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Optimal Bilinear Control Problems Governed by Evolution Partial Differential Equation

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Abstract

In this paper, we study a control problem of nonlinear evolution equation. The system is governed by a nonlinear operator and the control is acting multiplicatively on the state of the system. Under suitable hypothesis, it is shown that there exists an optimal control $\overline{u}$ for the standard cost function satisfying an appropriate optimality conditions. As application, we give an example involving the Laplacien operator

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The analysis and control of nonlinear distributed systems have received a systematic study in the literature, see for instance [7],[8],[9].The interest of these systems lies in the fact that many naturel and industrial process has

intrinsically nonlinear structure.
So, it is the question of minimizing the function:

$$J(u) = \int_0^T L(z(t), u(t))dt,$$

where z is the solution of the state equation:

$$\begin{cases} \dot{z}(t) + Az(t) = u(t)Bz(t) + f(t) \\ z(0) = z_0. \end{cases}$$

1 Setting of the problem

Throughout this paper, H denotes a separable Hilbert space and V a subspace of H having the structure of reflexive Banach space which is continuously and densely embedded in H.
Identifying H with its dual H'we have the Gelfand triplet $V \hookrightarrow H \hookrightarrow V'$ where V' is the dual of V. We suppose that all these embedding are compact. Let $<,>$ be the duality pairing between V and V'as well as the inner product on H. Let $\|.\|_V$,and $|.|$ and $\|.\|_{V'}$denote the norms on V,H and V'respectively. Given a fixed number $T > 0$ and $p = 2$ we introduce the spaces:
$L^2(V) = L(0, T; V), L^2(H) = L^2(0, T; H), L^2(V') = L^2(0, T; V')$,and $W = \{w \in L^2(V); \dot{w} \in L^2(V')\}$. Here the derivative is understood in the sense of vector valued distributions.
It is well known that if $w \in W$, then after eventual modification on a measure-zero set, w becomes continuous from $[0, T]$ into H and the embedding $W \hookrightarrow \mathcal{C}([0, T]; H)$ is continuous [6,7]. Furthermore,if $V \hookrightarrow H$ compactly, then also $W \hookrightarrow L^2(H)$compactly.
We study the control problem

$$\inf_u J(u) = \inf_u \int_0^T L(z(t), u(t))dt \tag{1.1}$$

subject to the state equation

$$\begin{cases} \dot{z}(t) + Az(t) = u(t)Bz(t) + f(t) \\ z(0) = z_0. \end{cases}$$

Our aim is to provide conditions under which the optimal solution (1.1)exists. By an optimal solution, we mean a control $\overline{u}$ on which the infimum is attained. For problem (1.1),we need the following hypothesis:
$(H)_1 A : V \to V'$is such that:
 (i) $\|A\varphi\|_{V'} \leq \alpha_1\|\varphi\|$ $\quad$ *with* $\alpha_1 > 0$.
 (ii) $< A\varphi, \varphi > \alpha_2\|\varphi\|^2$ $\quad$ *with* $\alpha_2 > 0$.

$(iii) < A\varphi_1 - A\varphi_2, \varphi_1 - \varphi_2 > \geq \alpha_3 \|\varphi_1 - \varphi_2\|^2$ *with* $\alpha_3 > 0$
(iv) $\varphi \to A\varphi$ is continuously Frechet differentiable.
$(H)_2$ B:$H \to H$ is linear and continuous with$|B(\varphi)| \leq b|\varphi|, b > 0$.
$(H)_3$ $u \in L^\infty(0,T)$
$(H)_4$ $f \in L^2(V')$.
$(H)_5$ $z_0 \in H$
$(H)_6$ $L : H \times \mathcal{R} \to \mathcal{R}$ is integrand convex such that:
(i)L is coercive :$\lim_{\|u\|_{L^\infty(T)} \to \infty} \int_0^T L(z(t), u(t))dt = +\infty$
$(ii)(x, u) \to L(x, u)$ is continuously Frechet differentiable.
for every $x \in \mathcal{C}([0,T], H)$ and every $u \in L^\infty(0,T)$, $J(u)$is finite.

