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Generalized CR-Submanifolds of Manifolds with a Sasakian 3-Structure

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Abstract. In this article, we study the generalized CR-submanifolds of manifolds with a Sasakian 3-structure. The Integrability conditions for the distributions, which are involved in the definition of such manifolds are obtained. Geometry of leaves of certain distributions are also studied.

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1. **Introduction**

A. Bejancu [1] defined and studied CR-submanifolds of Kählerian manifolds. Geometry of CR-submanifolds of manifolds equipped with Kähler, Sasakian, quasi-Sasakian and S-structures have been studied by a number of authors, including those of [1], [2], [10], [3].

Ion Mihai [6] introduced a new class of submanifold called "generalized CR-submanifold" of a Kähler manifold. This class contains both CR-submanifolds and slant submanifolds. Purpose of present work is to study the generalized CR-submanifolds of a manifold with a Sasakian 3-structures. Section-2 is devoted to some necessary preliminaries. In section-3, some basic results are given. Integrability conditions for certain distributions on the generalized CR-submanifolds are investigated in section-4. In the last section, the geometry of leaves of certain distributions are studied.

2. Preliminaries

In 1970, Y.Y.Kuo [10] studied manifolds with an almost contact 3-structure. Let $(\overline{M}\ , g)$ be a $(4n+3)$- dimensional differentiable manifold admitting three almost contact structures $(\varphi_a, \xi_a, \eta_a)$, $a = 1, 2, 3$ on $\overline{M}$, admits such that

$$\varphi_a^2 = -I + \eta_a \otimes \xi_a,\ \eta_a(\xi_a) = 1, \varphi_a(\xi_a) = 0, \eta_a o \varphi_a = 0, \tag{2.1}$$

where I is the identity morphism on the tangent bundle to $\overline{M}$. Let these three almost contact structures satisfy

$$\varphi_a o \varphi_b - \xi_a \otimes \eta_b = -\varphi_b o \varphi_a + \xi_b \otimes \eta_a = \varphi_c, \tag{2.2}$$

$$\varphi_a(\xi_b) = -\varphi_b(\xi_a) = \xi_c, \tag{2.3}$$

$$\eta_a o \varphi_b = -\eta_b o \varphi_a = \eta_c, \tag{2.4}$$

$$\eta_a(\xi_b) = 0, \quad a \neq b, \tag{2.5}$$

for any cyclic permutation (a, b, c) of $(1, 2, 3)$. Then we say that $\overline{M}$ is endowed with an almost contact 3-structurer[10] . If $\overline{M}$ is a Riemannian manifold, then there is always a Riemannian metric g on $\overline{M}$ such that

$$g(\varphi_a X, \varphi_a Y) = g(X, Y) - \eta_a(X)\eta_a(Y) \tag{2.6}$$

and

$$g(X, \xi_a) = \eta_a(X), \tag{2.7}$$

for any $X, Y \in T\overline{M}$ and $a = 1, 2, 3$. Then we say that $\overline{M}$ is endowed with an almost contact metric 3-structurer $(\varphi_a, \xi_a, \eta_a, g)$. From (2.7), it follows that ξ_1, ξ_2, ξ_3 are mutually orthogonal. We also have

$$g(\varphi_a X, Y) + g(X, \varphi_a Y) = 0, \tag{2.8}$$

for all $X, Y \in T\overline{M}$ and $a = 1, 2, 3$.

An almost contact metric structure (φ, ξ, η, g) is called a Sasakian structure [4] if and only if we have

$$(\overline{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.9}$$

where $\overline{\nabla}$ is Riemannian connection of $\overline{M}$. In this case, we have

$$\overline{\nabla}_X \xi = -\varphi X \tag{2.10}$$

If all the three almost contact metric structurers $(\varphi_a, \xi_a, \eta_a, g), a = 1, 2, 3$ are Sasakian structures i.e.,

$$(\overline{\nabla}_X \varphi_a)Y = g(X, Y)\xi_a - \eta_a(Y)X, \tag{2.11}$$

$$\overline{\nabla}_X \xi = -\varphi_a X, \tag{2.12}$$

then we say that the manifold $\overline{M}$ is endowed with a Sasakian 3-structure $(\varphi_a, \xi_a, \eta_a, g)$; $a = 1, 2, 3$. Y.Y. Kuo [10] proved that the Euclidean space R^{4n+3} and the sphere S^{4n+3} are examples of manifolds with a Sasakian 3-structure.

