpha - \beta \geq 0$, and*

$$\begin{aligned} -\frac{1}{2} < \beta \leq \mu & \quad \text{for } p = 1, \\ -\frac{1}{2p} < \beta < \mu + \frac{1}{2} - \frac{1}{2p} & \quad \text{for } 1 < p < \infty, \\ 0 \leq \beta < \mu + \frac{1}{2} & \quad \text{for } p = \infty. \end{aligned}$$

For $f(x) \in AD(p, \alpha, \beta)$ the following inequality holds true

$$E_n(f)_{p,\alpha,\beta} \leq \frac{C}{n^2} \|D_{x,\nu,\mu} f(x)\|_{p,\alpha,\beta},$$

where constant C does not depend on f and n .

Proof. We choose the positive integer q such that $q > \nu$. For every positive integer n we choose the positive integer m such that

$$\frac{n-1}{q+2} < m \leq \frac{n-1}{q+2} + 1.$$

In [6] and [7] it is proved that the function

$$Q(x) = \frac{1}{\gamma_m} \int_0^\pi S_t(f, x, \nu, \mu) \left(\frac{\sin \frac{mt}{2}}{\sin \frac{t}{2}} \right)^{2q+4} \left(\sin \frac{t}{2} \right)^{2\nu+1} \left(\cos \frac{t}{2} \right)^{2\mu+1} dt,$$

where

$$\gamma_m = \int_0^\pi \left(\frac{\sin \frac{mt}{2}}{\sin \frac{t}{2}} \right)^{2q+4} \left(\sin \frac{t}{2} \right)^{2\nu+1} \left(\cos \frac{t}{2} \right)^{2\mu+1} dt,$$

is an algebraic polynomial of degree not greater than $n - 1$. Applying the generalised Minkowski's inequality we get

$$\begin{aligned} E_n(f)_{p,\alpha,\beta} &\leq \|f - Q\|_{p,\alpha,\beta} \leq \frac{1}{\gamma_m} \int_0^\pi \|S_t(f, x, \nu, \mu) - f(x)\|_{p,\alpha,\beta} \\ &\quad \times \left(\frac{\sin \frac{mt}{2}}{\sin \frac{t}{2}} \right)^{2q+4} \left(\sin \frac{t}{2} \right)^{2\nu+1} \left(\cos \frac{t}{2} \right)^{2\mu+1} dt. \end{aligned}$$

In [9, p. 47] it is proved that under the conditions of the lemma we have

$$\|S_t(f, x, \nu, \mu) - f(x)\|_{p,\alpha,\beta} \leq C_1 t^2 \|D_{x,\nu,\mu} f(x)\|_{p,\alpha,\beta},$$

where constant C_1 does not depend on f and t . Hence we get

$$\begin{aligned} E_n(f)_{p,\alpha,\beta} &\leq C_1 \|D_{x,\nu,\mu} f(x)\|_{p,\alpha,\beta} \\ &\quad \times \frac{1}{\gamma_m} \int_0^\pi t^2 \left(\frac{\sin \frac{mt}{2}}{\sin \frac{t}{2}} \right)^{2q+4} \left(\sin \frac{t}{2} \right)^{2\nu+1} \left(\cos \frac{t}{2} \right)^{2\mu+1} dt. \end{aligned}$$

Applying a standard estimate of Jackson's kernel [4, p. 233–235] we obtain

$$E_n(f)_{p,\alpha,\beta} \leq \frac{C_2}{m^2} \|D_{x,\nu,\mu} f(x)\|_{p,\alpha,\beta} \leq \frac{C_3}{n^2} \|D_{x,\nu,\mu} f(x)\|_{p,\alpha,\beta}.$$

Lemma 2.5 is proved. □

Corollary 2.1. *Let numbers $p, \alpha, \beta, \nu,$ and μ satisfy the conditions of Lemma 2.5. For $f(x) \in AD(p, \alpha, \beta)$ the following inequality holds true*

$$E_n(f)_{p,\alpha,\beta} \leq \frac{C}{n^2} E_n(D_{x,\nu,\mu} f)_{p,\alpha,\beta},$$

where constant C_0 does not depend on f and n .

