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Domination Number of

Circular-Arc Overlap Graphs: A New Perspective

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Abstract

The paper puts forth the findings that characterize circular-arc overlap graphs(CAOG) as various complete partite graphs namely, $k_{m,n}$, $k_{m,n,p}$, ………….., $k_{m_1,m_2,....,m_p}$.

An attempt also has been made to promote the characterization of circular-arc overlap graphs, comprising a set consisting of any pair of adjacent vertices of the graph as a minimal dominating set, entitling the neighborhood of the arcs of the graph. Simultaneously, the paper proposes an additional feature of algorithms that check the nature of the CAOG. In our recent paper, Circular-arc overlap graphs have been individualized as $k_{2,2}$, $k_{3,3}$, $k_{n,n}$ and $k_{2,2,.....,2}$ graphs [5].

Mathematics Subject Classification: 05C69

Keywords: Circular-arc overlap graphs, complete partite graphs, dominating set, minimal dominating set, independent set

1 Introduction

Graphs presented in this article are all finite and contain no loops or multiple edges. Any undefined terms in this paper may be found in Harary [2] and for more details about the domination number one can refer to Walikar et al. [6].

Circular-arc overlap graphs are a new class of overlap graphs introduced by Kashiwa Bara and Masuda, defined for a set of arcs on a circle. A representation of a graph with arcs helps in the solving of combinatorial problems on the graph. A graph is a circular - arc graph, if it is the intersection graph of a finite set of arcs on a circle. A circular-arc overlap graph is a specific enclosure of circular - arc graph; it is an overlap graph defined for a set of arcs on a circle. That is**,** there exists one arc for each vertex of G and two vertices in G are adjacent in G, if and only if the corresponding arcs intersect and one is not contained in the other.

Let A= $\{A_1,A_2,\ldots,A_n\}$ be a circular - arc family, where each A_i is an arc on the circle. Two arcs A_i and A_j are considered to overlap each other, if they intersect each other but, neither of them comprises the other. A graph G is called a Circular-Arc Overlap Graph (CAOG), if there is a one-to-one correspondence between the vertex set V and the circular-arc family A and two vertices in V are adjacent to each other, if and only if the corresponding arcs in A overlap each other. Neighborhood of an arc A_i is defined as the set of all arcs belonging to A that overlap the arc A_i. The notion of dominating set is due to Ore [4]. Cockayne and Hedetnieme [1] contributed to the domination theory in graphs besides many others. A subset D of vertex set in a graph G (V, E) is called a dominating set of G, if every vertex of V- D is adjacent to some vertex of D. A dominating set is designated to be minimal, if no proper subset of the set is a dominating set. Whereas the domination number of G is defined as the minimum cardinality taken over the minimal dominating sets of G. With the help of the following theorems, Circular-Arc Overlap Graphs are characterized as complete partite graphs**.**

2 Theorems

Theorem 2.1: Let A = {1, 2, 3, 4, …., k} be a circular - arc family corresponding to a CAOG 'G' of order k. Then there exists arbitrary arcs m, n $\in$ A, such that m, n < k and m + n = k, satisfying the conditions

$$\text{nbd}[i] = \begin{cases} \{i, m+1, m+2, \ldots\ldots\ldots, k\} & \text{for } i = 1, 2, \ldots\ldots\ldots\ldots, m \quad \ldots\ldots(i) \\ \{1, 2, \ldots\ldots\ldots\ldots, m, i\} & \text{for } i = m+1, m+2, \ldots, m+n = k \quad \ldots\ldots(ii) \end{cases}$$

if and only if G is a $k_{m,\,n}$ graph.

