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Uniformly Stable Solution of a Non-autonomous Delayed System with Caputo Fractional Derivative

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Abstract

In this work, we are concerned with a class of nonlocal non-autonomous linear fractional-order delay differential equations with different delays, we prove the existence and uniqueness of the solution. Also we study the uniformly stability of the solution.

Keywords: Caputo fractional derivative; non-autonomous time delay system; nonlocal condition; stability analysis

1 Introduction

Let $x(t) = (x_1(t), x_2(t), ..., x_n(t))'$, where $'$ denoted the transpose of the matrix. Let $\alpha \in (0,1]$ and $i = 1,2,...,n$. Consider the nonlocal problem

$$^cD^\alpha x_i(t) = \sum_{j=1}^{n} a_{ij}(t)\, x_j(t) + \sum_{j=1}^{n} b_{ij}(t)\, x_j(t-r_j) + h_i(t) \quad , \quad t \in (0,1] \tag{1}$$

$$x_i(0) + \sum_{k=1}^{m} c_k\, x_k(t_k) = x_{i0} \tag{2}$$

$$x(t) = \Phi(t) \quad \text{for} \quad t < 0 \tag{3}$$

where $^cD^\alpha$ denoted the Caputo fractional-order derivative of order α, and $A(t) = (a_{ij}(t))_{n\times n}$, $B(t) = (b_{ij}(t))_{n\times n}$, $H(t) = (h_i(t))_{n\times 1}$ and $\Phi(t) = (\phi_i(t))_{n\times 1}$ are given matrices and $r_j \geq 0$ are constants.
Also $x_{i0} \in \Re$ and $0 < t_1 < t_2 < \ ... \ < t_m < 1$, and $c_k \neq 0$ for all $k = 1,2,...,m$.
The stability theory of fractional differential equations is of main interest in physical systems. Recently, considerable attention has given to the stability of fractional differential equations by many researchers (see [1]-[2], [4]-[15], and the references therein).
In this work, we discuss the existence, uniqueness and uniform stability of solution of the Caputo fractional non-autonomous time-varying delays system (1) with the m-point initial condition (2) and the initial condition (3).

2 Preliminaries

Let $L_1[a,b]$ be the space of Lebesgue integrable functions on the interval $[a,b]$, $0 \leq a < b < \infty$ with the norm $||x||_{L_1} = \int_a^b |x(t)|\, dt$

Definition 2.1 The fractional (arbitrary) order integral of the function $f(t) \in L_1[a,b]$ of order $\alpha \in R^+$ is defined by (see [14], [16]-[18])

$$I_a^\alpha \ f(t) = \int_a^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds.$$

where $\Gamma(\ . \ )$ is the gamma function.

Definition 2.2 The Caputo fractional (arbitrary) order derivative of order $\alpha, n < \alpha < n+1$ of the function $f(t)$ is defined by (see [16]-[18]),

$${}^cD_a^\alpha \ f(t) = I_a^{n-\alpha} \ D^n \ f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t - s)^{n-\alpha-1} f^{(n)}(s)\, ds \quad , \quad t \in [a,b]$$

Definition 2.3 The Riemann-liouville fractional (arbitrary) order derivatives of order α, $n < \alpha < n+1$ of the function $f(t)$ is defined by (see [14], [16]-[18])

$$D_a^\alpha \ f(t) = \frac{d^n}{dt^n} I_a^{n-\alpha} \ f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t - s)^{n-\alpha-1} f(s)\, ds \quad , \quad t \in [a,b]$$

The following theorem on the properties of fractional order integration can be easily proved
Theorem 2.1 Let $\alpha, \ \beta \in R^+$, then we have

(i) $I_a^\alpha : \ L_1 \longrightarrow L_1$, and if $f(t) \in L_1$, then $I_a^\alpha \ I_a^\beta \ f(t) = I_a^{\alpha+\beta} \ f(t)$

(ii) $\lim_{\alpha \to n} I_a^\alpha = I_a^n$, $n = 1, \ 2, \ 3, \ ...$ uniformly

(iii) If $f(t)$ is absolutely continuous on $[a,b]$, then

$$\frac{d}{dt} I_a^\alpha \ f(t) = \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} f(t)|_{t=a} + I_a^\alpha \frac{d}{dt} f(t)$$

