Int. Journal of Math. Analysis, Vol. 6, 2012, no. 45, 2227 - 2234

Limit Theorems for Delayed Averages of Random Sequence[1]

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Abstract. In this paper we introduce the asymptotic logarithmic delayed likelihood ratio $\gamma(\omega)$ as a measure of deviation of the joint distribution from the product of their marginals, and establish some generalized strong limit theorems of delayed average of random sequence.

Mathematics Subject Classification: 60F15

Keywords: delayed average; likelihood ratio; asymptotic logarithmic delayed likelihood ratio; strong deviation theorem

1. Introduction

Let $(\xi_n)_{n\in\mathbf{N}}$ be a sequence of random variables and let $(f(n))_{n\in\mathbf{N}}$ be a sequence of positive integers.The numbers

$$\rho_{n,f(n)} = \{\Sigma_{j=n+1}^{n+f(n)}\xi_j\}/f(n)$$

are called the (forward) delayed first arithmetic means (cf. [7]).In [4],Lai studied the analogues of the law of the iterated logarithm for delayed sums of independent random variables. Recently,Chen (cf. [1])has presented an accurate description the limiting behavior of delayed sums under a non-identically distribution setup,and has deduced Chover-type laws of hte iterated logarithm for them.Vasudeva,R.and Gooty Divanji (cf. [5])obtained a bivariate law of iterated logarithm for the vector of partial sums and delayed sums when the random variables were positive strictly stable.

The aim of this paper is to establish some generalized strong limit theorems of delayed average for arbitrarily dependent random variables. The crucial part of the proofing is the construction of random variables depending on a parameter,and then the application of the Borel-Cantelli lemma. We extend

[1]Foundation item: Supported by graduate student innovation fund of Anhui University of Technology (2011040).

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the analytic technique method proposed by Liu (cf. [3]) to the case of delayed sums.

Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space, and suppose that $(\xi_n)_{n\in\mathbf{N}}$ be a stochastic sequence defined on it.

Let the joint probability density function of random vector $(\xi_m, \cdots, \xi_n)$ be

$$p(x_m, \cdots, x_n), m \leq n, n \in N \tag{1.1}$$

Let $p_k(x_k)$ be the density function of $\xi_k, k \in \mathbf{N}, i.e.$ the marginal density functions of $(\xi_n)_{n\in\mathbf{N}}$, and let

$$q(x_m, \cdots, x_n) = \prod_{k=m}^{n} p_k(x_k) \tag{1.2}$$

In order to prove our main results, we first propose a new concept, which will play a fundamental role in the proofs of our results.

Definition 1. (cf. [6]) Let the quotient

$$\text{(1.3)}\ \Lambda_n(\omega) = \begin{cases} \frac{p_{n+1}(\xi_{n+1})\cdots p_{n+f(n)}(\xi_{n+f(n)})}{p(\xi_{n+1},\cdots,\xi_{n+f(n)})}, & if\ the\ denominator > 0 \\ 0, & otherwise \end{cases}$$

be called the delayed likelihood ratio, and let (cf. [3])

$$\gamma(\omega) = \liminf_{n} \frac{\log \Lambda_n(\omega)}{f(n)}. \tag{1.4}$$

which $\gamma(\omega)$ is called the asymptotic logarithmic delayed likelihood ratio, where $0\log 0 = 0$, ω is the sample point, and ξ_k stands for $\xi_k(\omega)$.

It is easy to see that $\Lambda_n(\omega) \equiv 1, a.s. n \geq 1$ if $(\xi_n)_{n\in\mathbf{N}}$ are independent.

An article which is important for this paper is that of Wang and Yang(2011). In the paper, they established a lemma that is an important tool for deriving results on strong deviation theorems. In order to prove our main results, we first give the lemma.

Lemma *(cf. [6]) Let $\Lambda_n(\omega)$ be defined as above, then*

$$\limsup_{n} \frac{1}{f(n)} \log \Lambda_n(\omega) \leq 0, \quad a.s.$$

(cf.[4])Let functions $\Psi_n : R^+ \to R^+$ be nonnegative, continuous and nondecreasing, which for constants $r_n \geq 0$, satisfy the following conditions

$$\frac{x_1^{r_n}}{x_2^{r_n}} \leq \frac{\Psi_n(x_1)}{\Psi_n(x_2)}, \quad x_1 < x_2 \tag{1.5}$$

2. The Main Results and Proofs

With the preliminaries accounted for, we can formulate and prove the main results of this paper. We emphasize that there are no independence or identical distribution assumptions on the original sequence of random variables $(\xi_n)_{n\in\mathbf{N}}$.