Remark 1.1 *If* $\varphi \in \mathrm{L}^2(\mathrm{V})$, *then* $\mathrm{A}\varphi(\mathrm{t}) \in \mathrm{V}'$. *On the other hand, for* $\mathrm{u} \in \mathrm{L}^\infty(0,T)$ *and* $\varphi \in \mathrm{L}^2(\mathrm{V})$, *we have* $\mathrm{uB}\varphi \in \mathrm{L}^2(V')$. *Therefore, the choice of control space is compatible with the equation(2.1) definite in the following section.*

2 Results on the evolution problem

We consider the evolution problem

$$\begin{cases} \dot{\mathrm{z}}(t) + \mathrm{Az(t)} = u(t)\mathrm{Bz(t)} + \mathrm{f}(t) \\ \mathrm{z}(0) = \mathrm{z}_0 \end{cases} \tag{2.1}$$

We recall that by the solution of the problem, we mean a function z $\in$ W that satisfies (2.1).

Theorem 2.1 *Under hypothesis* $(H_1)(i), (H_1)(ii), (H_1)(iii), (H_2), (H_3), (H_4) and (H_5)$*,the equation(1.3) has a unique solution* z *satisfying*

$$\mathrm{z} \in L^\infty(H) \cap \mathrm{W}.$$

Let us remark that by Theorem (2.1) $z \in \mathcal{C}([0,T]; H)$.

Proof.

For the uniqueness, we use the **Gronwall** Lemma; and the existence of the solution follows from a standard application of the **Galerkin** method and the priori estimates given in the following Lemma:

Lemma 2.2 *Under the hypothesis of Theorem (2.1), if z is a solution of the problem (1.3), then*

$$\|z\|_{\mathcal{C}[0,T],H} \leq (|z_0|^2 + \frac{C^2}{2\alpha_2} \|f\|_{L^2(V')})^{\frac{1}{2}} e^{bT\|u\|_{L^\infty(0,T)}}, \quad (E_1)$$

$$\|z\|_{L^2(V)} \leq (\frac{1}{2\alpha_2}|z_0|^2 + \frac{bm_1}{\alpha_2}T\|u\|_{L^\infty(0,T)} + \frac{m_1}{\alpha_2}\|f\|_{L^2(V')}\sqrt{T})^{\frac{1}{2}}, \qquad (E_2)$$

$$\|\dot{z}\|_{L^2(V)} \leq (\alpha_1^2 m_2^2 + 2\alpha_1 K m_2 + K)^{\frac{1}{2}}, \qquad (E_3)$$

where

$$K = \int_0^T a^2(t)dt,$$

$$m_1 = (|z_0|^2 + \frac{C^2}{2\alpha_2}\|f\|_{L^2(V')})^{\frac{1}{2}} e^{bT\|u\|_{L^\infty(0,T)}},$$

$$m_2 = (\frac{1}{2\alpha_2}|z_0|^2 + \frac{bm_1}{\alpha_2}T\|u\|_{L^\infty(0,T)} + \frac{m_1}{\alpha_2}\|f\|_{L^2(V')}\sqrt{T})^{\frac{1}{2}},$$

and

$$a(t) = bm_1\|u\|_{L^\infty(0,T)} + \|f(t)\|_{V'}.$$

Proof: See[1]

3 OPTIMAL CONTROL

The main purpose of this section is to prove the existence of optimal control for the problem (1.1).

3.1 Existence theorem for the control problem

Theorem 3.1 *If* $(H_1), (H_2), (H_3), (H_4), (H_5)$ *and* (H_6)*hold, then (1.3) has an optimal solution.*

Proof: See[1].