The concept of stability can be related to that of continuous dependence of solution on their initial value.
Consider the non-autonomous linear system

$$x'(t) = A(t)\ x(t) \tag{4}$$

with the initial condition

$$x(t_0) = x^0$$

Definition 2.5 (see [3]) The solution $x = 0$ of (4) will be called stable if for any $\epsilon > 0$, $t_0 \geq 0$, there exist $\delta(\epsilon, t_0) > 0$ such that $||x(t, t_0, x^0)|| < \epsilon$ for $t \geq t_0$ as soon as $||x^0|| < \delta$. And the solution $x = 0$ of (4) will be called uniformly stable if $\delta(\epsilon, t_0)$ can be chosen independent of t_0 : $\delta(\epsilon, t_0) \equiv \delta(\epsilon)$

3 Existence and Uniqueness

let $X = (C_n(I) , || . ||)$ where $C_n (I)$ be the class of all continuous column n-vectors function and for $x \in C_n[0,1]$ the norm is defined by $|| x || = \sum_{i=1}^{n} \sup_{t\in[0,1]} |x_i(t)|$. and define the norm of A by $|| A || = \sum_{i=1}^{n} |a_i| = \sum_{i=1}^{n} \sup_{t,\forall j} | a_{ij} |$, the norm of B by $|| B || = \sum_{i=1}^{n} |b_i| = \sum_{i=1}^{n} \sup_{t,\forall j} | b_{ij} |$.

Theorem 3.1: Let $a_{ij}(t)$, $b_{ij}(t)$, $h_i(t)$ and $\phi_i(t)$ are absolutely continuous functions and

$$1 + \sum_{k=1}^{m} c_k \neq 0 \quad \text{where} \quad (1 + \sum_{k=1}^{m} c_k)^{-1} = c$$

If
$$C = \frac{||A|| + ||B||}{\Gamma(\alpha+1)} (1 + |c| \sum_{k=1}^{m} |c_k|) < 1$$

Then there exist a unique solution $x(t) \in X$ of (1)-(3) with $x'(t) \in L_1[0,T]$.
Proof: For simplicity let

$$g(t, x_j(t), x_j(t-r_j)) = \sum_{j=1}^{n} a_{ij}(t)\, x_j(t) + \sum_{j=1}^{n} b_{ij}(t)\, x_j(t-r_j) + h_i(t).$$

For $t > 0$, $0 < \alpha \leq 1$ and from the properties of fractional 0f the fractional derivative, equation (1) can be written as

$$I^{1-\alpha} \frac{d}{dt} x_i(t) = g(t, x_j(t), x_j(t-r_j))$$

operating by I^{α} on both sides, we get

$$I^{\alpha}\, I^{1-\alpha} \frac{d}{dt} x_i(t) = I^{\alpha}\, g(t, x_j(t), x_j(t-r_j))$$

then
$$x_i(t) - x_i(0) = I^{\alpha}\, g(t, x_j(t), x_i(t-r_j))$$

by substituting for the value of $x_i(0)$ from (2), we get

$$x_i(t) = x_{i0} - \sum_{k=1}^{m} c_k\, x_i(t_k) + I^{\alpha}\, g(t, x_j(t), x_j(t-r_j)) \tag{5}$$

putting $t = t_k$ in (5), we obtain

$$x_i(t_k) = x_{i0} - \sum_{k=1}^{m} c_k\, x_i(t_k) + I^{\alpha}\, g(t, x_j(t), x_j(t-r_j))|_{t=t_k} \tag{6}$$

then subtract (5) from (6), we get

$$x_i(t_k) = x_i(t) + I^{\alpha}\, g(t, x_j(t), x_j(t-r_j))|_{t=t_k} - I^{\alpha}\, g(t, x_j(t), x_j(t-r_j)) \tag{7}$$

substitute from (7) in (5), we get

$$\begin{aligned} x_i(t) = {} & x_{i0} + I^{\alpha}\, g(t, x_j(t), x_j(t-r_j)) \\ & - \sum_{k=1}^{m} c_k\, \{x_i(t) + I^{\alpha}\, g(t, x_j(t), x_j(t-r_j))|_{t=t_k} - I^{\alpha}\, g(t, x_j(t), x_j(t-r_j))\} \end{aligned}$$

$$(1 + \sum_{k=1}^{m} c_k)\ x_i(t) = x_{i0} + (1 + \sum_{k=1}^{m} c_k)\ I^{\alpha}\ g(t, x_j(t), x_j(t-r_j)) - \sum_{k=1}^{m} c_k\ I^{\alpha}\ g(t, x_j(t), x_j(t-r_j))|_{t=t_k}$$

$$x_i(t) = c\ x_{i0} + I^{\alpha}\ g(t, x_j(t), x_j(t-r_j)) - c \sum_{k=1}^{m} c_k\ I^{\alpha}\ g(t, x_j(t), x_j(t-r_j))|_{t=t_k} \tag{8}$$

