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Estimating π Easily and Accurately

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Abstract

It is shown that π could be approximated easily and efficiently using only the function arctan and an elementary integration.

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1 Introduction.

For an approximate calculation of Archimede's constant π we have at our disposal a huge number of formulas. From the older ones the most known is the famous Machin's formula [6]. On the other hand, the most popular among the recently discovered ones is the miraculous and remarkably simple Bailey-Borwein-Pouffe (BBP) formula [2, 13, 15]. Many rapidly convergent methods are known, but they are more or less complicated. More than thirty years ago, R. Brent gave an algorithm which requires only the evaluation of a square root and a square, and gave Pi to 16,000,000 decimals in just 24 iterations [4]. However, it depends on several techniques including also the theory of elliptic integrals.

Recently, Hirschhorn [5] and Lucas [11] studied some rational approximations to π using an integral as the only tool. Here, we would like to show how to produce approximations to π achieving any desired accuracy, using only an easy and straightforward integration method. Moreover, we also want to show how π can be approximated very accurately, using slowly convergent Gregory-Leibniz series and the Euler-Maclaurin summation formula as the series convergence accelerating tool [8].

2 Series approximation to $\arctan x$.

Using any positive integer m and setting $q=-t^2$ in the equality

$$\sum_{k=0}^{m-1} q^k = \frac{1-q^m}{1-q}\,,$$

we obtain the identity

$$\frac{1}{1+t^2} = \sum_{k=0}^{m-1} (-1)^k\, t^{2k} \; + (-1)^m \frac{t^{2m}}{1+t^2}\,,$$

valid for all $t\in\mathbb{R}$. Integrating this equation from 0 to x the following expression results:

$$\arctan x = \sum_{k=0}^{m-1} (-1)^k \frac{x^{2k+1}}{2k+1} + (-1)^m \int_0^x \frac{t^{2m}\,\mathrm{d}t}{1+t^2}\,.$$

Using any $n\in\mathbb{N}$ and choosing $m=2n+1$ in this identity we obtain the equality

$$\arctan x = x + x\sum_{k=1}^{2n} (-1)^k \frac{x^{2k}}{2k+1} - r_n(x)\,, \tag{1}$$

valid for any $x\in\mathbb{R}$ with

$$r_n(x) = \int_0^x \frac{t^{4n+2}}{1+t^2}\,\mathrm{d}t = -\int_0^{-x} \frac{t^{4n+2}}{1+t^2}\,\mathrm{d}t\,. \tag{2}$$

Pairwise collecting the terms with odd and even indices, the sum figuring in (1) could be transformed as follows,

$$\begin{aligned}\sum_{k=1}^{2n} (-1)^k \frac{x^{2k}}{2k+1} &= \sum_{i=1}^{n} \frac{x^{4i}}{4i+1} - \sum_{i=1}^{n} \frac{x^{4i-2}}{4i-1}\\ &= -\sum_{i=1}^{n} x^{4i}\left(\frac{x^{-2}}{4i-1} - \frac{1}{4i+1}\right).\end{aligned}$$

Hence,

$$\arctan x = x\left(1 - \sum_{i=1}^{n} x^{4i}\left(\frac{x^{-2}}{4i-1} - \frac{1}{4i+1}\right)\right) - r_n(x)\,, \tag{3}$$

where, according to (2), the remainder $-r_n(x)$ is estimated as

$$0 < r_n(x) < \int_0^x \frac{t^{4n+2}}{1+0}\,\mathrm{d}t = \frac{x^{4n+3}}{4n+3}\,, \quad \text{for } x > 0, \tag{4}$$

and

$$0 < -r_n(x) < \int_0^{|x|} \frac{t^{4n+2}}{1+0}\,\mathrm{d}t = \frac{|x|^{4n+3}}{4n+3}\,, \quad \text{for } x < 0. \tag{5}$$

Consequently, using (3) we obtain the expansion

$$\arctan x = x\left(1 - \sum_{i=1}^{\infty} x^{4i}\left(\frac{x^{-2}}{4i-1} - \frac{1}{4i+1}\right)\right), \tag{6}$$

valid for[1] any $x \in [-1,1]$. Accordingly to (1), (4) and (5), this equality is equivalent to the expansion

$$\arctan x = \sum_{k=1}^{\infty}(-1)^{k-1}\,\frac{x^{2k-1}}{2k-1}\,, \tag{6a}$$

also true for $x \in [-1,1]$.