3.2 Optimality conditions

Before proceeding with investigation of the mapping $\Theta : u \to z$,where z is defined by(2.1),we introduce a technical lemma generalizing the Gronwall inequality

Lemma 3.2 *Let* $T > 0$ *and* $c \geq 0$*.Assume that* λ *and* m *are integrable in* $[0, T]$ *whith positive values.Let* $\varphi : [0, T] \to \mathcal{R}^+$ *be such that:*

$$i)\lambda\varphi \text{ and } \lambda\varphi^2 \text{ are integrable on } [0, T].$$

$$ii)\frac{1}{2}\varphi^2(t) \leq \frac{1}{2}c^2 + \int_0^T \lambda(s)\varphi(s)ds + \int_0^T m(s)\varphi^2(s)ds \qquad for\ t \geq 0.$$

Then

$$\varphi(t) \leq [c + \int_0^T \lambda(s)ds]e^{\int_0^T m(s)ds}$$

Proof: see [1].

Lemma 3.3 *Assume that the conditions $(H_1)(H_2)(H_3)(H_4)and(H_5)$ hold. Then the mapping $\Theta : L^\infty(0,T) \to L^\infty(H) \cap L^2(V)$ is locally Lipschitz.*

proof:See[1].

Theorem 3.4 *i)Suppose that the hypothesis of the lemma (2.2) are satisfied with $f = 0$,*
ii)$\forall \varphi, \phi \in \mathcal{C}([0,T], H)$ with $\|\phi\|_{\mathcal{C}([0,T],H)} \leq 1$ we have

$$\|A'(\varphi(t) + \phi(t)) - A'(\varphi(t))\|_{\mathcal{L}(H)} \leq \gamma(t)|\phi(t)|_H,$$

where $\gamma \in L^1(0,T)$.
Then the functionΘ:$L^\infty(0,T) \longrightarrow L^\infty(H) \cap L^2(V)$ is Frechet differentiable and $\Theta'_{\overline{u}}(h)$ is a solution of the equation

$$\begin{cases} \dot{y}(t) + A'_{\overline{z}(t)} y(t) = \overline{u} B y(t) + h(t) B \overline{z}(t), \\ y(0) = 0, \end{cases}$$

where $\overline{z} = \Theta(\overline{u})$.

Theorem 3.5 *Assume that the hypothesis of theorem 3.4 and (H_6) hold. Then an optimal control $\overline{u}$, a corresponding state $\overline{z}$, and its adjoint state p are necessarily tied by the optimality system:*

$$\begin{cases} \dot{\overline{z}}(t) + A\overline{z}(t) = \overline{u}(t) B \overline{z}(t), \\ \overline{z}(0) = z_0, \\ -\dot{p} + A'^{*}_{\overline{z}} p = \overline{u} B^* p + \partial_1 L(\overline{z}(t), \overline{u}(t)), \quad p(T) = 0 \\ < B\overline{z}(t), p(t) > + \partial_2 L(\overline{z}(t), \overline{u}(t)) = 0, \quad a.e \ in \ [0,T]. \end{cases}$$

proof:
Since L is Frechet differentiable, we deduce that the functional $J(u) = \int_0^T L(z(t), u(t))dt$ is Frechet differentiable on $L^\infty(0,T)$. Since $\overline{u}$ is a minimum point for $J(u)$, then

$$J'_{\overline{u}}.h = 0, \qquad \forall h \in L^\infty(0,T).$$

On the other hand,

$$J'_{\overline{u}}.(h) = \int_0^T < -\dot{p} + A'^{*}_{\overline{z}} p(t) - \overline{u} B^* p(t), y(t) > dt + \int_0^T h(t) \partial_2 L(\overline{z}(t), \overline{u}(t)) dt$$

$$= \int_0^T < -\dot{p}(t), y(t) > dt + \int_0^T < p(t), A'_{\overline{z}} y(t) > dt - \int_0^T < \overline{u}(t) B y(t), p(t) > dt$$

$$= \int_0^T < -\dot{p}(t), y(t) > dt + \int_0^T < (A'_{\overline{z}} - \overline{u}(t) B) y(t), p(t) > dt.$$