New, Let $F : X \to X$, defined by

$$F\ x_i(t) = c\ x_{i0} + I^{\alpha}\{ \sum_{j=1}^{n} a_{ij}(t)\ x_j(t) + \sum_{j=1}^{n} b_{ij}(t)\ x_j(t-r_j) + h_i(t)\} - c\sum_{k=1}^{m} c_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}\{\sum_{j=1}^{n} a_{ij}(s)x_j(s)ds + \sum_{j=1}^{n} b_{ij}(s)x_j(s-r_j) + h_i(s)\}\ ds$$

Let $x_i(t),\ y_i(t) \in X$, then

$$\begin{aligned}
|Fx_i(t) - Fy_i(t)| \leq\ & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{j=1}^{n} |a_{ij}(s)|\ |x_j(s) - y_j(s)|\ ds \\
& + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{j=1}^{n} |b_{ij}(s)|\ |x_j(s-r_j) - y_j(s-r_j)|\ ds \\
& + |c| \sum_{k=1}^{m} |c_k| \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{j=1}^{n} |a_{ij}(s)|\ |x_j(s) - y_j(s)|\ ds \\
& + |c| \sum_{k=1}^{m} |c_k| \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{j=1}^{n} |b_{ij}(s)|\ |x_j(s-r_j) - y_j(s-r_j|\ ds \\
\leq\ & \sup_{t,\forall j} |a_{ij}(t)| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{j=1}^{n} |x_j(s) - y_j(s)|\ ds \\
& + \sup_{t,\forall j} |b_{ij}(t)| \int_0^{r_j} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{j=1}^{n} |x_j(s-r_j) - y_j(s-r_j)|\ ds \\
& + \sup_{t,\forall j} |b_{ij}(t)| \int_{r_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{j=1}^{n} |x_j(s-r_j) - y_j(s-r_j)|\ ds \\
& + |c| \sum_{k=1}^{m} |c_k| \sup_{t,\forall j} |a_{ij}(t)| \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{j=1}^{n} |x_j(s) - y_j(s)|\ ds \\
& + |c| \sum_{k=1}^{m} |c_k| \sup_{t,\forall j} |b_{ij}(t)| \sum_{j=1}^{n} \int_0^{r_j} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} |x_j(s-r_j) - y_j(s-r_j|\ ds \\
& + |c| \sum_{k=1}^{m} |c_k| \sup_{t,\forall j} |b_{ij}(t)| \sum_{j=1}^{n} \int_{r_j}^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} |x_j(s-r_j) - y_j(s-r_j|\ ds \\
\leq\ & |a_i| \sum_{j=1}^{n} \sup_t |x_j(t) - y_j(t)| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ ds
\end{aligned}$$

$$+ |b_i| \sum_{j=1}^{n} \sup_t |x_j(t) - y_j(t)| \int_{r_j}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds$$
$$+ |a_i|\,|c| \sum_{k=1}^{m} |c_k| \sum_{j=1}^{n} \sup_t |x_j(t) - y_j(t)| \int_{0}^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} ds$$
$$+ |b_i|\,|c| \sum_{k=1}^{m} |c_k| \sum_{j=1}^{n} \sup_t \{ |x_j(t) - y_j(t)| \} \int_{r_j}^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} ds$$
$$\leq \frac{|a_i|\ t^{\alpha}}{\Gamma(\alpha+1)} ||x-y|| + |b_i| \sum_{j=1}^{n} \frac{(t-r_j)^{\alpha}}{\Gamma(\alpha+1)} \sup_t |x_j(t) - y_j(t)|$$
$$+ |a_i|\,|c| \sum_{k=1}^{m} |c_k| \frac{t_k^{\alpha}}{\Gamma(\alpha+1)} ||x-y||$$
$$+ |b_i|\,|c| \sum_{k=1}^{m} |c_k| \sum_{j=1}^{n} \frac{(t_k - r_j)^{\alpha}}{\Gamma(\alpha+1)} \sup_t |x_j(t) - y_j(t)|$$
$$\leq \frac{|a_i| + |b_i|}{\Gamma(\alpha+1)} (1 + |c| \sum_{k=1}^{m} |c_k|) ||x-y||$$