Proof. Let $P_n(x)$ be the algebraic polynomial of best approximation of the function $D_{x,\nu,\mu}f(x)$ of degree not greater than $n - 1$. It is obvious that the polynomial $P_n(x)$ may be written in the following form

$$P_n(x) = \sum_{k=0}^{n-1} \lambda_k P_k^{(\nu,\mu)}(x).$$

Put

$$g(x) = f(x) + \sum_{k=0}^{n-1} \frac{\lambda_k}{k(k + \nu + \mu + 1)} P_k^{(\nu,\mu)}(x).$$

From Lemma 2.5 it follows that [2, p. 171]

$$\begin{aligned} E_n(g)_{p,\alpha,\beta} &\leq \frac{C_1}{n^2} \|D_{x,\nu,\mu}g(x)\|_{p,\alpha,\beta} \\ &= \frac{C_1}{n^2} \left\| D_{x,\nu,\mu}f(x) + \sum_{k=0}^{n-1} \frac{\lambda_k}{k(k + \nu + \mu + 1)} D_{x,\nu,\mu}P_k^{(\nu,\mu)}(x) \right\|_{p,\alpha,\beta} \\ &= \frac{C_1}{n^2} \left\| D_{x,\nu,\mu}f(x) - \sum_{k=0}^{n-1} \lambda_k P_k^{(\nu,\mu)}(x) \right\|_{p,\alpha,\beta} = \frac{C_1}{n^2} E_n(D_{x,\nu,\mu}f)_{p,\alpha,\beta}. \end{aligned}$$

Thus, considering that the function $f(x) - g(x)$ is an algebraic polynomial of degree not greater than $n - 1$, we obtain

$$\begin{aligned} E_n(f)_{p,\alpha,\beta} &\leq E_n(f - g)_{p,\alpha,\beta} + E_n(g)_{p,\alpha,\beta} = E_n(g)_{p,\alpha,\beta} \\ &\leq \frac{C_1}{n^2} E_n(D_{x,\nu,\mu}f)_{p,\alpha,\beta}. \end{aligned}$$

The corollary is proved. □

Note that an analogue to the corollary is given in [10].

3. STATEMENTS OF RESULTS

Now we formulate and prove our results.

Theorem 3.1. *Let be given numbers $p, \alpha, \beta, \nu, \mu$ and r such that $1 \leq p \leq \infty$, $r \in \mathbb{N}$, $\nu \geq \mu \geq -\frac{1}{2}$;*

1. *if $\nu = \mu = -\frac{1}{2}$, then $\alpha = \beta = -\frac{1}{2p}$;*
2. *if $\nu = \mu > -\frac{1}{2}$, then $\alpha = \beta$, and*

$$\begin{aligned} -\frac{1}{2} < \alpha \leq \nu & \quad \text{for } p = 1, \\ -\frac{1}{2p} < \alpha < \nu + \frac{1}{2} - \frac{1}{2p} & \quad \text{for } 1 < p < \infty, \\ 0 \leq \alpha < \nu + \frac{1}{2} & \quad \text{for } p = \infty; \end{aligned}$$

3. if $\nu > \mu = -\frac{1}{2}$, then $\beta = -\frac{1}{2p}$, and

$$\begin{aligned} -\frac{1}{2} < \alpha \leq \nu & \quad \text{for } p = 1, \\ -\frac{1}{2p} < \alpha < \nu + \frac{1}{2} - \frac{1}{2p} & \quad \text{for } 1 < p < \infty, \\ 0 \leq \alpha < \nu + \frac{1}{2} & \quad \text{for } p = \infty; \end{aligned}$$

4. if $\nu > \mu > -\frac{1}{2}$, then $\nu - \mu > \alpha - \beta \geq 0$, and

$$\begin{aligned} -\frac{1}{2} < \beta \leq \mu & \quad \text{for } p = 1, \\ -\frac{1}{2p} < \beta < \mu + \frac{1}{2} - \frac{1}{2p} & \quad \text{for } 1 < p < \infty, \\ 0 \leq \beta < \mu + \frac{1}{2} & \quad \text{for } p = \infty. \end{aligned}$$