Proof: Let A = {1, 2, 3, 4, …………., k} be a circular - arc family corresponding to a CAOG 'G' of order k. There exists one - to - one correspondence between the vertex set V= {v_1, v_2 , v_3, … ,v_k} and the circular arc family A of the graph G. Let v_i be the vertex corresponding to the arc i for i=1, 2 ,…, k respectively. First, let us suppose that the family A satisfies the conditions (i) and (ii) mentioned in the theorem. Then the distinct edges of the graph are
{v_1, v_{m+1}}, {v_1, v_{m+2}}, ……, {v_1, v_{m+n} }; {v_2, v_{m+1}}, {v_2, v_{m+2}}, ……, {v_2, v_{m+n} };
………………………………………; {v_m , v_{m+1}}, {v_m, v_{m+2}},….., {v_m, v_{m+n} };
{v_1, v_{m+1}}, {v_2, v_{m+1}}, ….. , {v_m, v_{m+1}};{v_1, v_{m+2}}, {v_2, v_{m+2}},……, {v_m, v_{m+2}};
……………………………………….. ;{v_1, v_{m+n}},{v_2, v_{m+n}}, ……, {v_m, v_{m+n}}.
The sets U = {v_1, v_2, …. , v_m} and V = { v_{m+1}, v_{m+2}, ……. , v_{m+n} } are independent sets and there exists an edge between every vertex in the set U and every vertex in the set V. Therefore the sets U and V can be considered as the bipartites of the graph. Hence G is a $k_{m, n}$ graph with bipartites U and V. Conversely, let G = $k_{m,n}$ graph with two partites U = {v_1, v_2, … , v_m} and V = { v_{m+1}, v_{m+2}, … , v_{m+n} }. Then U and V are independent sets and moreover there exists an edge between every vertex in U to each vertex in V. This happens only when the circular arc family satisfies the conditions (i) and (ii) stated in the theorem.

Algorithm 2.1.1 :

Input : Circular - arc family A= {1, 2, …….. , k}
Output : G is a $k_{m, n}$ graph.

Step 1 : Find nbd [i], for i=1 to k
Step 2 : If nbd [i] ={i, m+1, m+2,…, k} for i= 1,2, ……, m for some integer m, with m < k go to step 3
Else
G is not a $k_{m,n}$ graph

Step 3 : If nbd [i] = {i, 1, 2, … , m} for i= m+1, m+2, … , m+n=k for some integer n, with m+n = k go to step 4
Else
G is not a $k_{m,n}$ graph
Step 4 : End

Illustration 2.1.2: Let the circular - arc family A = {1, 2, …, 5} corresponding to the circular-arc overlap graph G be as in Figure 1. Clearly, the circular arc family satisfies the conditions mentioned in the theorem for m=2 and n=3. Hence G is a $k_{2,3}$ graph

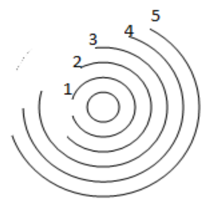

Figure 1

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Verification: In this case, the circular-arc overlap graph G can also be drawn as follows:

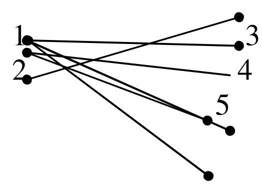

Figure 2

Clearly the sets of vertices {1, 2} and {3, 4, 5} are independent sets and moreover there exits an edge between every vertex in the first set to every vertex in the second set. Hence the two sets can be considered as the partites of the graph. Implies, $G = K_{2,3}$

Theorem 2.2: Let A = {1, 2, 3, 4, …, k} be a circular - arc family corresponding to a CAOG 'G' of order k. Then there exists arbitrary arcs m, n, p ∈ A, such that m, n, p < k and m + n + p = k, satisfying the conditions

$$\text{nbd}[i]= \begin{cases} \{i, m+1, m+2, \ldots\ldots, k\} & \text{for } i = 1, 2, \ldots, m \\ \{i,1,2,\ldots,m,m+n+1,m+n+2,\ldots,m+n+p=k\} & \text{for } i = m+1,m+2,\ldots,m+n \\ \{i, 1, 2, \ldots, m, m+1,\ldots, m+n\} & \text{for } i= m+n+1,m+n+2,\ldots,m+n+p \end{cases}$$

if and only if G is a $k_{m,n,p}$ graph.