and

$$||Fx - Fy|| = \sum_{i=1}^{n} \sup_t |Fx_i - Fy_i| \leq \sum_{i=1}^{n} \frac{|a_i| + |b_i|}{\Gamma(\alpha+1)} (1 + |c| \sum_{k=1}^{m} |c_k|) ||x-y||$$
$$\leq \frac{||A|| + ||B||}{\Gamma(\alpha+1)} (1 + |c| \sum_{k=1}^{m} |c_k|) ||x-y|| = C\ ||x-y||.$$

Then map $F : X \to X$ is a contraction and it has a fixed point $x = F\ x$ and hence, there exist a unique column vector $x(t) \in X$ which is the solution of the integral equation (8).

Now, differentiate (8) and using Theorem 2.1, we obtain

$$x_i'(t) = \frac{d}{dt} I^{\alpha} \{\sum_{j=1}^{n} a_{ij}(t)\ x_j(t) + \sum_{j=1}^{n} b_{ij}(t)\ x_j(t-r_j) + h_i(t)\}$$
$$= [\sum_{j=1}^{n} a_{ij}(t)\ x_j(t) + \sum_{j=1}^{n} b_{ij}(t)\ x_j(t-r_j) + h_i(t)]|_{t=0} \frac{t^{\alpha-1}}{\Gamma(\alpha)}$$
$$+ I^{\alpha} \frac{d}{dt} \{\sum_{j=1}^{n} a_{ij}(t)\ x_j(t) + \sum_{j=1}^{n} b_{ij}(t)\ x_j(t-r_j) + h_i(t)\}$$

$$= \frac{K\ t^{\alpha-1}}{\Gamma(\alpha)} + I^{\alpha} \{\sum_{j=1}^{n} a_{ij}'(t)\ x_j(t) + \sum_{j=1}^{n} a_{ij}(t)\ x_j'(t)$$
$$+ \sum_{j=1}^{n} b_{ij}'(t)\ x_j(t-r_j) + \sum_{j=1}^{n} b_{ij}(t)\ x_j'(t-r_j) + h_i'(t)\},$$