Let φ be a function of modulus of continuity type such that

$$\sum_{j=n+1}^{\infty} \frac{1}{j} \varphi\left(\frac{1}{j}\right) \leq C_1 \varphi\left(\frac{1}{n}\right),$$

where constant C_1 does not depend on n . Let $f(x) \in L_{p,\alpha,\beta}$. Necessary and sufficient condition for the function $f(x)$ to have the derivative of order $2r - 1$ inside of the interval $(-1, 1)$ and

$$E_n(D_{x,\nu,\mu}^r f)_{p,\alpha,\beta} \leq C_2 \varphi\left(\frac{1}{n}\right)$$

is that the following inequality is satisfied

$$E_n(f)_{p,\alpha,\beta} \leq C_3 n^{-2r} \varphi\left(\frac{1}{n}\right),$$

where constants C_2 and C_3 do not depend on f and n .

Proof. The necessity of the condition is implied by induction directly from Corollary 2.1. We prove that the condition is sufficient.

Let $P_n(x)$ be the algebraic polynomial of best approximation of the function f . We consider the sequence of polynomials $Q_k(x)$ given by

$$Q_k(x) = P_{2^k}(x) - P_{2^{k-1}}(x) \quad (k = 1, 2, \dots)$$

and $Q_0(x) = P_1(x)$. From the conditions of the theorem, considering the properties of the function φ for $k \geq 1$ it follows that

$$\begin{aligned} (2) \quad \|Q_k\|_{p,\alpha,\beta} &= \|P_{2^k} - P_{2^{k-1}}\|_{p,\alpha,\beta} \leq E_{2^k}(f)_{p,\alpha,\beta} + E_{2^{k-1}}(f)_{p,\alpha,\beta} \\ &\leq 2E_{2^{k-1}}(f)_{p,\alpha,\beta} \leq C_4 2^{-2(k-1)r} \varphi\left(\frac{1}{2^{k-1}}\right) \leq C_5 2^{-2kr} \varphi\left(\frac{1}{2^k}\right). \end{aligned}$$

Applying Lemma 2.1 twice we get

$$\begin{aligned} \|D_{x,\nu,\mu}Q_k(x)\|_{p,\alpha,\beta} &\leq \|Q_k''(x)\|_{p,\alpha+1,\beta+1} + (|\mu - \nu| + |\nu + \mu + 2|) \|Q_k'(x)\|_{p,\alpha,\beta} \\ &\leq C_6 2^{2k} \|Q_k\|_{p,\alpha,\beta}, \end{aligned}$$

where constant C_6 does not depend on k . Applying this inequality r times we obtain

$$\|D_{x,\nu,\mu}^r Q_k(x)\|_{p,\alpha,\beta} \leq C_7 2^{2kr} \|Q_k\|_{p,\alpha,\beta}.$$

Thus inequality (2) yields

$$\sum_{k=1}^N \|D_{x,\nu,\mu}^r Q_k(x)\|_{p,\alpha,\beta} \leq C_8 \sum_{k=1}^N \varphi\left(\frac{1}{2^k}\right).$$

Noting that

$$\sum_{j=2^k}^{2^{k+1}-1} \frac{1}{j} \varphi\left(\frac{1}{j}\right) \geq C_{\varphi,1}^{-1} C_{\varphi,2}^{-1} \varphi\left(\frac{1}{2^k}\right) \sum_{j=2^k}^{2^{k+1}-1} \frac{1}{j} \geq C_9 \varphi\left(\frac{1}{2^k}\right),$$

considering the conditions of the theorem we have

$$\sum_{k=1}^{\infty} \|D_{x,\nu,\mu}^r Q_k(x)\|_{p,\alpha,\beta} \leq C_{10} \sum_{k=1}^{\infty} \sum_{j=2^k}^{2^{k+1}-1} \frac{1}{j} \varphi\left(\frac{1}{j}\right) \leq C_{10} \sum_{k=1}^{\infty} \frac{1}{k} \varphi\left(\frac{1}{k}\right) < \infty.$$