Proof: : First, let us suppose that the circular- arc family 'A' corresponding to the CAOG 'G' satisfy the conditions mentioned in the theorem and v_i be the vertex corresponding to the arc i for i=1, 2 ,…, k. Then the distinct edges of the graph are

$\{v_1,v_{m+1}\},\{v_1,v_{m+2}\},\ldots,\{v_1,v_{m+n}\},\{v_1,v_{m+n+1}\},\ldots,\{v_1,v_{m+n+p}\}$;

$\{v_2,v_{m+1}\}\{v_2,v_{m+2}\},\ldots,\{v_2,v_{m+n}\},\{v_2,v_{m+n+1}\},\ldots,\{v_2,v_{m+n+p}\}$; …………………… ;

$\{v_m, v_{m+1}\}\{v_m, v_{m+2}\},\ldots\ldots, \{v_m, v_{m+n}\}\},\{v_m, v_{m+n+1}\},\ldots\ldots,\{v_m, v_{m+n+p}\}$;

$\{v_1,v_{m+1}\}$, $\{v_2,v_{m+1}\}$,…,$\{v_m, v_{m+1}\}$,$\{v_{m+1}, v_{m+n+1}\}$,$\{v_{m+1}, v_{m+n+2}\}$,…,$\{v_{m+1}, v_{m+n+p}\}$;
$\{v_1, v_{m+2}\}$, $\{v_2, v_{m+2}\}$,…,$\{v_m,v_{m+2}\}$,$\{v_{m+2},v_{m+n+1}\}$,$\{v_{m+2},v_{m+n+2}\}$,…..,$\{v_{m+2},v_{m+n+p}\}$;
…,$\{v_1,v_{m+n}\}$,$\{v_2, v_{m+n}\}$,…,$\{v_m, v_{m+n}\}$,$\{v_{m+n},v_{m+n+1}\}$,$\{v_{m+n},v_{m+n+2}\}$,...,$\{v_{m+n}, v_{m+n+p}\}$;
$\{v_1, v_{m+n+1}\}$, $\{v_2, v_{m+n+1}\}$…..,$\{v_m, v_{m+n+1}\}$, $\{v_{m+1}, v_{m+n+1}\}$,…………, $\{v_{m+n}, v_{m+n+1}\}$;
$\{v_1, v_{m+n+2}\}$, $\{v_2, v_{m+n+2}\}$,….,$\{v_m, v_{m+n+2}\}$, $\{v_{m+1}, v_{m+n+2}\}$,…………, $\{v_{m+n},v_{m+n+2}\}$;
…….; $\{v_1, v_{m+n+p}\}$,$\{v_2, v_{m+n+p}\}$,….,$\{v_m, v_{m+n+p}\}$, $\{v_{m+1}, v_{m+n+p}\}$,…, $\{v_{m+n}, v_{m+n+p}\}$.
are the distinct edges of the graph. The sets $U=\{v_1,v_2,\dots,v_m\}$, $V=\{v_{m+1},v_{m+2},\dots,v_{m+n}\}$ and $W=\{v_{m+n+1}, v_{m+n+2},\dots,v_{m+n+p}\}$ are independent sets and every vertex in one set is adjacent to every other vertex in the remaining sets . Therefore the sets U, V and W can be considered as the partites of the graph. Hence G is a $k_{m, n, p}$ graph with partites U, V and W.

Conversely, let $G = k_{m, n, p}$ with three partites $U = \{v_1,v_2, \dots ,v_m\}$, $V = \{ v_{m+1}, v_{m+2}, \dots,v_{m+n}\}$ and $W = \{v_{m+n+1}, v_{m+n+2},\dots,v_{m+n+p}\}$. Then U, V and W are independent sets and moreover there exists an edge between every vertex in one partite to each vertex in every other partite. This happens only when the circular arc family satisfies the conditions stated in the theorem.

Algorithm 2.2.1

Input : Circular arc family A= {1, 2, …, k}
Output : G is a $k_{m, n, p}$ graph.
Step 1 : Find nbd [i], for i=1 to k
Step 2 : If nbd [i] = {i, m+1, m+2, …., k} for i= 1,2, …………………….. , m;
for some integer m, with m < k, go to step 3
Else
G is not a $k_{m, n, p}$ graph.
Step 3 : If nbd [i] = {i,1,2,…, m, m+n+1, m+n+2,…,k } for i= m+1,m+2,…,m+n
for some integer n with m+n < k**,** go to step 4
Else
G is not a $k_{m, n, p}$ graph.
Step 4 : If nbd [i] = {i, 1, 2, … , m+n} for i= m+n+1,m+n+2,…,m+n+p = k;
for some integer p with m+n+p = k**,** go to step 5
Else
G is not a $k_{m, n, p}$ graph.
Step 5 : End

Illustration 2.2.2:

Let the circular arc family A = {1, 2, …, 10} corresponding to the circular-arc overlap graph G be as shown in the Figure 3.