$$\begin{aligned}
\int_0^1 |x_i'(t)|\,dt \;\leq\;& \frac{K\,t^{\alpha}}{\Gamma(\alpha+1)}\Big|_0^1 + \int_0^1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \{\sum_{j=1}^{n} |a_{ij}'(s)|\,|x_j(s)| \;+\; \sum_{j=1}^{n} |a_{ij}(s)|\,|x_j'(s)| \\
&+ \sum_{j=1}^{n} |b_{ij}'(s)|\,|x_j(s-r_j)| \;+\; \sum_{j=1}^{n} |b_{ij}(s)|\,|x_j'(s-r_j)| \;+\; |h_i'(s)|\}ds\,dt \\
\leq\;& \frac{K}{\Gamma(\alpha+1)} + \int_0^1 \{\sum_{j=1}^{n} |a_{ij}'(s)|\,|x_j(s)| \;+\; \sum_{j=1}^{n} |a_{ij}(s)|\,|x_j'(s)| \\
&+ \sum_{j=1}^{n} |b_{ij}'(s)|\,|x_j(s-r_j)| + \sum_{j=1}^{n} |b_{ij}(s)|\,|x_j'(s-r_j)| + |h_i'(s)|\} \int_s^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,dt\,ds \\
\leq\;& \frac{K}{\Gamma(\alpha+1)} \;+\; \int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} \{\sum_{j=1}^{n} |a_{ij}'(s)|\,|x_j(s)| \;+\; \sum_{j=1}^{n} |a_{ij}(s)|\,|x_j'(s)| \\
&+ \sum_{j=1}^{n} |b_{ij}'(s)|\,|x_j(s-r_j)| \;+\; \sum_{j=1}^{n} |b_{ij}(s)|\,|x_j'(s-r_j)| \;+\; |h_i'(s)|\}\,ds \\
\leq\;& \frac{K}{\Gamma(\alpha+1)} \;+\; \frac{1}{\Gamma(\alpha+1)} \{\sum_{j=1}^{n} \sup_t |x(t)| \int_0^1 |a_{ij}'(s)|\,ds \\
&+ \sup_{t,\forall j} |a_{ij}(t)| \int_0^1 \sum_{j=1}^{n} |x_j'(s)|\,ds \;+\; \sum_{j=1}^{n} \int_0^{r_j} |b_{ij}'(s)|\,|x_j(s-r_j)|\,ds \\
&+ \sum_{j=1}^{n} \int_{r_j}^1 |b_{ij}'(s)|\,|x_j(s-r_j)|\,ds \;+\; \sum_{j=1}^{n} \sup_{t,\forall j} |b_{ij}(t)| \int_0^{r_j} |x_j'(s-r_j)|ds \\
&+ \sum_{j=1}^{n} \sup_{t,\forall j} |b_{ij}(t)| \int_{r_j}^1 |x_j'(s-r_j)|ds + \int_0^1 |h_i'(s)|\,ds\} \\
\leq\;& \frac{K}{\Gamma(\alpha+1)} \;+\; \frac{1}{\Gamma(\alpha+1)} \{a^*\,||x|| \;+\; |a_i|\,||x'||_{L_1} \\
&+ \sum_{j=1}^{n} \sup_t |\phi_j(t)| \int_0^{r_j} |b_{ij}'(s)|\,ds \;+\; \sum_{j=1}^{n} \sup_t |x_j(t)| \int_{r_j}^1 |b_{ij}'(s)|\,ds \\
&+ |b_i| \sum_{j=1}^{n} \int_0^{r_j} |x_j'(s-r_j)|ds \;+\; |b_i| \sum_{j=1}^{n} \int_{r_j}^1 |x_j'(s-r_j)|ds + \int_0^1 |h_i'(s)|\,ds\} \\
\leq\;& \frac{K}{\Gamma(\alpha+1)} \;+\; \frac{1}{\Gamma(\alpha+1)} \{a^*\,||x|| \;+\; |a_i|\,||x'||_{L_1} \;+\; b^*\,||\Phi|| \\
&+ |b_i|\,||\Phi'||_{L_1} \;+\; |b_i|\,||x'||_{L_1} \;+\; \int_0^1 |h_i'(s)|\,ds\}
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{n} \int_0^1 |x_i'(t)|\,dt \;=\; ||x'||_{L_1} \;\leq\;& \frac{1}{\Gamma(\alpha+1)} \{nK \;+\; n\,a^*\,||x|| \;+\; ||A||\,||x'||_{L_1} \\
&+ n\,b^*\,||\Phi|| \;+\; ||B||\,||\Phi'||_{L_1} \;+\; ||B||\,||x'||_{L_1} \;+\; ||H'||_{L_1}\}
\end{aligned}$$

$$(1 - \frac{(||A|| + ||B||)}{\Gamma(\alpha+1)})||x'||_{L_1} \leq \frac{1}{\Gamma(\alpha+1)}\{nK \;+\; n\,a^*\,||x|| \;+\; n\,b^*\,||\Phi|| + ||B||\,||\Phi'||_{L_1} + ||H'||_{L_1}\}$$

$$||x'||_{L_1} \leq (1 - \frac{(||A|| + ||B||)}{\Gamma(\alpha+1)})^{-1} \left(\frac{1}{\Gamma(\alpha+1)} \{nK + n\ a^*\ ||x|| + n\ b^*\ ||\Phi|| + ||B||\ ||\Phi'||_{L_1} + ||H'||_{L_1}\} \right).$$

Therefore we obtain $x' \in L_1[0,1]$

We now prove the equivalence between the integral equation (8) and the nonlocal problem (1)- (3), differentiate both sides of (8), we get

$$\begin{aligned} x_i'(t) &= \frac{d}{dt} I^\alpha\ g(t, x_j(t), x_j(t-r_j)) \\ &= I^\alpha \frac{d}{dt} g(t, x_j(t), x_j(t-r_j)) + g(t, x_j(t), x_j(t-r_j))|_{t=0} \frac{t^{\alpha-1}}{\Gamma(\alpha)} \end{aligned}$$