Since

$$\sum_{k=0}^n Q_k(x) = P_{2^n}(x),$$

from the inequality (2) and the conditions of the theorem it follows that for every segment $[a, b] \subset (-1, 1)$ the series $\sum_{k=0}^{\infty} Q_k(x)$ converges in terms of $L_p[a, b]$ metrics to the function $f(x)$. Since the series

$$\sum_{k=0}^{\infty} D_{x,\nu,\mu}^r Q_k(x)$$

also converges in terms of $L_p[a, b]$ metrics, then [4, p. 202] these series converge to the function $D_{x,\nu,\mu}^r f(x)$. This way we showed that the function $f(x)$ has the derivative of order $2r - 1$ absolutely continuous on every segment $[a, b] \subset (-1, 1)$.

Now we estimate the expression

$$I = \|D_{x,\nu,\mu}^r f(x) - D_{x,\nu,\mu}^r P_{2^N}(x)\|_{p,\alpha,\beta}.$$

From what we said above it is obvious that

$$\begin{aligned}
 I &\leq \sum_{k=N+1}^{\infty} \|D_{x,\nu,\mu}^r Q_k(x)\|_{p,\alpha,\beta} \leq C_7 \sum_{k=N+1}^{\infty} 2^{2kr} \|Q_k\|_{p,\alpha,\beta} \\
 &\leq C_{11} \varphi\left(\frac{1}{2^k}\right) \leq C_{12} \sum_{k=2^{N+1}}^{\infty} \frac{1}{k} \varphi\left(\frac{1}{k}\right).
 \end{aligned}$$

Hence we conclude that

$$I \leq C_{13} \varphi\left(\frac{1}{2^{N+1}}\right) \leq C_{14} \varphi\left(\frac{1}{2^N}\right).$$

Put

$$R_n(x) = D_{x,\nu,\mu}^r P_{2^N}(x) \quad (2^N \leq n < 2^{N+1});$$

we have

$$\begin{aligned}
 E_n(D_{x,\nu,\mu}^r f)_{p,\alpha,\beta} &\leq \|D_{x,\nu,\mu}^r f(x) - R_n(x)\|_{p,\alpha,\beta} \\
 &\leq C_{14} \varphi\left(\frac{1}{2^N}\right) \leq C_{15} \varphi\left(\frac{1}{n}\right).
 \end{aligned}$$

Theorem 3.1 is proved. □

Note that for a power function $\varphi(\delta) = \delta^\lambda$, the assertion of the theorem is given in [12].

Theorem 3.2. *Let be given a function φ of modulus of continuity type and numbers p, α and β such that $1 \leq p \leq \infty$;*

$$\begin{aligned}
 \alpha \leq 2, \quad \beta \leq 2 &\quad \text{for } p = 1, \\
 \alpha < 3 - \frac{1}{p}, \quad \beta < 3 - \frac{1}{p} &\quad \text{for } 1 < p \leq \infty.
 \end{aligned}$$

Let $f \in L_{p,\alpha,\beta}$. If

$$\tilde{\omega}(f, \delta)_{p,\alpha,\beta} \leq M\varphi(\delta),$$

then

$$E_n(f)_{p,\alpha,\beta} \leq CM\varphi\left(\frac{1}{n}\right),$$

where constant C does not depend on f, M dhe n .

Proof. From the properties of the function φ it follows that there exists a constant γ such that for every $l > 0$ the following inequality is satisfied

$$\varphi(lt) \leq C_1(l + 1)^\gamma \varphi(t),$$

where constant C_1 does not depend on l and t .