3. **Generalized CR-submanifolds**

Let M be a $(m + 3)$-dimensional submanifold of a $(4n + 3)$-dimensional manifold $\overline{M}$ with a Sasakian 3-structure $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$, such that the structure vector fields $\xi_a, a = 1, 2, 3$ are tangential to M. Let T_xM (resp.$T_x^{\perp}M$) denotes the tangent (resp.normal) space to M at x. We say that M is a generalized CR-submanifold of $\overline{M}$ if

$$D^{\perp}_{ax} = T_x(M) \cap \varphi_a T_x^{\perp}(M); \qquad a = 1, 2, 3, \quad x \in M, \tag{3.1}$$

defines a differentiable subbundle of $T_x(M)$ [6]. Thus we have three distributions $D_a^{\perp}$ on M defined by

$$D_a^{\perp} : x \to D^{\perp}_{ax}; \qquad \forall x \in M, \ a = 1, 2, 3. \tag{3.2}$$

For any $X \in TM$, we have $g(\varphi_a X, \xi_a) = 0$. We put

$$\varphi_a X = B_a X + C_a X; \quad a = 1, 2, 3, \tag{3.3}$$

where $B_a X \in \{\xi_a\}^{\perp}$ and $C_a X \in T^{\perp}M$. For $X \in D_a^{\perp}$ we have $B_a X = 0$. So $\varphi_a(D_a^{\perp}) \subset T^{\perp}M$, $a = 1, 2, 3$.

Proposition 3.1. *Let M be a generalized CR-submanifold of $\overline{M}$, then the distributions $D_1^{\perp}, D_2^{\perp}, D_3^{\perp}$ are mutually orthogonal to each other.*

Proof. Let $X_a \in D_a^{\perp}$ & $X_b \in D_b^{\perp}$ with $a \neq b$. Then there exist $V_a, V_b \in T^{\perp}M$ such that

$$X_a = \varphi_a(V_a) \text{ and } X_b = \varphi_b(V_b).$$

Using (2.8), we obtain

$$\begin{aligned} g(X_a, X_b) &= g(\varphi_a(V_a), \varphi_b(V_b)) \\ &= -g(V_a, (\varphi_a o \varphi_b)(V_b)) \\ &= -g(V_a, \varphi_c(V_b) + \eta_b(V_b)\xi_a) \\ &= 0, \end{aligned}$$

where (a, b, c) is a cyclic permutation of $(1, 2, 3)$. Since ξ_a is tangent to M and $\varphi_c(T_x^{\perp}M)$ is orthogonal to $T_x^{\perp}M$, $D_a^{\perp}$ is orthogonal to $D_b^{\perp}$, $a \neq b$. $\square$

Now, we denote

$$D^{\perp} = D_1^{\perp} \oplus D_2^{\perp} \oplus D_3^{\perp} \quad \text{and} \quad \langle \xi \rangle = \langle \xi_1 \rangle + \langle \xi_2 \rangle + \langle \xi_3 \rangle .$$

Proposition 3.2. *The distribution $D^{\perp}$ is orthogonal to the distribution $\langle \xi \rangle$.*

Proof. Let $Y_a \in D_a^{\perp}, a = 1, 2, 3$, then by (3.1), there exist a normal vector field U_a such that $Y_a = \varphi_a(U_a)$. Using (2.8), we get

$$\begin{aligned} g(Y_a, \xi_b) &= g(\varphi_a(U_a), \xi_b) \\ &= -g(U_a, \varphi_a(\xi_b)) \\ &= -g(U_a, \xi_c) \\ &= 0, \end{aligned}$$

where (a, b, c) is a cyclic permutation of $(1, 2, 3)$. □

We denote by D, the complementary orthogonal subbundle to $D^{\perp} \oplus \{\xi\}$ in TM. Thus we have

$$TM = D \oplus D^{\perp} \oplus \{\xi\}, \tag{3.4}$$

where distributions $D, D^{\perp}$ and $\{\xi\}$ are mutually orthogonal to each other.