Operating by $I^{1-\alpha}$ on both sides, we obtain

$$\begin{aligned} I^{1-\alpha}\ Dx_i(t) &= I^\alpha\ I^{1-\alpha} \frac{d}{dt} g(t, x_j(t), x_j(t-r_j)) + g(t, x_j(t), x_j(t-r_j))|_{t=0} \\ {}^cD^\alpha x_i(t) &= I \frac{d}{dt} g(t, x_j(t), x_j(t-r_j)) + g(t, x_j(t), x_j(t-r_j))|_{t=0} \end{aligned}$$

and we have

$${}^cD^\alpha x_i(t) = g(t, x_j(t), x_j(t-r_j))$$

which prove the equivalence of (8) and (1).
We want to prove that $\lim_{t\to 0^+} x_i(t) = x_{i0} - \sum_{k=1}^m c_k\ x_i(t_k)$.
Let $t \to 0$ in (8) we find the nonlocal condition (2) is satisfied, which prove the equivalence.

4 Stability

In this section we study the stability of the solution of the nonlocal problem (1)-(3)
Definition 4.1 The solution of the non-autonomous linear system (1) is stable if for any $\epsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that for any two solutions $x(t) = (x_1(t), x_2(t), ..., x_n(t))'$ and $\widetilde{x}(t) = (\widetilde{x}_1(t), \widetilde{x}_2(t), ..., \widetilde{x}_n(t))'$ with the initial conditions (2)-(3) and $\tilde{x}_i(0) + \sum_{k=1}^m c_k\ \tilde{x}_i(t_k) = \tilde{x}_{i0}$, $\tilde{x}(t) = \tilde{\Phi}(t)$ for $t < 0$ respectively, one has $||x_0 - \widetilde{x}_0(t)|| \leq \delta_1$ and $||\Phi(t) - \widetilde{\Phi}(t)|| \leq \delta_2$, then $||x(t) - \widetilde{x}(t)|| < \epsilon$ for all $t \geq 0$.

Theorem 4.2: The solution of the nonlocal delay system (1)-(3) is uniformly stable.
Proof: Let $x(t)$ and $\widetilde{x}(t)$ be two solutions of the system (1) under the conditions (2) - (3) and $\tilde{x}_i(0) + \sum_{k=1}^m c_k\ \tilde{x}_i(t_k) = \tilde{x}_{i0}$, $\tilde{x}(t) = \tilde{\Phi}(t)$ for $t < 0$ respectively,
then from (8), for $t > 0$,

$$\begin{aligned} x_i(t) &= c\ x_{i0} + I^\alpha\{ \sum_{j=1}^n a_{ij}(t)\ x_j(t) + \sum_{j=1}^n b_{ij}(t)\ x_j(t-r_j) + h_i(t)\} \\ &\quad - c\sum_{k=1}^m c_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} \{\sum_{j=1}^n a_{ij}(s)x_j(s)ds + \sum_{j=1}^n b_{ij}(s)x_j(s-r_j) + h_i(s)\}\ ds, \end{aligned}$$