3 Irrational approximations to π.

Setting $x = \frac{1}{\sqrt{3}}$ in (3), we obtain the equality

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \sum_{i=1}^{n}\frac{1}{9^i}\left(\frac{3}{4i-1} - \frac{1}{4i+1}\right)\right) - r_n\left(\tfrac{1}{\sqrt{3}}\right).$$

Thus, considering (4), we obtain the following theorem.

Theorem 1. *For every positive integer n there holds the equality*

$$\pi = \pi_n - \varepsilon_n\,, \tag{7}$$

where

$$\pi_n = 2\sqrt{3}\left(1 - 4\sum_{i=1}^{n}\frac{2i+1}{(16i^2-1)9^i}\right). \tag{7a}$$

and the error term $-\varepsilon_n := -6\,r_n\left(1/\sqrt{3}\right)$ is estimated, according to (4), as

$$0 < \varepsilon_n < \varepsilon_n^* := \frac{2\sqrt{3}}{3(4n+3)\,9^n}\,. \tag{7b}$$

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[1] The expansion for $|x| > 1$ can be obtained using the identities $\arctan x = \frac{\pi}{2} - \arctan\left(\frac{1}{x}\right)$, for $x > 1$, and $\arctan x = -\frac{\pi}{2} - \arctan\left(\frac{1}{x}\right)$, for $x < -1$.

Figure 1 illustrates, on the left, the convergence $\lim_{n\to\infty} \pi_n = \pi$ by plotting the graph of the sequence $n \mapsto E_n := 9^n \varepsilon_n$ and, on the right, it shows the graph of the sequence $n \mapsto q_n := \varepsilon_n^*/\varepsilon_n$. Since $q_n \approx 1$ the approximation $\varepsilon_n \approx \varepsilon_n^*$ is not bad. Here we used the program Mathematica [16].

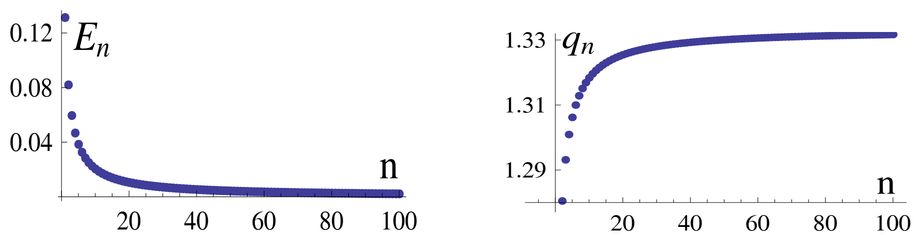

Figure 1: The graphs of the sequences $n \mapsto E_n$ and $n \mapsto q_n$.

Although the convergence $\lim_{n\to\infty} \varepsilon_n = 0$ is linear, the formulas above enable an easy computation of π. Indeed, considering (7a), $\pi_{10} = 3.141592653595\ldots$ and thanks to the inequality $\varepsilon_{10} < 8 \times 10^{-12}$, due to (7b), we have the estimate $3.141592653587 < \pi_{10} < 3.141592653595$. So, π is determined to ten decimal places: $\pi = 3.1415926535\ldots$. We remark that

$$\frac{223}{71} < \pi_2 - \varepsilon_2^* < \pi_6 - \varepsilon_6^* < \pi < \pi_6 < \frac{355}{113} < \frac{22}{7}\,.$$

4 Rational approximations to π.

All the terms π_n presented in the preceding section are irrational. However, we can also produce fast converging π-sequences with rational terms. For example, setting $\alpha := \arctan\frac{1}{2}$ and $\beta := \arctan\frac{1}{3}$, we have $\alpha, \beta \in (0, \frac{\pi}{4})$, consequently $0 < \alpha + \beta < \frac{\pi}{2}$, and $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{1}{3}$. Moreover,

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\cdot\tan\beta} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}} = 1.$$

Thus

$$\frac{\pi}{4} = \arctan 1 = \alpha + \beta = \arctan\tfrac{1}{2} + \arctan\tfrac{1}{3}. \tag{8}$$

This is the Euler's arctan–formula, see [12] and [14]. The famous Machin's formula [14]

$$\frac{\pi}{4} = 4\arctan\tfrac{1}{5} - \arctan\tfrac{1}{239}\,,$$

or even further more sophisticated Machin–like formulas, see [12] and [14], which all enable relatively good convergence, can be derived similarly.

