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New Subclasses of Analytic Functions Defined by Subordination

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Abstract. In this paper, we introduce and study certain subclasses of analytic functions which are defined by differential subordination. Coefficient inequalities, some properties of neighborhoods, distortion and covering theorems for these subclasses are given.

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1. INTRODUCTION

Let $\mathcal{T}(j)$ be the class of analytic functions f of the form

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k, \quad (a_k \geq 0,\ j \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

defined in the open unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.
Let Ω be the class of functions ω analytic in $\mathcal{U}$ such that $\omega(0) = 0,\ |\omega(z)| < 1$. For any two functions f and g in $\mathcal{T}(j)$, f is said to be subordinate to g that is denoted $f \prec g$, if there exists an analytic functions $\omega \in \Omega$ such that $f(z) = g(\omega(z))$ [3].

Definition 1.1. [1]*: For $n \in \mathbb{N}$ and $\lambda \geq 0$, the Al - Oboudi operator $D_\lambda^n : \mathcal{T}(j) \longrightarrow \mathcal{T}(j)$ is defined as $D_\lambda^0 f(z) = f(z)$, $D_\lambda^1 f(z) = (1-\lambda)f(z) + \lambda z f'(z) = D_\lambda f(z)$ and $D_\lambda^n f(z) = D_\lambda(D_\lambda^{n-1} f(z))$.*

For $\lambda = 1$, we get Sălăgean differential operator [5].

Further, if $f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k$, then

$$D_\lambda^n f(z) = z - \sum_{k=j+1}^{\infty} [1 + (k-1)\lambda]^n a_k z^k, \quad (a_k \geq 0). \tag{1.2}$$

For any function $f \in \mathcal{T}(j)$ and $\delta \geq 0$, the (j, δ) - neighborhood of f is defined as,

$$\mathcal{N}_{j,\delta}(f) = \left\{ g(z) = z - \sum_{k=j+1}^{\infty} b_k z^k \in \mathcal{T}(j) : \quad \sum_{k=j+1}^{\infty} k|a_k - b_k| \leq \delta \right\}. \tag{1.3}$$

In particular, for the identity function $e(z) = z$, we see that,

$$\mathcal{N}_{j,\delta}(e) = \left\{ g(z) = z - \sum_{k=j+1}^{\infty} b_k z^k \in \mathcal{T}(j) : \quad \sum_{k=j+1}^{\infty} k|b_k| \leq \delta \right\}. \tag{1.4}$$

The concept of neighborhoods was first introduced by Goodman [3] and then generalized by Ruscheweyh [4] .

Definition 1.2. *A function $f \in \mathcal{T}(j)$ is said to be in the class $\mathcal{T}_j(\beta, n, m, A, B, \lambda)$ if*

$$\frac{D_\lambda^{n+m} f(z)}{D_\lambda^n f(z)} \prec \frac{1 + \gamma z}{1 + Bz}, \quad z \in \mathcal{U}, \tag{1.5}$$

where, $\gamma = (1-\beta)A + \beta B$, $0 \leq \beta < 1$, $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, $\lambda \geq 1$ *and* $-1 \leq B < A \leq 1$.

We observe that $\mathcal{T}_j(0, n, m, 1-2\alpha, -1, 1) \equiv \mathcal{T}_j(n, m, \alpha)$ [2], $\mathcal{T}_j(0, 0, 1, 1-2\alpha, -1, 1) \equiv \mathcal{S}_j^\star(\alpha)$ [3], the class of starlike functions of order α and $\mathcal{T}_j(0, 1, 1, 1-2\alpha, -1, 1) \equiv \mathcal{C}_j(\alpha)$ [3], the class of convex functions of order α.

2. Neighborhoods for the class $\mathcal{T}_j(\beta, n, m, A, B, \lambda)$

Theorem 2.1. *A function $f \in \mathcal{T}(j)$ belongs to the class $\mathcal{T}_j(\beta, n, m, A, B, \lambda)$ if and only if*

$$\sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^n \left\{(1-B)[1+(k-1)\lambda]^m - (1-\gamma)\right\} a_k \leq \gamma - B \tag{2.1}$$

where, $\gamma = (1-\beta)A + \beta B$, $0 \leq \beta < 1$, $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, $\lambda \geq 1$ and $-1 \leq B < A \leq 1$.

