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Stability of Quadratic Functional Equations in Fuzzy Normed Spaces

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Abstract

The aim of this paper is to investigate fuzzy Hyers - Ulam - Rassias stability of the quadratic functional equation

$$f(x+y+z)+f(x-y)+f(y-z)+f(z-x)=3f(x)+3f(y)+3f(z)$$

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1 Introduction

The stability problem of functional equations was raised by S. M. Ulam [1] in 1940 .Ulam stated the problem as follows:
"Given a group G_1, a metric group (G_2, d), a number $\epsilon > 0$ and a mapping $f : G_1 \to G_2$ which satisfies $d(f(xy), f(x)f(y)) < \epsilon$ for all $x, y \in G_1$, does there exist an homomorphism $h : G_1 \to G_2$ and a constant $k > 0$, depending only on G_1 and G_2 such that $d(f(x), h(x)) \leq k\epsilon$ for all x in G_1?"

In fact he posed this question : Assume that a function satisfies a functional equation approximately according to some convention. Is it then possible to find near this function a function satisfying the equation accurately? In 1941 D. H. Hyers gave a significant partial solution to this problem in his paper [2]. Hyer's result was generalized by T. Aoki [3] for additive mappings . In 1978, Th.M. Rassias [4] generalized the Hyer's result, a fact which rekindled interest in the field. Such type of stability is now called the Ulam-Hyers-Rassias stability of functional equations.We refer the curious readers for further information on such problems to e.g. [5, 6, 7].

In 1984, Katrasas [8] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space. Later, some mathematicians have defined fuzzy norms on a linear space from various points of view [9, 10]. In particular, in 2003, Bag and Samanta [11], following Cheng and Mordeson [12], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [13]. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces.

Recently, considerable attention has been increasing to the problem of fuzzy stability of functional equations. Several various fuzzy stability results concerning Cauchy, Jensen, simple quadratic, and cubic functional equations have been investigated [14, 15, 16, 17].

2 Preliminary Notes

Before obtaining the main result , we firstly introduce some useful concepts a fuzzy norm .

Definition 2.1 *Let X be a real vector space. A function $N : X \times R \to [0, 1]$ is called fuzzy normed on X if for all $x, y \in X$ all $s, t \in R$*

(N_1) $N(x, t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x, t) = 1$ for all t> 0 ;
(N_3) $N(cx, t) = N(x, t/|c|)$ if c≠ 0 ;
(N_4) $N(x + y, t + s) \geq \min \{N(x, t), N(y, s)\}$;
(N_5) $N(x, .)$ is a non-decreasing function of R and

$$lim_{t\to\infty} N(x, t) = 1 ;$$

(N_6) for $x \neq 0, N(x, .)$ is continuous on R ;
The pair (X, N) is called a *fuzzy normed vector space.*

Example 2.2 *Let (X, $\|.\|$) be a normed linear space. One can be easily verify that for each $k > 0$,*

$$N_k(x,t) = \begin{cases} \dfrac{t}{t+k\|x\|} & \text{if } t > 0 \\ \\ 0 & \text{if } t \le 0 \end{cases}$$

defines a fuzzy norm on X.

Example 2.3 *Let (X, $\|.\|$) be a normed linear space.We define function N by*

$$N(x,t) = \begin{cases} \dfrac{t^2 - \|x\|^2}{t^2 + \|x\|^2} & \text{if } t > \|x\| \\ \\ 0 & \text{if } t \le \|x\| \end{cases}$$

then N defines a fuzzy norm on X .

Definition 2.4 *Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $lim_{n\to\infty} N(x_n - x, t) = 1$ for all t >0. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by*

$$N - lim_{n\to\infty} N(x_n - x, t) = x.$$

Definition 2.5 *Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be cauchy if for each $\epsilon > 0$ and each $\delta > 0$ there exists an $n_0 \in N$ such that :*

$$N(x_m - x_n, \delta) > 1 - \epsilon \ (m, n \ge n_0)$$

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

Theorem 2.6 *Let (X, N) be a fuzzy normed linear space. Assume further that,*
(N_6) $\forall t > 0$ $N(x,t) > 0$ *implies* $x = 0$.
Define $\|x\|_\alpha = \inf\{t > 0 : N(x,t) \ge \alpha\}, \alpha \in (0,1)$.
Then $\{\| \ \|_\alpha : \alpha \in (0,1)\}$is an ascending family of norms on X and they are called α norms on X corresponding to the fuzzy norm N on X.