For example, considering (8), (3) and (4), a simple calculation yields the following theorem.

Theorem 2. *For any $n \in \mathbb{N}$ we have*

$$\pi = P_n - \rho_n, \tag{9}$$

where

$$P_n = \frac{1}{3}\left[10 - \sum_{i=1}^{n} \frac{1}{16\,i^2 - 1}\left(\frac{72\,i + 30}{16^i} + \frac{128\,i + 40}{81^i}\right)\right] \tag{9a}$$

and

$$0 < \rho_n < \rho_n^* := \frac{1}{8n \cdot 16^n}\,. \tag{9b}$$

Proof. Referring to (4), we estimate

$$\begin{aligned} 0 < \rho_n &= 4\left(r_n(\tfrac{1}{2}) + r_n(\tfrac{1}{3})\right) \\ &< 4\left(\frac{\left(\frac{1}{2}\right)^{4n+3}}{4n+3} + \frac{\left(\frac{1}{3}\right)^{4n+3}}{4n+3}\right) \\ &= \frac{4}{4n+3}\left(\frac{1}{8 \cdot 16^n} + \frac{1}{27 \cdot 81^n}\right) \\ &< \frac{4}{4n+3}\left(\frac{1}{8 \cdot 16^n} + \frac{1}{3 \cdot 4^n \cdot 8 \cdot 16^n}\right) \\ &= \frac{4}{4n+3}\left(1 + \frac{1}{3 \cdot 4^n}\right)\frac{1}{8 \cdot 16^n}\,. \end{aligned}$$

The relation (9b) verifies the following equivalences, valid for $n \in \mathbb{N}$,

$$\frac{4}{4n+3}\left(1 + \frac{1}{3 \cdot 4^n}\right) < \frac{1}{n} \iff \frac{1}{3 \cdot 4^n} < \frac{3}{4n} \iff 3 \cdot 4^n > \frac{4n}{3} \iff 4^{n-1} > \frac{n}{9}\,,$$

where the last inequality can be confirmed by induction or using the inequalities $4^{n-1} \geq e^{n-1} \geq 1 + (n-1) > \frac{n}{9}$, true for $n \geq 1$. □

The estimate (9b) is rather good as is evident from Figure 2 where the sequence $n \mapsto E_n^* := 8n \cdot 16^n \cdot \rho_n$ is depicted. This figure was also plotted by Mathematica [16]. Since $\rho_{840} < 6 \times 10^{-1016}$ we could compute one thousand decimals of π summing 840 terms of P_{840}.

5 Approximations on the basis of Gregory-Leibniz series.

From (6) and (4) we obtain Gregory-Leibniz expansion

$$\frac{\pi}{4} = 1 - \sum_{i=1}^{\infty}\left(\frac{1}{4i-1} - \frac{1}{4i+1}\right),$$

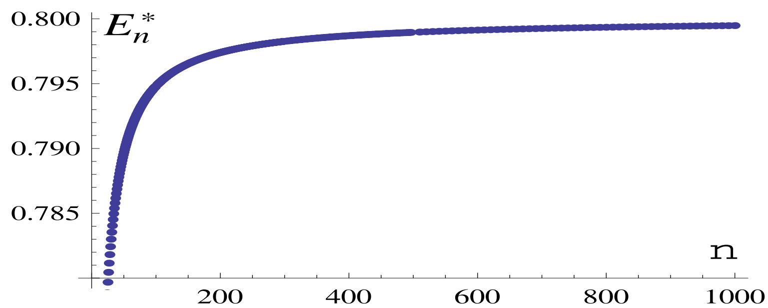

Figure 2: The graph of the sequences $n \mapsto E_n^*$.

i.e.

$$\pi = 4 - \sum_{i=1}^{\infty} \left(\frac{1}{i - \frac{1}{4}} - \frac{1}{i + \frac{1}{4}} \right) . \tag{10}$$