Indeed, if $l < 1$, then

$$\varphi(lt) \leq C_{\varphi,1} \varphi(t),$$

i.e. we get $\gamma \geq 0$. If $l \geq 1$, then choosing the positive integer m such that

$$2^{m-1} \leq l < 2^m$$

we have

$$\varphi(lt) \leq C_{\varphi,1}\varphi(2^m t) \leq C_{\varphi,1}C_{\varphi,2}^m\varphi(t).$$

We choose the positive integer N such that

$$2^{N-1} \leq C_{\varphi,2} < 2^N,$$

getting

$$\varphi(lt) \leq C_{\varphi,1}2^{Nm}\varphi(t) = C_{\varphi,1}2^N 2^{N(m-1)}\varphi(t) \leq C_2(l+1)^N\varphi(t),$$

i.e. $\gamma \geq N$.

We choose a $\gamma > 0$ and a positive integer q such that $2q > \gamma$, and for every positive integer n we choose the positive integer m satisfying the condition

$$(3) \quad \frac{n-1}{q+2} < m \leq \frac{n-1}{q+2} + 1.$$

It is easy to prove that under the condition of the theorem we have $f \in L_{1,2,2}$. Thus, for those q and m the algebraic polynomial $Q(x)$ defined in Lemma 2.3 is an algebraic polynomial of degree not greater than $n-1$. Hence

$$\begin{aligned} E_n(f)_{p,\alpha,\beta} &\leq \|f(x) - Q(x)\|_{p,\alpha,\beta} \\ &= \left\| \frac{1}{\gamma_m} \int_0^\pi (f(x) - \tilde{T}_t(f, x)) \left(\frac{\sin \frac{mt}{2}}{\sin \frac{t}{2}} \right)^{2q+4} \sin^3 t dt \right\|_{p,\alpha,\beta}. \end{aligned}$$

Applying the generalised Minkowski's inequality we obtain

$$E_n(f)_{p,\alpha,\beta} \leq \frac{1}{\gamma_m} \int_0^\pi \left\| \tilde{T}_t(f, x) - f(x) \right\|_{p,\alpha,\beta} \left(\frac{\sin \frac{mt}{2}}{\sin \frac{t}{2}} \right)^{2q+4} \sin^3 t dt.$$

There from by the conditions of the theorem we get

$$E_n(f)_{p,\alpha,\beta} \leq \frac{M}{\gamma_m} \int_0^\pi \varphi(t) \left(\frac{\sin \frac{mt}{2}}{\sin \frac{t}{2}} \right)^{2q+4} \sin^3 t dt.$$

Since

$$\varphi(t) = \varphi\left(nt \cdot \frac{1}{n}\right) \leq C_1(1+nt)^\gamma \varphi\left(\frac{1}{n}\right),$$

we have

$$\begin{aligned} E_n(f)_{p,\alpha,\beta} &\leq C_1 \frac{M}{\gamma_m} \varphi\left(\frac{1}{n}\right) \int_0^\pi (1+nt)^\gamma \left(\frac{\sin \frac{mt}{2}}{\sin \frac{t}{2}} \right)^{2q+4} \sin^3 t dt \\ &\leq C_3 M \varphi\left(\frac{1}{n}\right) \left\{ 1 + \frac{n^\gamma}{\gamma_m} \int_0^\pi t^\gamma \left(\frac{\sin \frac{mt}{2}}{\sin \frac{t}{2}} \right)^{2q+4} \sin^3 t dt \right\}. \end{aligned}$$

Applying now the standard estimate of Jackson's kernel and the inequality (3) we obtain

$$E_n(f)_{p,\alpha,\beta} \leq C_4 M\varphi\left(\frac{1}{n}\right) (1 + n^\gamma m^{-\gamma}) \leq C_5 M\varphi\left(\frac{1}{n}\right).$$

Theorem 3.2 is proved. \square

Theorem 3.3. *Let be given numbers p , α and β such that $1 \leq p \leq \infty$;*

$$\begin{aligned} \alpha > 1 - \frac{1}{2p}, \quad \beta > 1 - \frac{1}{2p} \quad \text{for } 1 \leq p < \infty, \\ \alpha \geq 1, \quad \beta \geq 1 \quad \text{for } p = \infty. \end{aligned}$$