4. Integrability of distributions on a generalized CR-submanifold

Let M be a $(m + 3)$-dimensional generalized CR-submanifold of a $(4n + 3)$-dimensional manifold $\overline{M}$ with a Sasakian 3-structure $(\varphi_a, \xi_a, \eta_a, g)$ $a = 1, 2, 3$. Let us denote by P, Q_1, Q_2 and Q_3, the projection morphism of TM to distributions $D, D_1^{\perp}, D_2^{\perp}, D_3^{\perp}$ respectively. Then for any vector field $X \in TM$, we have

$$X = PX + \sum_{d=1}^{3} \{Q_d X + \eta_d(X)\xi_d\}. \tag{4.1}$$

Applying φ_a to (4.1) and taking account of (2.2), (2.3), (2.4) and (2.5), we obtain

$$\varphi_a X = \varphi_a PX + \varphi_a Q_b X + \varphi_a Q_c X + \eta_b(X)\xi_c - \eta_c(X)\xi_b + \varphi_a Q_a X, \tag{4.2}$$

where (a, b, c) is a cyclic permutation of $(1, 2, 3)$. Let us put

$$\varphi_a X = F_a X + U_a X, \tag{4.3}$$

where $F_a X, U_a X$ are tangential and normal parts of $Q_a X$ respectively. Then from (4.2), we have

$$F_a X = \varphi_a PX + \varphi_a Q_b X + \varphi_a Q_c X + \eta_b(X)\xi_a - \eta_c(X)\xi_b, \tag{4.4}$$

$$U_a X = \varphi_a Q_a X. \tag{4.5}$$

For $X \in V_a^{\perp}$, $a = 1, 2, 3$,

$$\varphi_a X = U_a X. \tag{4.6}$$

Now, Let us denote by ∇, the Levi-Civita connection of M and by $\nabla^{\perp}$, the Riemannian connection on the normal bundle $T^{\perp}M$. Then the equations of Gauss and Weingarten are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{4.7}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{4.8}$$

for all $X, Y \in TM$ and $V \in T^{\perp}M$, where h is the second fundamental form of M and A_V is the fundamental tensor of Weingarten with respect to the normal section V. We also have

$$g(h(X,Y),V) = g(A_V X, Y). \tag{4.9}$$

For any $V \in T^{\perp}M$, we get

$$\varphi_a V = B_a V + C_a V, a = 1,2,3, \tag{4.10}$$

where $B_a V \in TM$ and $C_a V \in T^{\perp}M$.

Lemma 4.1. *Let M be a generalized CR-submanifold of $\overline{M}$. Then we have*

$$\begin{aligned} \nabla_Y F_a X - F_a(\nabla_Y X) &= A_{U_a X} Y + B_a h(X,Y) \\ &\quad + g(X,Y)\xi_a - \eta_a(X) Y \end{aligned} \tag{4.11}$$

and

$$h(Y, F_a X) + \nabla^{\perp}_Y U_a X = U_a(\nabla_Y X) + C_a h(X,Y), \tag{4.12}$$

for all $X, Y \in TM$.

Proof. On differentiating covariantly (4.3) and using (2.11), (4.7) & (4.8), we obtain

$$\begin{aligned} & \nabla_Y F_a X + h(Y, F_a X) - AU_{aX} Y + \nabla^{\perp}_Y U_a X \\ = \; & F_a(\nabla_Y X) + U_a(\nabla_Y X) + B_a h(X,Y) + C_a h(X,Y) \\ & + g(X,Y)\xi_a - \eta_a(X) Y, \quad a = 1,2,3. \end{aligned} \tag{4.13}$$

Then (4.11), (4.12) follow from (4.13), by taking tangential and normal components respectively. □

Lemma 4.2. *Let M be a generalized CR-submanifold of $\overline{M}$. Then we have*

$$\nabla_X \xi_a = F_a X; \; h(X, \xi_a) = U_a X, \qquad \forall X \in D, \tag{4.14}$$

$$\nabla_Y \xi_a = 0; \; h(Y, \xi_a) = -\varphi_a Y, \qquad \forall Y \in D_a^{\perp}, \; a = 1,2,3, \tag{4.15}$$

$$\nabla_Z \xi_a = -\varphi_a Z; \; h(Z, \xi_a) = 0 \qquad \forall Z \in D_b^{\perp}, \; a \neq b. \tag{4.16}$$

Proof. Using (2.10) and (4.7) and then taking tangential and normal components, the proof of the Lemma follows. □

Theorem 4.3. *Let M be a generalized CR-submanifold of $\overline{M}$. Suppose $D+\{\xi\}$ is integrable and $\varphi_a D = D$, $a = 1, 2, 3$. Then we have*

$$h(X, \varphi_a Y) = h(Y, \varphi_a X), \tag{4.17}$$

for any $a = 1, 2, 3$ and $X, Y \in D$. Moreover, M is D-geodesic.