The series on the right does not converge fast. Fortunately, this convergence can be accelerated strongly using the Euler-Maclaurin summation formula (shortened the EM formula). This was already done in [8], where the standard EM summation rule of order four has been applied. Now we shall use more general EM formula [7, Theorem 1 on p.114]. According to this theorem, setting $h = 1$, $a = m$, $b = N$, $n = N - m$, $p = 2q$ ($m, N, q \in \mathbb{N}$, $m < N$) and

$$f(x) := \frac{1}{x - \frac{1}{4}} - \frac{1}{x + \frac{1}{4}}, \tag{11}$$

we have[2]

$$\sum_{i=m}^{N} f(i) - f(N) - \int_m^N f(x)\,\mathrm{d}x \tag{12}$$
$$= -\frac{1}{2}\left[f(N) - f(m)\right] + \sum_{i=1}^{q} \left[\frac{B_{2i}}{(2i)!} f^{(2i-1)}(N) - \frac{B_{2i}}{(2i)!} f^{(2i-1)}(m) \right] + \rho_q(m, N),$$

with the remainder

$$\rho_q(m, N) := r_{2q}(m, N, N - m) = -\frac{1}{(2q)!} \int_m^N P_{2q}(m - x) f^{(2q)}(x)\,\mathrm{d}x\,. \tag{13}$$

The functions $B_k(x)$ are the Bernoulli polynomials [1, 23.1.1] and $B_k := B_k(0)$ are named Bernoulli coefficients. For any $k \geq 0$, $P_k(x)$ are 1-periodic functions: $P_k(x) = B_k(x)$ for $x \in [0, 1)$ and $P_k(x + 1) \equiv P_k(x)$.

[2]considering that $B_1 = -\frac{1}{2}$ and $B_{2k+1} = 0$, for $k \in \mathbb{N}$.

According to (11), the derivatives are given as

$$f^{(k)}(x) := (-1)^k (k!) \left(\left(x - \frac{1}{4} \right)^{-(k+1)} - \left(x + \frac{1}{4} \right)^{-(k+1)} \right), \quad k \geq 0. \tag{14}$$

Therefore, $\lim_{N\to\infty} f^{(k)}(N) = 0$, for $k \geq 0$, and, due to the boundedness of periodic functions $P_k(x)$, the integrals $\int_m^\infty P_k(m-x) f^{(k)}(x)\,\mathrm{d}x$ are absolutely convergent for every $k \geq 0$. Hence, letting $N \to \infty$ in (12) we obtain the equality[3]

$$\begin{aligned}\sum_{i=m}^{\infty} f(i) &= \int_m^\infty f(x)\,\mathrm{d}x + \frac{1}{2} f(m) - \sum_{i=1}^{q-1} \frac{B_{2i}}{(2i)!} f^{(2i-1)}(m) \\ &\qquad - \frac{B_{2q}}{(2q)!} f^{(2q-1)}(m) - \frac{1}{(2q)!} \int_m^\infty P_{2q}(m-x)\, f^{(2q)}(x)\,\mathrm{d}x \\ &= \int_m^\infty f(x)\,\mathrm{d}x + \frac{1}{2} f(m) - \sum_{i=1}^{q-1} \frac{B_{2i}}{(2i)!} f^{(2i-1)}(m) - R_q(m),\end{aligned} \tag{15}$$

where the error term

$$-R_q(m) = \frac{1}{(2q)!} \int_m^\infty \big[B_{2q} - P_{2q}(m-x)\big] f^{(2q)}(x)\,\mathrm{d}x\,. \tag{16}$$

According to [10, items (13), (14) and (20)] and Stirling's formula [1, 6.1.38] we have, for $m, q \in \mathbb{N}$,

$$0 < \frac{(-1)^{q-1}}{(2q)!} \big[B_{2q}(0) - P_{2q}(x)\big] < \frac{1 - 4^{-q}}{1 - 2 \cdot 4^{-q}} \cdot \frac{4}{(2\pi)^{2q}} \leq \frac{6}{(2\pi)^{2q}}\,, \tag{17}$$

for $m, q \geq 1$ and $x \in \mathbb{R} \setminus \mathbb{Z}$. Consequently, considering (14), we estimate

$$0 < (-1)^q R_q(m) < \frac{6}{(2\pi)^{2q}} \cdot \big[- f^{(2q-1)}(m) \big]\,. \tag{18}$$

Referring to (14) and using the finite increment theorem, we have

$$\begin{aligned} f^{(2q-1)}(m) &= -(2q-1)! \cdot \left(\left(m - \frac{1}{4} \right)^{-2q} - \left(m + \frac{1}{4} \right)^{-2q} \right) \\ &= -(2q-1)! \cdot (-2q) \xi^{-2q-1} \big(-\tfrac{1}{4} - \tfrac{1}{4} \big), \end{aligned} \tag{19}$$

for some $\xi \in \big(m - \frac{1}{4}, m + \frac{1}{4}\big)$. Thus,

$$-f^{(2q-1)}(m) < \frac{(2q-1)!}{2} \cdot (2q) \big(m - \tfrac{1}{4}\big)^{-2q-1} = \frac{(2q)!}{2} \big(m - \tfrac{1}{4}\big)^{-2q-1}.$$