Proof. Let $f \in \mathcal{T}_j(\beta, n, m, A, B, \lambda)$. Then,

$$\frac{D_\lambda^{n+m} f(z)}{D_\lambda^n f(z)} \prec \frac{1+\gamma z}{1+Bz}, \quad z \in \mathcal{U}. \tag{2.2}$$

Therefore, there exists an analytic function ω such that

$$\omega(z) = \frac{D_\lambda^n f(z) - D_\lambda^{n+m} f(z)}{B D_\lambda^{n+m} f(z) - \gamma D_\lambda^n f(z)} \tag{2.3}$$

Hence,

$$\begin{aligned} |\omega(z)| &= \left| \frac{D_\lambda^n f(z) - D_\lambda^{n+m} f(z)}{B D_\lambda^{n+m} f(z) - \gamma D_\lambda^n f(z)} \right| \\ &= \left| \frac{\sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^n \left\{[1+(k-1)\lambda]^m - 1\right\} a_k z^k}{(\gamma - B)z + \sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^n \left\{B[1+(k-1)\lambda]^m - \gamma\right\} a_k z^k} \right| < 1. \end{aligned}$$

Thus,

$$\Re \left\{ \frac{\sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^n \left\{[1+(k-1)\lambda]^m - 1\right\} a_k z^k}{(\gamma - B)z + \sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^n \left\{B[1+(k-1)\lambda]^m - \gamma\right\} a_k z^k} \right\} < 1. \tag{2.4}$$

Taking $|z| = r$, for sufficiently small r with $0 < r < 1$, the denominator of (2.4) is positive and so it is positive for all r with $0 < r < 1$, since $\omega(z)$ is analytic for $|z| < 1$. Then, the inequality (2.4) yields

$$\sum_{k=j+1}^{\infty}[1+(k-1)\lambda]^n\left\{[1+(k-1)\lambda]^m-1\right\}a_kr^k$$

$$<(\gamma-B)r+B\sum_{k=j+1}^{\infty}[1+(k-1)\lambda]^{n+m}a_kr^k-\gamma\sum_{k=j+1}^{\infty}[1+(k-1)\lambda]^{n}a_kr^k.$$

Equivalently,

$$\sum_{k=j+1}^{\infty}[1+(k-1)\lambda]^n\left\{(1-B)[1+(k-1)\lambda]^m-(1-\gamma)\right\}a_kr^k\le(\gamma-B)r$$

and (2.1) follows upon letting $r\to1$.
Conversely, for $|z|=r,\ 0<r<1$, we have $r^k<r$. That is,

$$\sum_{k=j+1}^{\infty}[1+(k-1)\lambda]^n\left\{(1-B)[1+(k-1)\lambda]^m-(1-\gamma)\right\}a_kr^k$$

$$\le\sum_{k=j+1}^{\infty}[1+(k-1)\lambda]^n\left\{(1-B)[1+(k-1)\lambda]^m-(1-\gamma)\right\}a_kr\le(\gamma-B)r.$$

From (2.1), we have

$$\left|\sum_{k=j+1}^{\infty}[1+(k-1)\lambda]^n\left\{[1+(k-1)\lambda]^m-1\right\}a_kz^k\right|$$

$$\le\sum_{k=j+1}^{\infty}[1+(k-1)\lambda]^n\left\{[1+(k-1)\lambda]^m-1\right\}a_kr^k$$

$$<(\gamma-B)r+\sum_{k=j+1}^{\infty}\left\{B[1+(k-1)\lambda]^m-\gamma\right\}[1+(k-1)\lambda]^na_kr^k$$

$$<\left|(\gamma-B)z+\sum_{k=j+1}^{\infty}\left\{B[1+(k-1)\lambda]^m-\gamma\right\}[1+(k-1)\lambda]^na_kz^k\right|.$$

This proves that

$$\frac{D_\lambda^{n+m}f(z)}{D_\lambda^nf(z)}\prec\frac{1+\gamma z}{1+Bz},\quad z\in\mathcal{U}$$

and hence $f\in\mathcal{T}_j(\beta,n,m,A,B,\lambda)$. □

Theorem 2.2. *If*

$$\delta=\frac{(\gamma-B)}{(1+\lambda j)^{n-1}\left[(1-B)(1+\lambda j)^m-(1-\gamma)\right]},\tag{2.5}$$

then $\mathcal{T}_j(\beta,n,m,A,B,\lambda)\subset N_{j,\delta}(e)$.