Therefore, integrating by party with $p(T) = 0$ and $y(0) = 0$, we deduce

$$J'_{\overline{u}}.h = \int_0^T < p(t), \dot{y}(t) + (A'_{\overline{z}}y(t) - \overline{u}(t)By(t) > dt + \int_0^T h(t)\partial_2 L(\overline{z}(t), \overline{u}(t))dt$$

$$= \int_0^T [< p(t), B\overline{z}(t) > +\partial_2 L(\overline{z}(t), \overline{u}(t))]h(t)dt.$$

Thus, we achieve the proof.

4 Example

In this section,we present an example which illustrates the application of the results of the theory developed in the previous section. Let Ω be a bounded domain in $\mathcal{R}^n$ whit smooth boundary $\Gamma = \partial\Omega$. We need to optimize the critter definite with :

$$J(u) = \int_Q |z(t,x)|^2 dtdx + \int_0^T |u(t)|^2 dt| = \|z\|_{L^2(H)} + \|u\|_{L^2(0,T)},$$

where z is the solution of the evolution equation:

$$\begin{cases} \dot{z} - \triangle z = \sum_{i=1}^n u_i(t)\frac{\partial z}{\partial x_i}(x,t), & \\ z = 0, & z \in \partial\Omega \times]0,T[\\ z(x,0) = z_0(x). & \end{cases} \quad (5.1)$$

Setting $V = H_0^1(\Omega)$, $H = L^2(\Omega)$ and $V' = H^{-1}(\Omega)$, where Ω is an open bounded domain in $\mathcal{R}^\setminus$ with smooth boundary $\Gamma = \partial\Omega$. We obtain $V \hookrightarrow H \hookrightarrow V'$continuously ,densely and compactly.The equation (5.1) can be written in the form

$$\begin{cases} \dot{z}(t) + Az(t) = u(t)Bz(t), \\ z(0) = z_0, \end{cases}$$

where

$$A : V \longrightarrow V'$$

$$\varphi \mapsto -\triangle\varphi$$

$$B_i \quad : V \to H$$

$$\varphi \mapsto B\varphi = \frac{\partial\varphi}{\partial x_i}$$

$$u \in L^\infty(0,T)$$

$$u(t)Bz(t) = \sum_{i=1}^n u_i(t)B_i z(t).$$

The operator A is linear continuous and we have
(H_1)

$$i) \qquad \|A\varphi\|_{V'} \leq \alpha_1 \|\varphi\|_V \qquad \alpha_1 > 0$$

$$ii) \forall \varphi \in V \qquad < A\varphi, \varphi > \geq \alpha_2 \|\varphi\|_V^2 \qquad \alpha_2 > 0$$

$$iii) < A\varphi_1 - A\varphi_2, \varphi_1 - \varphi_2 > \geq \alpha_3 \|\varphi_1 - \varphi_2\|_V^2 \qquad \alpha_3 > 0$$

$$iiii) \varphi \mapsto A\varphi = -\triangle\varphi$$

is continous and Frechet differentiable.
(H_2)

B is linear continuous and

$$\forall \varphi \in H \qquad \|B\varphi\| \leq \alpha \|\varphi\|_H$$

(H_3)

$$u \in L^\infty(0, T)$$

(H_4)

$$L \qquad : H \times \mathcal{R} \longrightarrow \mathcal{R}^+$$

definite as

$$(z(t), u(t)) \mapsto \int_\Omega |z(x,t)|^2 + |u(t)|^2$$

$$\forall z \in L_2(H) \text{ } and \text{ } \forall u \in L^\infty(0, T).$$

i)$L(z(t), u(t)) > 0 > -\infty$
ii)$D(L) \neq \emptyset$.
Thus, L is proper and

$$L(z(t), .) : H \longrightarrow \mathcal{R}$$

$$z(t) \mapsto \int_\Omega |z(x,t)|^2 + |u(t)|^2 dt$$

is convex and lower semi continuous.
i)L is coercive

$$\lim_{\|u\|_{L^\infty(0,T)} \to +\infty} \int_0^T L(z(t), u(t)) dt = +\infty.$$

ii)$(z, u) \mapsto L(z, u)$ is continuous and Frechet differentiable.
Denote by $\mathcal{E}(z_0, u, 0)$ the above equation. Thus, we obtain the following result:

Lemma 4.1

$$\|z\|_{(C[0,T],H)} \leq |z_0| e^{bT\|u\|_{L_\infty}(0,T)} \qquad (E_1)$$

$$\|z\|_{L^2(V)} \leq \left(\frac{1}{2\alpha_1}|z_0| + \frac{bm_1^2}{\alpha_1}\|u\|_{L_\infty(0,T)}\right)^{\frac{1}{2}} \qquad (E_2)$$

$$\|\dot{z}\|_{L^2(V')} \leq \alpha_1 m_1 + 2\alpha_1 m_1 \sqrt{K} + K \qquad (E_3),$$

where $K = \int_0^T a^2(t)dt$ *and* $a(t) = bm_1 \in L^\infty(0,T)$.

Proposition 4.2 *If* $u \in L^\infty(0,T)$ *and* $z_0 \in H$, *then the equation* $\mathcal{E}(z_0, u, 0)$ *has an unique solution* z *in* $L^2(V) \cap L^\infty(H)$ *and* $\dot{z} \in L_2(V')$.

proof: We use lemma 4.1.

Proposition 4.3 *The mapping*

$$\theta : \mathcal{U} \rightarrow \mathrm{C}([0,T], H)$$

$$u \mapsto z,$$

where z *is the solution of the* $\mathcal{E}(z_0, u, 0)$ *is Frechet differentiable,and* $\theta'_u(h)$ *satisfies the equation:*

$$\begin{cases} \dot{y(t)} + Ay(t) = u(t)By(t) + h(t)B\overline{z}(t), \\ y(0) = 0, \end{cases}$$

where $\overline{z} = \theta(\overline{u})$.

Proof:
i)Let us remark that the map

$$L : \mathcal{U} \rightarrow (\mathcal{C}([0,T], H)$$

$$h \rightarrow y,$$

where y is the solution of the equation $\mathcal{E}(0, \overline{u}, hB\overline{z})$ is linear. Since y is the solution of $\mathcal{E}(0, \overline{u}, hB\overline{z})$,then priori estimates E_1 gives

$$\|y\|_{\mathcal{C}([0,T],H)} \leq (\frac{c^2}{2\alpha_2} \|hB\overline{z}\|^2_{L^2(V')})^{\frac{1}{2}} e^{bT\|u\|_{L^\infty(0,T)}}.$$

Thus,

$$\|y\|_{\mathcal{C}([0,T],H)} \leq (\frac{c}{\sqrt{2\alpha_2}} \|h\|_{L^\infty(0,T)}) \|B\overline{z}\|_{L^2(V')}) e^{bT\|u\|_{L^\infty(0,T)}}.$$

Consequently,

$$\|y\|_{\mathcal{C}([0,T],H)} \leq (\frac{c}{\sqrt{2\alpha_2}} \|h\|_{L^\infty(0,T)}) \alpha \|\overline{z}\|_H e^{bT\|u\|_{L^\infty(0,T)}},$$

which implies

$$\|y\|_{\mathcal{C}([0,T],H)} \leq K\|h\|_{L^\infty(0,T)}.$$

Hence,

$$\|y\|_{\mathcal{C}([0,T],H)} \leq K\|h\|_{L^\infty(0,T)}.$$

Therefore, L is continuous.
ii)Set

$$z_h = \theta(\overline{u} + h)$$
$$z = z_h - \overline{z}.$$

Thus,

$$\begin{cases} \dot{z_h}(t) + Az_h(t) = (\overline{u} + h)(t)Bz_h(t), \\ z_h(0) = z_0. \end{cases}$$

