Int. Journal of Math. Analysis, Vol. 6, 2012, no. 48, 2357 - 2360

Integral Representations of Exton's Triple Hypergeometric Series

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Abstract

Exton has obtained twenty integral representations for the Exton's functions $X_i(i = 1 \text{ to } 20)$ whose integrands involved confluent hypergeometric functions and Humbert functions. In this paper some interesting integral representations for the Exton's functions $X_i(i = 1 \text{ to } 20)$ have been established, which involve Exton functions itself in their integrands with the help of known results of integration. Special cases have also been obtained.

Keywords: Triple Hypergeometric Functions, Eulerian Integral, Generalized Hypergeometric Functions

1. INTRODUCTION

Exton [2] has obtained twenty integral representation of Laplace type for the triple hypergeometric functions of the second order $X_1, X_2, \ldots, X_{20}$ which include confluent hypergeometric functions ${}_0F_1$, ${}_1F_1$ and Humbert functions Φ_2, Ψ_2 in their Kernels.

An interesting integral due to Mac Robert [4] is given below

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[1+\lambda t+\mu(1-t)]^{-\alpha-\beta}\,dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)(1+\lambda)^{\alpha}(1+\mu)^{\beta}} \qquad (1.1)$$

Provided that $\mathrm{Re}\,(\alpha) > 0, \mathrm{Re}\,(\beta) > 0$ and λ and μ are such that none of the Expression $1+\lambda,, 1+\mu$ and$[1+\lambda t+\mu(1-t)]$, where $0 \leq t \leq 1$ are zero.

2. INTEGRAL REPRESENTATIONS OF EULER TYPE FOR EXTON'S FUNCTIONS

$$X_3(a_1,a_2;\ c_1,c_2;\ x,y,z) = \frac{\Gamma(c_1)(1+\lambda)^d(1+\mu)^{c_1-d}}{\Gamma(c_1-d)\Gamma(d)}\int_0^1 t^{d-1}(1-t)^{c_1-d-1}[1+\lambda t+\mu(1-t)]^{-c_1} \times\left[X_3\left(a_1,a_2\,;\ d,c_2;\ \frac{xt(1+\lambda)}{[1+\lambda t+\mu(1-t)]},\frac{yt(1+\lambda)}{[1+\lambda t+\mu(1-t)]},z\right)\right] dt, \tag{2.1}$$

$R(c_1) > R(d) > 0$

$$X_4(a_1,a_2;\ c_1,c_2,c_3;\ x,y,z) = \frac{\Gamma(b)(1+\lambda)^{a_2}(1+\mu)^{b-a_2}}{\Gamma(b-a_2)\Gamma(a_2)}\int_0^1 t^{a_2-1}(1-t)^{b-a_2-1}[1+\lambda t+\mu(1-t)]^{-b} \times\left[X_4\left(a_1,b;\ c_1,c_2,c_3;\ x,\frac{yt(1+\lambda)}{[1+\lambda t+\mu(1-t)]},\frac{zt(1+\lambda)}{[1+\lambda t+\mu(1-t)]}\right)\right] dt, \tag{2.2}$$

$R(b) > R(a_2) > 0$

3. PROOF OF THE RESULTS

By denoting the right hand side of (2.1) as I, we have

$$I = \frac{\Gamma(c_1)(1+\lambda)^d(1+\mu)^{c_1-d}}{\Gamma(c_1-d)\Gamma(d)}\int_0^1 t^{d-1}(1-t)^{c_1-d-1}[1+\lambda t+\mu(1-t)]^{-c_1} \times\left[X_3\left(a_1,a_2\,;\ d,c_2;\ \frac{xt(1+\lambda)}{[1+\lambda t+\mu(1-t)]},\frac{yt(1+\lambda)}{[1+\lambda t+\mu(1-t)]},z\right)\right] dt \tag{3.1}$$

Using result (1.3) [Exton [2]], and changing the order of integration and summation, we have

$$I = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a_1)_{2m+n+p}\,(a_2)_{n+p}x^m\,y^n\,z^p}{(d)_{m+n}\,(c_2)_p\ \ m!\ \ n!\ \ p!}\left[\frac{\Gamma(c_1)(1+\lambda)^{d+m+n}(1+\mu)^{c_1-d}}{\Gamma(c_1-d)\Gamma(d)} \times\int_0^1 t^{d+m+n-1}(1-t)^{c_1-d-1}[1+\lambda t+\mu(1-t)]^{-(c_1+m+n)}\,dt\right] \tag{3.2}$$

On using (1.1), in (3.2), we obtain

$$I=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a_1)_{2m+n+p}\,(a_2)_{n+p}x^m\,y^n\,z^p}{(c_1)_{m+n}\,(c_2)_p\;m!\;\;n!\;\;p!} \tag{3.3}$$

(3.3) can be expressed as X_3 [Exton [2]], which is the required result (2.1) Similarly result (2.2) can also be obtained with the help of (1.1).

4. SPECIAL CASES

If we take $\lambda = \mu$ in the results (2.1) to (2.2) we have obtained the following results.

$$X_3(a_1, a_2;\; c_1, c_2;\; x, y, z)$$
$$=\frac{\Gamma(c_1)}{\Gamma(d)\Gamma(c_1-d)}\int_0^1 t^{d-1}\,(1-t)^{c_1-d-1}X_3(a_1, a_2\;;\; d, c_2;\; xt, yt, z)\,dt, \tag{4.1}$$

$R(c_1) > R(d) > 0$

$$X_4(a_1, a_2;\; c_1, c_2, c_3;\; x, y, z)$$
$$=\frac{\Gamma(b)}{\Gamma(b-a_2)\Gamma(a_2)}\int_0^1 t^{a_2-1}\,(1-t)^{b-a_2-1}X_4(a_1, b;\; c_1, c_2, c_3;\; x, yt, zt)\,dt, \tag{4.2}$$

$R(b) > R(a_2) > 0$

5. CONCLUSION

We conclude it by remarking that the remaining Exton's functions can be expressed in Euler type integral representations. Also, many other integral representations exist which are applicable to the function X_1 to X_{20} in addition to those listed above.

ACKNOWLEDGEMENT

The author is indebted to Dr. Rachana Mathur and Dr. A.K. Rathie for making valuable suggestions and constant encouragement towards the improvement of the paper.

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Received: April, 2012