Theorem 2.7 *Let $\{\| \ \| : \alpha \in (0,1)\}$ be an ascending family of norms on a linear space X. Now we define a function $N' : X \times R \to [0,1]$ as:*

$$N'(x,t) = \begin{cases} \sup\{\alpha : \|x\|_\alpha \le t\} & (x,t) \ne (0,0) \\ 0 & (x,t) = (0,0) \end{cases}$$

Then N' is a fuzzy norm on X.

We can transform fuzzy normed spaces in α-normed spaces and inverse by above theorems.

3 Results and Discussion

Theorem 3.1 *Let f be a mapping from fuzzy normed space (X, N_1) to fuzzy Banach normed space (Y, N_2) with $f(0) = 0$ such that*

$$N_2(f(x+y+z)+f(x-y)+f(y-z)+f(z-x)-3f(x)-3f(y)-3f(z), t+s+u)$$
$$\geq \min\{N_1(x, t^q), N_1(y, s^q), N_1(z, u^q)\}$$

for all $x, y, z \in X - \{0\}$ and positive real numbers t, s, u with $q > \frac{1}{2}$. Then

(1) There is a quadratic mapping $Q : X \longrightarrow Y$ such that

$$Q(x) = N - \lim_{n\to\infty} \frac{f(3^n x)}{3^{2n}}$$

and

$$N_2(\frac{f(3^n x)}{3^{2n}} - f(x), \sum_{k=1}^{n} 3^{k(p-2)}.3^{(1-p)}.t^p) \geq N_1(x, t) \qquad (x \in X, t > 0)$$

where $p = \frac{1}{q}$.

(2) Q is an unique functional equation.

Putting $x = y = z$ and $t = s = u$ in (3.1), so we get

$$N_2(f(3x) + 3f(0) - 3^2 f(x), 3t) \geq N_1(x, t^q) \tag{1}$$

for all $x \in X$ and $t > 0$. Replacing x by $3^{n-1}x$ in (1), and by the property fuzzy normed, we obtain

$$N_2(f(3^n x) - 3^2 f(3^{n-1}x), 3t) \geq N_1(3^{n-1}x, t^q) = N_1(x, \frac{t^q}{3^{n-1}}) \tag{2}$$

Therefore

$$N_2(f(3^n x) - 3^2 f(3^{n-1}x), 3.3^{(n-1)p}.t^p) \geq N_1(x, t) \tag{3}$$

where $p = \frac{1}{q}$. It follows that

$$N_2(\frac{f(3^n x)}{3^{2n}} - \frac{f(3^{n-1}x)}{3^{2(n-1)}}, 3^{(p-2)n}.3^{(1-p)}.t^p) \geq N_1(x, t) \tag{4}$$

for all $x \in X$, $t > 0$ and $n \geq 0$. If $n > m \geq 0$, then

$$N_2(\frac{f(3^n x)}{3^{2n}} - \frac{f(3^m x)}{3^{2m}}, \sum_{k=m+1}^{n} 3^{k(p-2)}.3^{(1-p)}.t^p) \geq N_1(x,t) \tag{5}$$

Let $c > 0$ and ϵ be given. Since $\lim_{t\to\infty} N_1(x,t) = 1$, there is some $t_0 > 0$ such that

$$|N_1(x,t_0) - 1| < \epsilon \Rightarrow N_1(x,t_0) > 1 - \epsilon \qquad (N_1(x,t_0) \in [0,1])$$

for $t > t_0$. Since $p > 2$ series $\sum_{n=1}^{\infty} 3^{n(p-2)}.3^{(1-p)}.t^p$ are convergence, so there exists some $n_0 \geq 0$ such that for each $n > m \geq n_0$, the inequality

$$\sum_{k=m+1}^{n} 3^{k(p-2)}.3^{(1-p)}.t^p < c. \tag{6}$$

Thus

$$\begin{aligned} N_2(\frac{f(3^n x)}{3^{2n}} - \frac{f(3^m x)}{3^{2m}}, c) &\geq N_2(\frac{f(3^n x)}{3^{2n}} - \frac{f(3^m x)}{3^{2m}}, \sum_{k=m+1}^{n} 3^{k(p-2)}.3^{(1-p)}.t^p) \\ &\geq N_1(x,t) > 1 - \epsilon \end{aligned} \tag{7}$$