Theorem 1. *Let $(\xi_n), (\Psi_n(x)), n \in \mathbf{N}, \gamma(\omega)$ be defined as above and $(a_n)_{n\in\mathbf{N}}$ be a increasing sequences of positive numbers, for $n \in \mathbf{N}$, let*

$$\eta_n = \xi_n \mathbf{1}_{(|\xi_n|\le a_n)}, \tag{2.1}$$

$$q_n = \begin{cases} 1, & 0 \le r_n \le 2 \\ \frac{2}{r_n}, & r_n > 2 \end{cases} \tag{2.2}$$

and

$$\mathcal{J} = \{\omega : \gamma(\omega) > -\infty\}. \tag{2.3}$$

If

$$\limsup_n \frac{1}{f(n)} \sum_{k=n+1}^{n+f(n)} \{E[\frac{\Psi_k(|\xi_k|)}{\Psi_k(a_k)}]\}^{q_k} = c < \infty, \tag{2.4}$$

then

$$\liminf_n \frac{1}{f(n)} \sum_{k=n+1}^{n+f(n)} \frac{\eta_k - E\eta_k}{a_k} \ge \alpha(\gamma(\omega), c), \quad a.s.\ \omega \in \mathcal{J}. \tag{2.5}$$

$$\limsup_n \frac{1}{f(n)} \sum_{k=n+1}^{n+f(n)} \frac{\eta_k - E\eta_k}{a_k} \le \beta(\gamma(\omega), c), \quad a.s.\ \omega \in \mathcal{J}. \tag{2.6}$$

where

$$\alpha(x, y) = \sup\{\varphi(s, x, y), s < 0\}, \quad x \le 0, y > 0. \tag{2.7}$$

$$\beta(x, y) = \inf\{\varphi(s, x, y), s > 0\}, \quad x \le 0, y > 0. \tag{2.8}$$

$$\varphi(s, x, y) = -\frac{x}{s} + \frac{1}{2} s e^{2|s|} y, \quad x \le 0, y > 0, s \ne 0. \tag{2.9}$$

and

$$\alpha(x, y) \le 0,\ \ \beta(x, y) \ge 0, \tag{2.10}$$

$$\alpha(0, y) = \alpha(x, 0) = 0,\ \ \beta(0, y) = \beta(x, 0) = 0. \tag{2.11}$$

Proof. Let $\mathbf{1}_{(\cdot)}$ denote the indicator function, and Let s be a nonzero real number, define random variables

$$B_k(s) = E\left[\exp\frac{s(\eta_k - E\eta_k)}{a_k}\right] = \int_{|x_k|\le a_k} p_k(x_k)\exp[\frac{s(x_k - E\eta_k)}{a_k}]dx_k. \tag{2.12}$$

$$p_k(s;x_k) = \frac{p_k(x_k)\exp s[x_k\mathbf{1}_{(|x_k|\le a_k)} - E\eta_k]/a_k}{B_k(s)}. \tag{2.13}$$

and

$$p(s;x_{n+1},\cdots,x_{n+f(n)}) = \prod_{k=n+1}^{n+f(n)} p_k(s;x_k). \tag{2.14}$$

Define random variables as follows:

$$\Lambda_n(s;\omega) = \begin{cases} \frac{p(s;\xi_{n+1},\cdots,\xi_{n+f(n)})}{p(\xi_{n+1},\cdots,\xi_{n+f(n)})}, & if\ the\ denominator > 0 \\ 0, & otherwise \end{cases} \tag{2.15}$$

The lemma can be rewrite as

$$\limsup_n \frac{1}{f(n)}\log\Lambda_n(s;\omega) \le 0, \quad a.s. \tag{2.16}$$

Since

$$\log\Lambda_n(s;\omega) = s\sum_{k=n+1}^{n+f(n)}\frac{\eta_k - E\eta_k}{a_k} + \log\Lambda_n(\omega) - \sum_{k=n+1}^{n+f(n)}\log B_k(s). \tag{2.17}$$