Let φ be a function of modulus of continuity type such that inequality (1) for

$$\lambda_0 = \max \left\{ |\alpha - \beta|, \alpha - \frac{3}{2} + \frac{1}{2p}, \beta - \frac{3}{2} + \frac{1}{2p} \right\},$$

and inequality

$$(4) \quad \sum_{j=1}^n j\varphi\left(\frac{1}{j}\right) \leq C_{\varphi,4} n^2 \varphi\left(\frac{1}{n}\right)$$

are satisfied, where constant $C_{\varphi,4}$ does not depend on n . Let $f \in L_{p,\alpha,\beta}$. If

$$E_n(f)_{p,\alpha,\beta} \leq M\varphi\left(\frac{1}{n}\right),$$

then

$$\tilde{\omega}(f, \delta)_{p,\alpha,\beta} \leq CM\varphi(\delta),$$

where constant C does not depend on f , M and δ .

Proof. Let $P_n(x)$ be the algebraic polynomial of best approximation of degree not greater than $n - 1$ of the function f . Let the polynomials $Q_k(x)$ be given by

$$Q_k(x) = P_{2^k}(x) - P_{2^{k-1}}(x) \quad (k = 1, 2, \dots)$$

and $Q_0(x) = P_1(x)$. Since for $k \geq 1$ we have

$$\|Q_k\|_{p,\alpha,\beta} \leq E_{2^k}(f)_{p,\alpha,\beta} + E_{2^{k-1}}(f)_{p,\alpha,\beta},$$

considering the conditions of the theorem we have

$$(5) \quad \|Q_k\|_{p,\alpha,\beta} \leq C_1 M\varphi\left(\frac{1}{2^k}\right).$$

We estimate the expression

$$I = \left\| \tilde{T}_t(f, x) - f(x) \right\|_{p,\alpha,\beta}.$$

Let $0 < |t| \leq \delta$. Since the operator $\tilde{T}_t(f, x)$ is linear, for every positive integer N we have

$$I \leq \left\| \tilde{T}_t(f - P_{2^N}, x) - (f(x) - P_{2^N}(x)) \right\|_{p,\alpha,\beta} + \left\| \tilde{T}_t(P_{2^N}, x) - P_{2^N}(x) \right\|_{p,\alpha,\beta}.$$

Since $P_{2^N}(x) = \sum_{k=0}^N Q_k(x)$, we get

$$\begin{aligned} I &\leq \left\| \tilde{T}_t(f - P_{2^N}, x) - (f(x) - P_{2^N}(x)) \right\|_{p,\alpha,\beta} + \sum_{k=0}^N \left\| \tilde{T}_t(Q_k, x) - Q_k(x) \right\|_{p,\alpha,\beta} \\ &= J + \sum_{k=1}^N I_k. \end{aligned}$$

Let N be chosen so that

$$(6) \quad \frac{\pi}{2^N} < \delta \leq \frac{\pi}{2^{N-1}}.$$

We prove that the following inequalities are satisfied

$$(7) \quad J \leq C_2 M \varphi(\delta)$$

and

$$(8) \quad I_k \leq C_3 M \delta^2 2^{2k} \varphi\left(\frac{1}{2^k}\right),$$

where constants C_2 and C_3 do not depend on f , M , δ and k .

First we consider J . Applying Lemma 2.2 to the function $\Phi(x) = f(x) - P_{2^N}(x)$, considering that $|t| \leq \delta$ we obtain

$$\begin{aligned} J &\leq \left\| \tilde{T}_t(\Phi, x) \right\|_{p,\alpha,\beta} + \|\Phi(x)\|_{p,\alpha,\beta} \\ &\leq C_4 \left(\|\Phi\|_{p,\alpha,\beta} + \delta^{2(\gamma_1+\gamma_2)} \|\Phi\|_{p,\alpha-\gamma_1,\beta-\gamma_2} + \delta^{2\gamma_3} \|\Phi\|_{p,\alpha-\gamma_3,\beta-\gamma_3} \right. \\ &\quad \left. + \delta^{2(\gamma_1+\gamma_2+\gamma_3)} \|\Phi\|_{p,\alpha-\gamma_1-\gamma_3,\beta-\gamma_2-\gamma_3} \right), \end{aligned}$$