On the other hand, $\theta(\overline{u}) = \overline{z}$. Consequently,

$$\begin{cases} \dot{z}(t) + A\overline{z}(t) = \overline{u}(t)B\overline{z}(t), \\ \overline{z}(0) = z_0, \end{cases}$$

which means that

$$\begin{cases} \dot{z}(t) + Az(t) = (\overline{u}(t)Bz(t) + h(t)Bz_h(t), \\ z(0) = 0. \end{cases}$$

Hence,z is the solution of the equation $\mathcal{E}(0, \overline{u}, hBz_h)$. It follows also from estimates E_1 that

$$\|z\|_{\mathcal{C}([0,T],H)} \leq (\frac{c^2}{2\alpha_2}\|hB\overline{z_h}\|^2_{L^2(V')})^{\frac{1}{2}} e^{bT\|u\|_{L^\infty(0,T)}}.$$

Therefore,

$$\|z\|_{\mathcal{C}([0,T],H)} \leq K'\|h\|_{L^\infty(0,T)}.$$

iii) Setting $\omega = z - y$, we can easily see that ω is the solution of the equation $E(0, \overline{u}, hBz)$. Thus,

$$\begin{cases} \dot{\omega}(t) + A\omega(t) = (\overline{u}(t)B\omega(t) + h(t)Bz(t), \\ \omega(0) = 0, \end{cases}$$

and since $\omega \in \mathcal{C}([0,T];H)$, we deduce that

$$\|\omega\|_{\mathcal{C}([0,T],H)} \leq K\|h\|^2_{L^2(0,T)}.$$

Proposition 4.4 *The function $J(u)$ is differentiable and $\forall h \in \mathcal{U}$*

$$dJ(u).h = 2 < hBz, p > + < u, h >,$$

where p is the solution of adjoint equation:

$$\begin{cases} -\frac{dp(t)}{dt} + Ap(t) = -u(t)B^*p(t) + z(t), \\ p(T) = 0, p \in L^2(V), \dot{p} \in L^2(V'). \end{cases}$$

proof:
we have

$$J(u) = \|z\|^2_{L^2(H)} + \|u\|^2_{L^\infty(0,T)}$$
$$= \|\theta((u)\|^2_{L^2(H)} + \|u\|^2_{L^\infty(0,T)}.$$

Since the function θ is differentiable, we deduce

$$dJ(u).h = 2 < \theta'_u(h), z(t) > +2 < u, h >$$

$$= 2\int_0^T < y(t), -p(t) + Ap(t) - u(t)B^*p(t) > dt + 2 < u, h >$$

$$= 2\int_0^T < y(t), -\dot{p}(t) > dt + 2\int_0^T < y(t, Ap(t) - u(t)B * p(t) > dt + 2 < u, h >$$

$$= 2[< y(T), p(T) > + < y(0), p(0) >] + 2\int_0^T < \dot{y}(t), p(t) > dt$$

$$+2\int_0^T < y(t), Ap(t) - u(t)B^*p(t) > dt + 2 < u, h >$$

$$= 2\int_0^T < \dot{y}(t), A^*p(t) - u(t)By(t), p(t) > dt + 2 < u, h >$$

$$= 2\int_0^T < h(t)Bz(t), p(t) > dt + 2 < u, h > .$$

Thus, we achieve the proof.

Corollary 4.5 *An optimal control $\overline{u}$, the state $\overline{z}$ and the corresponding adjoint state $\overline{p}$ are necessarily dregs by the following equations: for $t \in [0, T]$*

$$\begin{cases} \dot{\overline{z}} + A\overline{z} = \overline{u}B\overline{z}, & \overline{z}(0) = z_0 \\ -\dot{p}(t) + A^*p(t) = \overline{u}B^*p(t). & \\ p(T) = 0. & \end{cases}$$

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