$$\begin{aligned}
|x_i(t)-\tilde{x}_i(t)| \;\le\;& |c|\;|x_{i0}-\tilde{x}_{i0}| \;+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\;\sum_{j=1}^{n}\;|a_{ij}(s)|\;|x_j(s)-\tilde{x}_j(s)|\;ds \\
&+\;\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\;\sum_{j=1}^{n}\;|b_{ij}(s)|\;|x_j(s-r_j)\;-\;\tilde{x}_j(s-r_j)|\;ds \\
&+|c|\;\sum_{k=1}^{m}\;|c_k|\;\int_0^{t_k}\;\frac{(t_k\;-\;s)^{\alpha-1}}{\Gamma(\alpha)}\;\sum_{j=1}^{n}\;|a_{ij}(s)|\;|x_j(s)-\tilde{x}_j(s)|\;ds \\
&+|c|\;\sum_{k=1}^{m}\;|c_k|\;\int_0^{t_k}\;\frac{(t_k\;-\;s)^{\alpha-1}}{\Gamma(\alpha)}\;\sum_{j=1}^{n}\;|b_{ij}(s)|\;|x_j(s-r_j)-\tilde{x}_j(s-r_j)|\;ds \\
\le\;& |c|\;|x_{i0}-\tilde{x}_{i0}|\;+\;\frac{|a_i|\,t^{\alpha}}{\Gamma(\alpha+1)}\;||x-\tilde{x}|| \\
&+|b_i|\;\sum_{j=1}^{n}\;\sup_t\;\{|\phi_j(t)-\tilde{\phi}_j(t)|\}\int_0^{r_j}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\;ds \\
&+|b_i|\;\sum_{j=1}^{n}\;\sup_t\;\{|x_j(t)-\tilde{x}_j(t)|\}\;\int_{r_j}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\;ds \\
&+|a_i|\;|c|\sum_{k=1}^{m}\;|c_k|\;\frac{t_k^{\alpha}}{\Gamma(\alpha+1)}\;||x-\tilde{x}|| \\
&+|b_i|\,|c|\;\sum_{k=1}^{m}\;|c_k|\;\sum_{j=1}^{n}\;\sup_t\{|\phi_j(t)-\tilde{\phi}_j(t)|\}\int_0^{r_j}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds \\
&+|b_i|\,|c|\;\sum_{k=1}^{m}\;|c_k|\sum_{j=1}^{n}\;\sup_t\{|x_j(t)-\tilde{x}_j(t)|\}\int_{r_j}^{t_k}\;\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha+1)}\;ds \\
\le\;& |c|\;|x_{i0}-\tilde{x}_{i0}|\;+\;\frac{|a_i|\;+\;|b_i|}{\Gamma(\alpha+1)}\;(1\;+\;|c|\;\sum_{k=1}^{m}\;|c_k|)\;||x-\tilde{x}|| \\
&+\;\frac{|b_i|}{\Gamma(\alpha+1)}\;(1\;+\;|c|\;\sum_{k=1}^{m}\;|c_k|)\;||\Phi-\tilde{\Phi}||
\end{aligned}$$

$$\begin{aligned}
||x-\tilde{x}|| \;=\;& \sum_{i=1}^{n}\;\sup_t|x_i-\tilde{x}_i|\le\;|c|\;\sum_{i=1}^{n}\;|x_{i0}-\tilde{x}_{i0}| \\
&+\sum_{i=1}^{n}\;\frac{|a_i|\;+\;|b_i|}{\Gamma(\alpha+1)}\;(1\;+\;|c|\;\sum_{k=1}^{m}\;|c_k|)\;||x-\tilde{x}|| \\
&+\sum_{i=1}^{n}\;\frac{|b_i|}{\Gamma(\alpha+1)}\;(1\;+\;|c|\;\sum_{k=1}^{m}\;|c_k|)\;||\Phi-\tilde{\Phi}|| \\
\le\;& |c|\;||x_0-\tilde{x}_0||_{R^n}\;+\;\frac{||A||\;+\;||B||}{\Gamma(\alpha+1)}\;(1\;+\;|c|\;\sum_{k=1}^{m}\;|c_k|)\;||x-\tilde{x}|| \\
&+\frac{||B||}{\Gamma(\alpha+1)}\;(1\;+\;|c|\;\sum_{k=1}^{m}\;|c_k|)\;||\Phi-\tilde{\Phi}||
\end{aligned}$$

and we have

$$(1 - C)\ ||x(t) - \tilde{x}(t)|| \leq |c|\ ||x_0 - \tilde{x}_0||_{R^n} + \frac{||B||}{\Gamma(\alpha+1)} (1 + |c| \sum_{k=1}^{m} |c_k|)\ ||\Phi - \tilde{\Phi}||.$$

Therefore for δ_1 , $\delta_2 > 0$ such that $||x_0 - \tilde{x}_0|| < \delta_1$ and $||\Phi - \tilde{\Phi}|| < \delta_2$ we can find

$$||x(t) - \tilde{x}(t)|| < \epsilon = (1 - C)^{-1}(|c|\ \delta_1 + \frac{||B||\ (1 + |c| \sum_{k=1}^{m} |c_k|)}{\Gamma(\alpha+1)} \delta_2),$$

where $||x_0 - \tilde{x}_0||_{R^n} = \sum_{i=1}^{n} |x_{i0} - \tilde{x}_{i0}|$

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