Proof. Using equation (4.12) and taking account $D+\{\xi\}$ is integrable, we have

$$h(X, F_a Y) - h(Y, F_a X) + \nabla^{\perp}_{X}(U_a Y) - \nabla^{\perp}_{Y}(U_a X) = U_a[X, Y] = 0. \tag{4.18}$$

Thus, (4.17) follows from (4.18). Now using (2.2), (2.3), (2.4), (2.5) & (4.18), we get

$$\begin{aligned} h(\varphi_a X, Y) &= h(X, \varphi_a Y) \\ &= h(X, \varphi_b o \varphi_c Y) \\ &= h(\varphi_b X, \varphi_c Y) \\ &= h((\varphi_c o \varphi_b) X, Y) \\ &= -h(\varphi_a X, Y), \end{aligned}$$

where (a, b, c) is a cyclic permutation of $(1, 2, 3)$, for any $X, Y \in D$.Thus we have

$$h(\varphi_a X, Y) = 0 \qquad \text{for any } X, Y \in D$$

implies M is D-geodesic, since φ_a is an automorphism of D. □

Lemma 4.4. *Let M be a generalized CR-submanifold of $\overline{M}$. Then we have*

$$A_{\varphi_a X} Y = A_{\varphi_a Y} X, \tag{4.19}$$

$$g([X, Y], \xi_b) = 0, \quad \textit{for any } X, Y \in D_a^{\perp}, \quad a, b = 1, 2, 3 \tag{4.20}$$

and

$$g([X, \xi_a], Y) = 0, \quad \forall \;\; X \in D_a^{\perp} \textit{ and } Y \in D_b^{\perp} \oplus D_c^{\perp} \oplus \{\xi_b\} \oplus \{\xi_c\}. \tag{4.21}$$

Proof. By using (2.1), (2.6), (2.7), (4.8), (4.9), (4.10), we have

$$\begin{aligned} g(A_{\varphi a X} Y, Z) &= g(h(Y, Z), \varphi_a X) \\ &= g(\overline{\nabla}_Z Y, \varphi_a X) \\ &= -g(\overline{\nabla}_Z (\varphi_a Y), X) \\ &= g(A_{\varphi a Y} Z, X) \\ &= g(A_{\varphi a Y} X, Z), \quad a = 1, 2, 3, \end{aligned}$$

for any $Z \in TM$ and $X, Y \in D_a^\perp$. This proves (4.19). Again, by using (2.6), (2.7), (2.10), we get

$$\begin{aligned} g([X,Y],\xi_b) &= g(\nabla_X Y - \nabla_Y X,\ \ \xi_b) \\ &= g(\overline{\nabla}_Y \xi_b, X) - g(\overline{\nabla}_X \xi_b, Y) \\ &= g(\varphi_b X, Y) - g(\varphi_b Y, X) \\ &= 2g((\varphi_b X, Y) \\ &= 0, \end{aligned}$$

for any $X, Y \in D_a^\perp$, $a, b = 1, 2, 3$. The equation (4.21) follows in similar way. □

Proposition 4.5. *Let M be a generalized CR-submanifold of $\overline{M}$. Then the distributions $D_a^\perp$, $a = 1, 2, 3$ are integrable.*

Proof. **Proof**. By using equation (4.3) and (4.11), we have

$$F_a(\nabla_Y X) = -A_{\varphi_a X} Y - B_a h(X,Y) - g(X,Y)\xi_a \tag{4.22}$$

for all $a = 1, 2, 3$ and $X, Y \in D_a^\perp$. By interchanging X &Y in (4.21) and then substraction the obtained relation from (4.21) and using (4.7), we get

$$F_a([Y,X]) = 0, \quad \forall X, Y \in D_a^\perp. \tag{4.23}$$

By (4.20) and (4.22), it follows that $[Y,X] \in D_a^\perp \quad \forall X, Y \in D_a^\perp$. Thus the distribution $D_a^\perp, a = 1, 2, 3$ are integrable. □

From (4.21) and theorem 4.3, we have

Corollary 4.6. *Let M be a generalized CR-submanifold of $\overline{M}$.Then the distributions $D_a^\perp \oplus \{\xi_a\}, a = 1, 2, 3$ are integrable and we have*

$$g(\nabla_{\varphi_a V_1}(\varphi_a V_2), X) = g(\nabla_{\varphi_a V_2}(\varphi_a V_1), X), \tag{4.24}$$

for all $V_1, V_2 \in \varphi_a\ (D_a^\perp)$ and $X \in D,\ a = 1, 2, 3$.