Hence, we have

$$\limsup_n \frac{1}{f(n)}[\sum_{k=n+1}^{n+f(n)} s\frac{\eta_k - E\eta_k}{a_k} + \log\Lambda_n(\omega) - \sum_{k=n+1}^{n+f(n)}\log B_k(s)] \le 0, \quad a.s. \tag{2.18}$$

This together with (1.4), we have

$$\limsup_n \frac{1}{f(n)}\sum_{k=n+1}^{n+f(n)} s\frac{\eta_k - E\eta_k}{a_k} \le -\gamma(\omega) + \limsup_n \frac{1}{f(n)}\sum_{k=n+1}^{n+f(n)}\log B_k(s), a.s. \tag{2.19}$$

(2.1) and (2.12) together with the inequalities

$$|\frac{\eta_k - E\eta_k}{a_k}| \le 2, \quad 0 \le e^x - 1 - x \le \frac{1}{2}x^2 e^{|x|} \quad for\ all \quad x \in R \tag{2.20}$$

yield the inequalities

$$\begin{aligned} 0 &\le B_k(s) - 1 \\ &= E\{\exp\left[\frac{s(\eta_k - E\eta_k)}{a_k}\right] - 1 - \frac{s(\eta_k - E\eta_k)}{a_k}\} \\ &\le \frac{1}{2}s^2 e^{2|s|}E[\frac{\eta_k - E\eta_k}{a_k}]^2 \le \frac{1}{2}s^2 e^{2|s|}E[\frac{\eta_k}{a_k}]^2. \end{aligned} \tag{2.21}$$

If $0 < r_k \leq 2$, by (2.1),(1.5), we have

$$E[\frac{\eta_k}{a_k}]^2 = \int_{|x_k|\leq a_k} p_k(x_k)\frac{x_k^2}{a_k^2}dx_k \leq \int_{|x_k|\leq a_k} p_k(x_k)(\frac{|x_k|}{a_k})^{r_k}dx_k$$

$$\leq \int_{|x_k|\leq a_k} p_k(x_k)\frac{\Psi_k(|x_k|)}{\Psi_k(a_k)}dx_k \leq E[\frac{\Psi_k(|\xi_k|)}{\Psi_k(a_k)}]. \tag{2.22}$$

If $r_k > 2$, by the Hölder's inequality, we have

$$E[\frac{\eta_k}{a_k}]^2 = \int_{|x_k|\leq a_k} p_k(x_k)\frac{x_k^2}{a_k^2}dx_k \leq |\int_{|x_k|\leq a_k} p_k(x_k)[\frac{x_k^2}{a_k^2}]^{r_k/2}dx_k|^{2/r_k}$$

$$= |\int_{|x_k|\leq a_k} p_k(x_k)\frac{|x_k|^{r_k}}{a_k^{r_k}}dx_k|^{2/r_k} \leq |\int_{|x_k|\leq a_k} p_k(x_k)\frac{\Psi_k(|x_k|)}{\Psi_k(a_k)}dx_k|^{2/r_k}. \tag{2.23}$$

and thus

$$E[\frac{\eta_k}{a_k}]^2 \leq [E\frac{\Psi_k(|\eta_k|)}{\Psi_k(a_k)}]^{q_k} \leq [E\frac{\Psi_k(|\xi_k|)}{\Psi_k(a_k)}]^{q_k}. \tag{2.24}$$

From (2.21), (2.24), we have

$$0 \leq B_k(s) - 1 \leq \frac{1}{2}s^2e^{2|s|}[E\frac{\Psi_k(|\xi_k|)}{\Psi_k(a_k)}]^{q_k}. \tag{2.25}$$

This and (2.4) we have

$$0 \leq \limsup_n \frac{1}{f(n)}\sum_{k=n+1}^{n+f(n)}[B_k(s) - 1]$$

$$\leq \frac{1}{2}s^2e^{2|s|}\limsup_n \frac{1}{f(n)}\sum_{k=n+1}^{n+f(n)}[E\frac{\Psi_k(|\xi_k|)}{\Psi_k(a_k)}]^{q_k}$$

$$= \frac{1}{2}s^2e^{2|s|}c, \quad a.s. \tag{2.26}$$

This together with the inequality $0 \leq \log x \leq x - 1 \ \ (x > 1)$ yields

$$0 \leq \limsup_n (f(n))^{-1}\sum_{k=n+1}^{n+f(n)}\log B_k(s) \leq \frac{1}{2}s^2e^{2|s|}c, \quad a.s. \tag{2.27}$$