where numbers γ_1 , γ_2 and γ_3 are chosen by the conditions of Lemma 2.2. Applying Lemma 2.4, considering the conditions of the theorem we obtain

$$\begin{aligned} J &\leq C_5 M \varphi\left(\frac{1}{2^N}\right) \left(1 + \delta^{2(\gamma_1+\gamma_2)} 2^{-2N(\gamma_1+\gamma_2)} \right. \\ &\quad \left. + \delta^{2\gamma_3} 2^{-2N\gamma_3} + \delta^{2(\gamma_1+\gamma_2+\gamma_3)} 2^{-2N(\gamma_1+\gamma_2+\gamma_3)} \right) \end{aligned}$$

for $\lambda > \lambda_0 + \varepsilon$, where constant C_5 does not depend on f , M and δ , and either $\varepsilon = 0$ or ε is an arbitrary number from the interval $0 < \varepsilon < \frac{1}{2}$. Hence this inequality holds true for every $\lambda > \lambda_0$. Finally, applying the inequality (6) and the properties of the function φ we obtain

$$J \leq C_6 M \varphi\left(\frac{1}{2^N}\right) \leq C_7 M \varphi(\delta).$$

Thus inequality (7) is proved.

Now we prove the inequality (8). It can be proved that [13]

$$I_k \leq C_8 \delta^2 2^{2k} \|Q_k\|_{p,\alpha,\beta},$$

where constant C_8 does not depend on M , δ and k . Hence inequality (5) yields

$$I_k \leq C_9 M \delta^2 2^{2k} \varphi\left(\frac{1}{2^k}\right).$$

Inequality (8) is proved.

Inequalities (7) and (8) imply

$$I \leq C_{10} M \left\{ \varphi(\delta) + \delta^2 \sum_{k=1}^N 2^{2k} \varphi\left(\frac{1}{2^k}\right) \right\}.$$

Note that

$$\sum_{j=2^k}^{2^{k+1}-1} j \varphi\left(\frac{1}{j}\right) \geq C_{\varphi,1}^{-1} C_{\varphi,2}^{-1} \varphi\left(\frac{1}{2^k}\right) \sum_{j=2^k}^{2^{k+1}-1} j \geq C_{11} 2^{2k} \varphi\left(\frac{1}{2^k}\right).$$

Hence considering the inequality (4) we have

$$\begin{aligned} \sum_{k=1}^N 2^{2k} \varphi\left(\frac{1}{2^k}\right) &\leq C_{12} \sum_{k=1}^N \sum_{j=2^k}^{2^{k+1}-1} j \varphi\left(\frac{1}{j}\right) \leq C_{12} \sum_{k=1}^{2^{N+1}} k \varphi\left(\frac{1}{k}\right) \\ &\leq C_{13} 2^{2(N+1)} \varphi\left(\frac{1}{2^{N+1}}\right) \leq C_{14} 2^{2N} \varphi\left(\frac{1}{2^N}\right). \end{aligned}$$

There from, applying the inequality (6) we get

$$I \leq C_{15} M \left(\varphi(\delta) + \delta^2 2^{2N} \varphi\left(\frac{1}{2^N}\right) \right) \leq C_{16} M \varphi(\delta).$$

This way for $0 < |t| \leq \delta$ we proved that

$$\left\| \tilde{T}_t(f, x) - f(x) \right\|_{p,\alpha,\beta} \leq C_{16} \varphi(\delta),$$

where constant C_{16} does not depend on f and t . Taking into consideration that $\tilde{T}_0(f, x) = f(x)$, we conclude that this inequality also holds for $t = 0$. Thus the last inequality implies

$$\tilde{\omega}(f, \delta)_{p,\alpha,\beta} \leq C_{16} M \varphi(\delta).$$

Theorem 3.3 is proved. □

Theorem 3.4. *Let be given numbers p, α and β such that $1 \leq p \leq \infty$;*

$$\begin{aligned} \frac{1}{2} < \alpha \leq 2, & \quad \frac{1}{2} < \beta \leq 2 & \text{for } p = 1, \\ 1 - \frac{1}{2p} < \alpha < 3 - \frac{1}{p}, & \quad 1 - \frac{1}{2p} < \beta < 3 - \frac{1}{p} & \text{for } 1 < p < \infty, \\ 1 \leq \alpha < 3, & \quad 1 \leq \beta < 3 & \text{for } p = \infty. \end{aligned}$$