We say that M is a proper generalized CR-submanifold if we have

$$D \neq \{0\}.$$

Proposition 4.7. *Let M be a proper generalized CR-submanifold of $\overline{M}$.Then the distribution D is not integrable.*

Proof. Using (2.8) and (2.10), we get

$$\begin{aligned} g([X,Y],\xi_a) &= g(X, \overline{\nabla}_Y \xi_a) - g(Y, \overline{\nabla}_X \xi_a) \\ &= g(Y, \varphi_a X) - g(X, \varphi_a Y) \\ &= 2g(Y, \varphi_a X) \\ &= 2g(Y, B_a X), \quad \forall X, Y \in D \textit{ and } a = 1, 2, 3. \end{aligned}$$

Taking $X \neq 0\ \ \&\ \ Y = B_a X$ in above equation, it follows that D is not integrable. □

Proposition 4.8. *Let M be a generalized CR-submanifold of $\overline{M}$. Then the distribution $D^{\perp}$ is not integrable.*

Proof. Let us take a non-zero vector field $U \in T^{\perp}M$ and $X = \varphi_1 U$, $Y = \varphi_2 U$. Then using (2.2), (2.3), (2.4), (2.5) and (2.10), we obtain

$$\begin{aligned} g([X,Y],\xi_3) &= g(\overline{\nabla}_X Y - \overline{\nabla}_Y X, \xi_3) \\ &= g(X, \overline{\nabla}_Y \xi_3) - g(Y, \overline{\nabla}_X \xi_3) \\ &= g(Y, \varphi_3 X) - g(X, \varphi_3 Y) \\ &= 2g(Y, \varphi_3 X) \\ &= 2g(U,U) \neq 0. \end{aligned}$$

Thus $D^{\perp}$ is not integrable. □

Proposition 4.9. *Let M be a generalized CR-submanifold of $\overline{M}$. Suppose the distribution D is invariant by each φ_a, then the distribution $D^{\perp} \oplus \{\xi\}$ is integrable if and only if*

$$B_a h(X,Y) = 0, \qquad a = 1,2,3, \tag{4.25}$$

for any $X \in D$ & $Y \in D^{\perp}$.

Proof. Let (a,b,c) be any cyclic permutation of $(1,2,3)$. Let $Y \in D_a$, $Z \in D_b$ and $U \in D$. Then there exist $V_1, V_2 \in \varphi_a(D_a^{\perp}), a = 1,2,3$ and $X \in D$ such that $Y = \varphi_a V_1$, $Z = \varphi_b V_2$ and $U = \varphi_a X$. By using (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), (2.11), (4.7) and (4.24), we obtain

$$g([Y,Z],U) = g((\nabla_{\varphi_a V_1}(\varphi_a V_2) + \nabla_{\varphi_b V_1}(\varphi_b V_2)), X). \tag{4.26}$$

In view of equation (4.25) and proposition 4.5, we obtain that the distribution $D^{\perp} \oplus \{\xi\}$ is integrable if and only if, we have

$$g(\nabla_{\varphi_a V_1}(\varphi_a V_2), X) = 0, \tag{4.27}$$

for all $a = 1,2,3$, $V_1, V_2 \in \varphi_a(D_a^{\perp})$ and $X \in D$. On the other hand by using (2.6), (2.7), (2.11), (4.7), (4.8) and (4.10), we obtain

$$g(\nabla_{\varphi_a V_1}(\varphi_a V_2), X) = g(B_a h(\varphi_a V_1, \varphi_a X), \varphi_a V_2). \tag{4.28}$$

Thus the proposition 4.9 follows from (4.27) & (4.28). □

5. **Geometry of leaves**

In this section, we investigate conditions under which leaves of distributions on a generalized CR-submanifold M of $\overline{M}$ are totally geodesic immersed in M or $\overline{M}$.

Proposition 5.1. *Let M be a generalized CR-submanifold of $\overline{M}$. If the distribution $D^{\perp} \oplus \{\xi\}$ is integrable and D is invariant by each φ_a, $a = 1,2,3$, then each leaf of $D^{\perp} \oplus \{\xi\}$ is totally geodesic immersed in M, but it is never totally geodesic immersed in $\overline{M}$.*