Proof. It follows from (2.1), that if $f \in \mathcal{T}_j(\beta, n, m, A, B, \lambda)$, then

$$(2.6) \qquad (1+\lambda j)^{n-1}\left[(1-B)(1+\lambda j)^m - (1-\gamma)\right]\sum_{k=j+1}^{\infty} k a_k \leq (\gamma - B),$$

which implies,

$$(2.7) \qquad \sum_{k=j+1}^{\infty} k a_k \leq \frac{(\gamma - B)}{(1+\lambda j)^{n-1}\left[(1-B)(1+\lambda j)^m - (1-\gamma)\right]} = \delta.$$

Using (1.4), we get the result. □

3. Distortion and covering Theorems

Theorem 3.1. *If $f \in \mathcal{T}_j(\beta, n, m, A, B, \lambda)$,then*

$$r - \frac{\gamma - B}{(1+j\lambda)^n\left\{(1-B)(1+j\lambda)^m - (1-\gamma)\right\}} r^{j+1} \leq |f(z)| \leq$$

$$r + \frac{\gamma - B}{(1+j\lambda)^n\left\{(1-B)(1+j\lambda)^m - (1-\gamma)\right\}} r^{j+1} \quad (0 < |z| = r < 1),$$

with equality for

$$(3.1) \qquad f(z) = z - \frac{\gamma - B}{(1+j\lambda)^n\left\{(1-B)(1+j\lambda)^m - (1-\gamma)\right\}} r^{j+1} \quad (z = \pm r)$$

Proof. In view of Theorem 2.1, we have

$$(1+j\lambda)^n\left\{(1-B)(1+j\lambda)^m - (1-\gamma)\right\}\sum_{k=j+1}^{\infty} a_k$$

$$\leq \sum_{k=j+1}^{\infty}[1+(k-1)\lambda]^n\left\{(1-B)[1+(k-1)\lambda]^m - (1-\gamma)\right\} a_k \leq \gamma - B.$$

Hence

$$|f(z)| \leq r + \sum_{k=j+1}^{\infty} a_k r^k \leq r + r^{j+1}\sum_{k=j+1}^{\infty} a_k$$

$$\leq r + \frac{\gamma - B}{(1+j\lambda)^n\left\{(1-B)(1+j\lambda)^m - (1-\gamma)\right\}} r^{j+1}$$

and

$$|f(z)| \geq r - \sum_{k=j+1}^{\infty} a_k r^k \geq r - r^{j+1}\sum_{k=j+1}^{\infty} a_k$$

$$\geq r - \frac{\gamma - B}{(1+j\lambda)^n \{(1-B)(1+j\lambda)^m - (1-\gamma)\}} r^{j+1}.$$

This completes the proof. □

Theorem 3.2. *If $f \in \mathcal{T}_j(\beta, n, m, A, B, \lambda)$, then*

$$1 - \frac{(\gamma - B)}{(1+j\lambda)^{n-1} \{(1-B)(1+j\lambda)^m - (1-\gamma)\}} r^j \leq |f'(z)| \leq$$

$$1 + \frac{\gamma - B}{(1+j\lambda)^{n-1} \{(1-B)(1+j\lambda)^m - (1-\gamma)\}} r^j \quad (0 < |z| = r < 1),$$

with equality for

$$f(z) = z - \frac{\gamma - B}{(1+j\lambda)^{n-1} \{(1-B)(1+j\lambda)^m - (1-\gamma)\}} z^{j+1} \quad (z = \pm r) \tag{3.2}$$

Proof. We have

$$|f'(z)| \leq 1 + \sum_{k=j+1}^{\infty} k\, a_k\, |z|^{k-1} \leq 1 + r^j \sum_{k=j+1}^{\infty} k\, a_k.$$

In view of Theorem 2.1,

$$\sum_{k=j+1}^{\infty} k\, a_k \leq \frac{\gamma - B}{(1+j\lambda)^{n-1} \{(1-B)(1+j\lambda)^m - (1-\gamma)\}}$$

Thus

$$|f'(z)| \leq 1 + \frac{\gamma - B}{(1+j\lambda)^{n-1} \{(1-B)(1+j\lambda)^m - (1-\gamma)\}} r^j.$$

On the other hand,

$$|f'(z)| \geq 1 - \sum_{k=j+1}^{\infty} k\, a_k\, |z|^{k-1} \geq 1 - r^j \sum_{k=j+1}^{\infty} k\, a_k$$

$$\geq 1 - \frac{\gamma - B}{(1+j\lambda)^{n-1} \{(1-B)(1+j\lambda)^m - (1-\gamma)\}} r^j.$$

This completes the proof. □

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