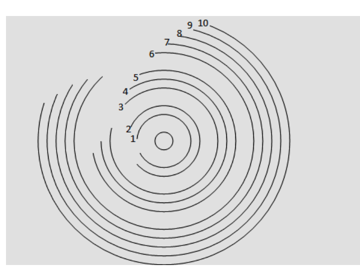

Figure 3

Clearly, the circular-arc family satisfies the conditions mentioned in the theorem for m=2, n=3 and p=5. Hence G is a $k_{2,3,5}$ graph.

Verification: The circular-arc overlap graph G corresponding to the given circular-arc family can also be drawn as in Figure 4 which shows that the graph considered is a complete partite graph $k_{2,3,5}$ graph with partites {1, 2}, {3, 4, 5} and {6, 7, 8, 9, 10}

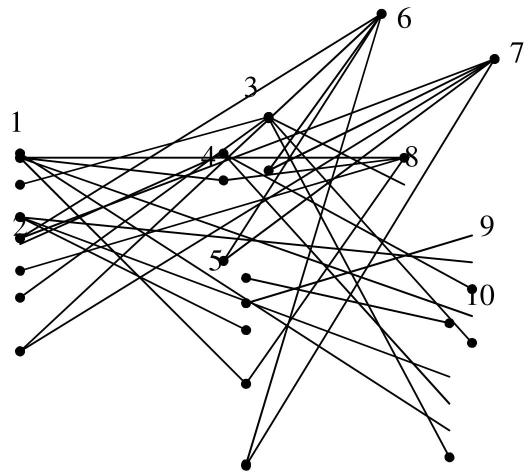

Figure 4

Theorem ss2.3: Let A = {1, 2, 3, 4,…,k} be a circular arc family corresponding to a CAOG 'G' order k. Then there exists arbitrary arcs $m_1, m_2, \ldots, m_p \in A$, such that $m_1, m_2, \ldots, m_p < k$ and $m_1+ m_2+ \ldots+ m_p = k$ satisfying the conditions

$$
nbd[i]= \begin{cases} \{i, m_1+1, m_1+2, \ldots, k\} & \text{for } i = 1, 2, \ldots, m_1 \\ \{i, 1, 2, \ldots, m_1, m_1+m_2+1, m_1+m_2+2, \ldots, k\} & \text{for } i=m_1+1, m_1+2, \ldots, m_1+m_2 \\ \ldots \\ \{i, 1, 2, \ldots, m_1, m_1+1, \ldots, m_1+m_2, m_1+m_2+1, \ldots, m_1+m_2+\ldots+m_{p-1}\} & \\ \quad \text{for } i= m_1+m_2+\ldots+m_{p-1}+1, m_1+m_2+\ldots+m_{p-1}+2, \ldots, m_1+m_2+\ldots+m_p & \end{cases}
$$

if and only if G is a complete partite graph $K_{m_1, m_2, \ldots, m_p}$ with p partites.

Proof: Let A = {1, 2, 3, 4, …, k} be a circular arc family corresponding to a CAOG 'G' of order k and v_i be the vertex corresponding to the arc i, for i= 1, 2, 3,4,…,k. First, let us suppose that the family A satisfies the conditions mentioned in the theorem. Then there exists 'p' independent sets, say $U_1, U_2, \ldots, U_P$ of cardinality $m_1, m_2, \ldots, m_p$ respectively of the graph G such that every vertex in one set is adjacent to every other vertex in the remaining sets. Therefore the sets U_1, U_2………….., U_P can be considered as the partites of the graph. Hence G is a complete partite graph $Km_1, m_2, \ldots, m_p$ with p partites U_1, U_2…………..U_P. Conversely, let G be a complete partite graph $Km_1, m_2, \ldots, m_p$ with p partites U_1,U_2…………..U_P. Then the sets U_1,U_2…………..U_P are independent sets and moreover there exists an edge between every vertex in one partite to each vertex in every other partite. This happens only when the circular arc family satisfies the conditions stated in the theorem.