Then $\{\frac{f(3^n x)}{3^{2n}}\}$ is a Cauchy sequence in fuzzy Banach normed space (Y, N_2) so this sequence converges to some $Q(x) \in Y$. Hence, we can define a mapping $Q(x) : X \longrightarrow Y$ by

$$Q(x) = N_2 - \lim_{n\to\infty} \frac{f(3^n x)}{3^{2n}}. \tag{8}$$

Now, if we put $m = 0$ in (5) we observe that

$$N_2(\frac{f(3^n x)}{3^{2n}} - f(x), \sum_{k=1}^{n} 3^{k(p-2)}.3^{(1-p)}.t^p) \geq N_1(x,t) \qquad (x \in X, t > 0)$$

so

$$N_2(\frac{f(3^n x)}{3^{2n}} - f(x), t) \geq N_1(x, \frac{t^q}{(\sum_{k=1}^{n} 3^{k(p-2)}.3^{(1-p)})^q} \qquad (x \in X, t > 0)$$

Next we will show that Q is quadratic. Let $x, y, z \in X$, then we have

$$N_2(Q(x+y+z)+Q(x-y)+Q(y-z)+Q(x-z)-3Q(x)-3Q(y)-3Q(z),t)$$
$$\geq \min\{N_2(Q(x+y+z)-\frac{f(3^n(x+y+z))}{3^{2n}},\frac{t}{8}),(N_2(Q(x-y)-\frac{f(3^n(x-y))}{3^{2n}},\frac{t}{8})$$
$$,(N_2(Q(y-z)-\frac{f(3^n(y-z))}{3^{2n}},\frac{t}{8}),(N_2(Q(x-z)-\frac{f(3^n(x-z))}{3^{2n}},\frac{t}{8})$$
$$,(N_2(3Q(x)-\frac{3f(3^n(x))}{3^{2n}},\frac{t}{8})),(N_2(3Q(y)-\frac{3f(3^n(y))}{3^{2n}},\frac{t}{8}))$$
$$,(N_2(3Q(z)-\frac{3f(3^n(z))}{3^{2n}},\frac{t}{8}))$$
$$,N_2(\frac{f(3^n(x+y+z))}{3^{2n}}+\frac{f(3^n(x-y))}{3^{2n}}+\frac{f(3^n(y-z))}{3^{2n}}+\frac{f(3^n(x-z))}{3^{2n}}+\frac{3f(3^n(x))}{3^{2n}}$$
$$+\frac{3f(3^n(y))}{3^{2n}}+\frac{3f(3^n(z))}{3^{2n}},\frac{t}{8})$$

The first seven terms on the right hand side of the above inequality tend to 1 as $n \to \infty$ and the eighth term by (3.1) is greater than or equal to

$$\min\{N_1(3^n x,(\frac{3^{2n}t}{8})^q,N_1(3^n y,(\frac{3^{2n}t}{8})^q,N_1(3^n z,(\frac{3^{2n}t}{8})^q\}.$$

Thus

$$=\min\{N_1(x,(\frac{3^{(2q-1)n}}{8^q}.t^q),N_1(y,(\frac{3^{(2q-1)n}}{8^q}.t^q),N_1(z,(\frac{3^{(2q-1)n}}{8^q}.t^q)\}.$$

which tends to 1 as $n \to \infty$. Therefore

$$N_2(Q(x+y+z)+Q(x-y)+Q(y-z)+Q(x-z)$$
$$-3Q(x)-3Q(y)-3Q(z),t)=1.$$

So

$$Q(x+y+z)+Q(x-y)+Q(y-z)+Q(x-z)$$
$$=3Q(x)+3Q(y)+3Q(z) \quad x,y \in X.$$

Thus Q is a quadratic mapping.

□

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