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[3] $\sum_{i=1}^{0} x_i := 0$

Hence, using (18) and Stirling's formula [1, 6.1.38],

$$\begin{aligned} 0 < (-1)^q R_q(m) &< \frac{3}{(2\pi)^{2q}\big(m-\frac14\big)^{2q+1}}\cdot(2q)! \\ &< \frac{3}{(2\pi)^{2q}\big(m-\frac14\big)^{2q}\big(m-\frac14\big)}\cdot\sqrt{2\pi\cdot 2q}\left(\frac{2q}{e}\right)^{2q}\exp\left(\frac{1}{12\cdot 2q}\right) \\ &\le \frac{6\sqrt{\pi}\,\exp(1/24)\sqrt{q}}{m-\frac14}\left(\frac{2q}{2\pi e\big(m-\frac14\big)}\right)^{2q}. \end{aligned} \tag{20}$$

From (10) and (14) we obtain

$$\begin{aligned} \pi =& 4-\left[\sum_{i=1}^{m-1}\left(\frac{1}{i-\frac14}-\frac{1}{i+\frac14}\right)+\frac12\left(\frac{1}{m-\frac14}-\frac{1}{m+\frac14}\right)+\ln\left(\frac{m+\frac14}{m-\frac14}\right)\right. \\ &\left.-\sum_{i=1}^{q-1}\frac{B_{2i}}{(2i)!}(2i-1)!\left(\left(m-\frac14\right)^{-2i}-\left(m+\frac14\right)^{-2i}\right)-R_q(m)\right]. \end{aligned}$$

Thus, invoking (20), we obtain the following theorem.

Theorem 3. *For any positive integers q and m the equality equality*

$$\pi = S_q(m) + R_q(m), \tag{21}$$

holds with

$$\begin{aligned} S_q(m) &= 4-\left[\sum_{i=1}^{m-1}\frac{8}{16i^2-1}+\frac{4}{16m^2-1}+\ln\left(\frac{4m+1}{4m-1}\right)\right. \\ &\left.+\sum_{i=1}^{q-1}\frac{B_{2i}}{2i}\left(\left(m-\frac14\right)^{-2i}-\left(m+\frac14\right)^{-2i}\right)\right] \end{aligned} \tag{22}$$

and

$$0 < (-1)^q R_q(m) < R_q^*(m) := \frac{45\sqrt{q}}{4m-1}\left(\frac{4q}{e\pi\big(4m-1\big)}\right)^{2q}. \tag{23}$$

Since $0 < -R_{19}(900) < 5\times 10^{-101}$, we could compute π to one hundred decimal places summing $m+q+1 = 920$ summands. Figure 3, showing the graph of the sequence $m \to (-1)^q R_q(m)$ and the graph of continuous function $m \to R_q^*(m)$, confirms that the estimate (23) is rather good.

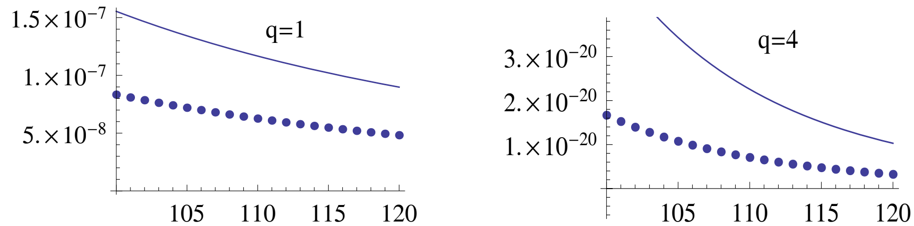

Figure 3: The graph of the sequence $m \mapsto (-1)^q R_q(m)$ and the graph of continuous function $m \mapsto R_q^*(m)$.

Remark 1. Unfortunately, Theorem 3 is less suitable for the calculation of π than are the Theorems 1 and 2 above. However, the convergence acceleration method just presented can be applied also for the series appearing in the two theorems mentioned. This way we should obtain much faster convergence than are given in these theorems.

Remark 2. We could derive accurate rational approximations to π (including no logarithms), similar to that presented in Theorem 3, if we would apply the Euler-Boole summation formula for alternating series [9, Theorem 3.15 or 3.16].

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