Let φ be a function of modulus of continuity type such hat inequality (1) for

$$\lambda_0 = \max \left\{ |\alpha - \beta|, \alpha - \frac{3}{2} + \frac{1}{2p}, \beta - \frac{3}{2} + \frac{1}{2p} \right\},$$

and inequality (4) are satisfied. Let $f \in L_{p,\alpha,\beta}$. For

$$E_n(f)_{p,\alpha,\beta} \leq C_1 \varphi \left(\frac{1}{n} \right),$$

it is necessary and sufficient that

$$\tilde{\omega}(f, \delta)_{p,\alpha,\beta} \leq C_2 \varphi(\delta),$$

where constants C_1 and C_2 do not depend on f, n and δ .

Theorem 3.4 is implied directly by Theorems 3.2 and 3.3.

Theorem 3.5. *Let be given numbers $p, \alpha, \beta, \nu, \mu, r, \nu_0$ and μ_0 such that $1 \leq p \leq \infty, r \in \mathbb{N} \cup \{0\}, \nu \geq \mu \geq -\frac{1}{2}$,*

$$\nu_0 = \min \left\{ \nu, \frac{5}{2} - \frac{1}{2p} \right\}, \quad \mu_0 = \min \left\{ \mu, \frac{5}{2} - \frac{1}{2p} \right\};$$

1. *if $\nu = \mu > \frac{1}{2}$, then $\alpha = \beta$, and*

$$\begin{aligned} \frac{1}{2} < \alpha \leq \nu_0 & \quad \text{for } p = 1, \\ 1 - \frac{1}{2p} < \alpha < \nu_0 + \frac{1}{2} - \frac{1}{2p} & \quad \text{for } 1 < p < \infty, \\ 1 \leq \alpha < \nu_0 + \frac{1}{2} & \quad \text{for } p = \infty; \end{aligned}$$

2. *if $\nu > \mu > \frac{1}{2}$, then $\nu - \mu > \alpha - \beta \geq 0$, and*

$$\begin{aligned} \frac{1}{2} < \beta \leq \mu_0 & \quad \text{for } p = 1, \\ 1 - \frac{1}{2p} < \beta < \mu_0 + \frac{1}{2} - \frac{1}{2p} & \quad \text{for } 1 < p < \infty, \\ 1 \leq \beta < \mu_0 + \frac{1}{2} & \quad \text{for } p = \infty; \end{aligned}$$

Let φ be a function of modulus of continuity type such that inequality (1) for

$$\lambda_0 = \max \left\{ |\alpha - \beta|, \alpha - \frac{3}{2} + \frac{1}{2p}, \beta - \frac{3}{2} + \frac{1}{2p} \right\},$$

and inequality (4) are satisfied. Let $f \in L_{p,\alpha,\beta}$. Necessary and sufficient condition for

$$E_n(f)_{p,\alpha,\beta} \leq \frac{C_1}{n^{2r}} \varphi \left(\frac{1}{n} \right)$$

is that the function $f(x)$ has the derivative of order $2r$ inside of the interval $(-1, 1)$ satisfying the condition

$$\tilde{\omega} (D_{x,\nu,\mu}^r f, \delta)_{p,\alpha,\beta} \leq C_2 \varphi(\delta),$$

where constants C_1 and C_2 do not depend on f , n and δ , while $D_{x,\nu,\mu}^0 f(x) = f(x)$.

Theorem 3.5 is implied by Theorems 3.4 and 3.1.

Note that for $\varphi(\delta) = \delta^\lambda$, $2\lambda_0 < \lambda < 2$ and $r = 0$ Theorem 3.5 is proved in [13].

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