Proof. Let M' be a leaf of $D^\perp \oplus \{\xi\}$ and ∇' be the Levi-Civita connection on M'. Then the Gauss formula for the immersion of M' in M is given by

$$\nabla_X Y = \nabla'_X Y + h'(X,Y), \tag{5.1}$$

for all $X, Y \in D^\perp \oplus \{\xi\}$, where h' is the second fundamental form of the immersion of M' in M. Let $Y \in D_a^\perp$, then $\exists V \in T^\perp M$, such that $Y = \varphi_a V$ By using (2.6), (2.7), (4.7), (4.9) and (5.1), we obtain

$$\begin{aligned} g(h'(X,Y),Z) &= g(h(X,\varphi_a Z),V) \\ &= g(B_a h(X,\varphi_a Z),Y), \end{aligned} \tag{5.2}$$

for all $X \in D^\perp$ and $Z \in D$. Taking account that $D^\perp \oplus \{\xi\}$ is integrable, using (4.25) and (5.2), we get

$$g(h'(X,Y),Z) = 0, \tag{5.3}$$

for any $X \in D^\perp$, $Y \in D_a^\perp$ and $Z \in D$. On the other hand

$$\begin{aligned} g(h'(X,\xi_a),Z) &= g(\nabla_X \xi_a, Z) \\ &= g(\overline{\nabla}_X \xi_a, Z) \\ &= -g(\varphi_a X, Z) \\ &= 0, \end{aligned} \tag{5.4}$$

for all $X, Y \in D^\perp \oplus \{\xi\}$ and $Z \in D$. In view of (5.3) and (5.4), we say that M' is totally geodesic immersed in M. Now, let $\overline{h}$ be the second fundamental form of the immersion M' in $\overline{M}$. Then we have

$$\overline{h}(X,Y) = h(X,Y) + h'(X,Y), \quad \forall\ X, Y \in D^\perp \oplus \{\xi\}. \tag{5.5}$$

Let us take a non-zero vector field $X = \varphi_a W$, where $W \in T^\perp M$, and $Y = \xi_a$ in (5.5). Then taking account of the fact that M' is totally geodesic immersed in M, we have

$$\overline{h}(X,Y) = h(\varphi_a W, \xi_a) = W \neq 0.$$

Hence M' is not totally geodesic immersed in $\overline{M}$. □

Proposition 5.2. *Let M be a generalized CR-submanifold of $\overline{M}$. Suppose the distribution $D \oplus \{\xi\}$ is integrable and $\varphi_a(D) = (D)$, $\forall a = 1, 2, 3, .$Then each leaf of $D \oplus \{\xi\}$ is totally geodesic immersed in $\overline{M}$.*

Proof. Let M^* be a leaf of $D \oplus \{\xi\}$.Let A^*_V denote the fundamental tensor of Weingarten of M^* in $\overline{M}$ with respect to normal section V. We remark that the normal bundle to M^* is $D_1^\perp \oplus D_2^\perp \oplus D_3^\perp \oplus TM^\perp$. Let $Z_a \in D_a^\perp$, $X \in$

$D \oplus \{\xi\}, Y \in D$ and taking account, $D + \{\xi\}$ is integrable, we have

$$\begin{aligned} g(A^*_{Z_a}X, Y) &= -g(\nabla_X Z_a, Y) \\ &= g(Z_a, \overline{\nabla}_X Y) \\ &= g(Z_a, \nabla_X Y + h(X, Y)) \\ &= g(Z_a, \nabla_X Y) \\ &= 0. \end{aligned} \tag{5.6}$$

On the other hand,

$$\begin{aligned} g(A^*_Z X, \xi_b) &= -g(\overline{\nabla}_X Z_a, \xi_b) \\ &= g(Z_a, \overline{\nabla}_X \xi_b) \\ &= -g(Z_a, \varphi_b X) \\ &= 0, \quad a, b = 1, 2, 3. \end{aligned} \tag{5.7}$$

Using the fact that $\varphi_a(D) = D$, we have

$$h(X, \xi_a) = 0, \qquad \nabla_X \xi_a = -\varphi_a X, \quad \text{for any } X \in D. \tag{5.8}$$

Now using the equations of Weingarten for both immersions of M^* in M and of M in $\overline{M}$, we have

$$\begin{aligned} g(A^*_V X, Y) &= -g(\nabla_X V, Y) \\ &= g(A_V X, Y) \\ &= g(h(X, Y), V). \end{aligned} \tag{5.9}$$

Finally, using (5.8), (5.9) and theorem 4.3, we have

$$g(A^*_V X, Y) = 0 \tag{5.10}$$

for any $X, Y \in D + \{\xi\}$ and $V \in T^\perp M$. In view of (5.6), (5.7) and (5.10), we have $A^*_V = 0$ for each V normal to M^*. Thus M^* is totally geodesic immersed in $\overline{M}$. □

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