Theorem 2.4: Let A={1, 2, 3, 4,…, k} be a circular arc family corresponding to a CAOG 'G' of order k.Then there exists arbitrary arcs $m_1, m_2,\ldots,m_p \in A$, such that $m_1, m_2, \ldots, m_p < k$ and $m_1+ m_2+ \ldots + m_p = k$ satisfying the conditions

$$
nbd[i]= \begin{cases} \{i, m_1+1, m_1+2, \ldots\ldots, k\} \text{ for } i = 1, 2, \ldots, m_1 \\ \{i, 1, 2,\ldots, m_1, m_1+m_2+1, m_1+m_2+2,\ldots,k\} \text{ for } i=m_1+1, m_1+2,\ldots,m_1+m_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \{i, 1, 2,\ldots,m_1, m_1+1,\ldots,m_1+m_2, m_1+m_2+1,\ldots, m_1+m_2+\ldots\ldots+m_{p-1},\} \\ \text{for } i= m_1+m_2+\ldots+m_{p-1}+1, m_1+m_2+\ldots+m_{p-1}+2,\ldots,m_1+m_2+\ldots+m_p \end{cases}
$$

if and only if every pair of adjacent vertices is a minimal dominating set of G

Proof: Let A = {1, 2, 3, 4,…, k} be a circular arc family corresponding to a circular-arc overlap graph G of order k. First, let us suppose that the family A satisfies the conditions mentioned in the theorem. Then by theorem 2.3, the graph G is a complete partite graph with 'p' partites. Hence any two adjacent vertices form a minimal dominating set of G. Conversely, let every pair of adjacent vertices in G form a minimal dominating set of G. Then G is a complete partite graph [3] and hence satisfies the conditions mentioned in the theorem.

Illustration 2.4.1: Let the circular arc family A = {1, 2,…, 10} corresponding to the circular-arc overlap graph G be as in Figure 5. From Figure 5**,** it is clear that the circular arc family satisfies the conditions mentioned in the theorem for m_1=1, m_2=2, m_3=3 and m_4=4. So, every pair of adjacent vertices forms a minimal dominating set of G.

Verification: The graph G can also be drawn as shown in the Figure 6. It is clear from the diagram that the given graph is a $k_{1,2,3,4}$ graph and every pair of adjacent vertices is a minimal dominating set of G.

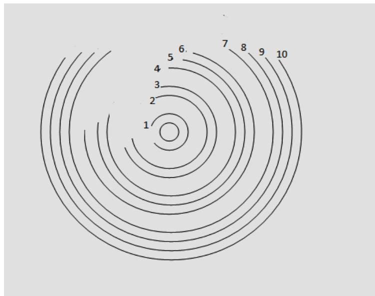

Figure 5

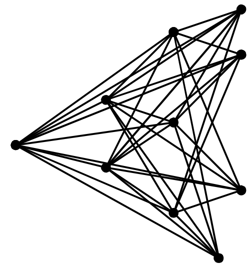

Figure 6

References

[1] E. J. Cokayne and S. T. Hedetnieme, Networks 7(1977), 247-61

[2] F. Harary, Graph Theory, Addison- A Wesley Reading Mass (1969).

[3] S. R. Jayaram, Minimal Dominating Sets of Cardinality Two in a Graph, Indian J. pure appl. Math, 28(1), January 1997, 43-46**.**

[4] O. Ore, Theory of Graphs, Ann. Math. Soc. Colloq. Publ. 38, Providence, 1962.

[5] A.Sudhakaraiah and V. Raghava Lakshmi, Characterization of Circular-Arc Overlap Graphs with Domination Number Two, International Journal of Computer Applications, volume 41-No.3, March 2012, 6-9.

[6] H. B. Walikar, B. D. Acharya and E. Sampathkumar, Recent Developments in the Theory of Domination in Graphs, Mehta Research Institute, Allahabad, MRI Lecture Notes in Math. 1 (1979).

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