By (2.19) and (2.27), we have

$$\limsup_n \frac{s}{f(n)}\sum_{k=n+1}^{n+f(n)}\frac{\eta_k - E\eta_k}{a_k} \leq -\gamma(\omega) + \frac{1}{2}s^2e^{2|s|}c, \quad a.s.\ \omega \in \mathcal{J} \tag{2.28}$$

Thus for $s > 0$, we have

$$\limsup_n \frac{1}{f(n)}\sum_{k=n+1}^{n+f(n)}\frac{\eta_k - E\eta_k}{a_k} \leq -\frac{\gamma(\omega)}{s} + \frac{1}{2}se^{2|s|}c, \quad a.s.\ \omega \in \mathcal{J} \tag{2.29}$$

By (2.8), (2.9) and (2.29), we have

$$\limsup_{n} \frac{1}{f(n)} \sum_{k=n+1}^{n+f(n)} \frac{\eta_k - E\eta_k}{a_k} \le \beta(\gamma(\omega), c), \quad a.s.\ \omega \in \mathcal{J} \tag{2.30}$$

(2.6) follows.

From (2.28), we analogously find for $s < 0$

$$\liminf_{n} \frac{1}{f(n)} \sum_{k=n+1}^{n+f(n)} \frac{\eta_k - E\eta_k}{a_k} \ge -\frac{\gamma(\omega)}{s} + \frac{1}{2} s e^{2|s|} c, \quad a.s.\omega \in \mathcal{J} \tag{2.31}$$

Similarly, we can prove :

$$\liminf_{n} \frac{1}{f(n)} \sum_{k=n+1}^{n+f(n)} \frac{\eta_k - E\eta_k}{a_k} \ge \alpha(\gamma(\omega), c), \quad a.s.\ \omega \in \mathcal{J}. \tag{2.32}$$

(2.5) follows from (2.32). □

Corollary 1. *Under the conditions of Theorem 1, if $c = 0$ or $\gamma(\omega) = 0, a.s.$, then*

$$\lim_{n} \frac{1}{f(n)} \sum_{k=n+1}^{n+f(n)} \frac{\eta_k - E\eta_k}{a_k} = 0, \quad a.s.\ \omega \in \mathcal{J}. \tag{2.33}$$

Proof. Since $\alpha(x, 0) = \beta(x, 0) = 0$ and $\alpha(0, y) = \beta(0, y) = 0$, if $c = 0$ or $\gamma(\omega) = 0$, $a.s.$, applying Theorem 1, the result follows. □

Theorem 2. *Let $(\xi_n)_{n\in\mathbf{N}}$ be a random sequence with $|\xi_n| \le 1, n \ge 1$ and $\gamma(\omega), \mathcal{J}$ be defined as above, if*

$$\limsup_{n} \frac{1}{f(n)} \sum_{k=n+1}^{n+f(n)} Var\xi_k = \sigma < \infty, \tag{2.34}$$

then

$$\liminf_{n} \frac{1}{f(n)} \sum_{k=n+1}^{n+f(n)} [\xi_k - E\xi_k] \ge \gamma(\omega) - c\sigma, \quad a.s.\ \omega \in \mathcal{J}. \tag{2.35}$$

$$\limsup_{n} \frac{1}{f(n)} \sum_{k=n+1}^{n+f(n)} [\xi_k - E\xi_k] \le -\gamma(\omega) + c\sigma, \quad a.s.\ \omega \in \mathcal{J}. \tag{2.36}$$

where c is the minimum value satisfying the inequality $0 \le e^x - 1 - x \le cx^2$ for $|x| < 2$.

Proof. Put $t = \pm 1$ and let

$$D_k(t) = E\exp t(\xi_k - E\xi_k) = \int p_k(x_k)\exp[t(x_k - E\xi_k)]dx_k. \tag{2.37}$$

and

$$p_k^{'}(t;x_k) = \frac{p_k(x_k)\exp t(x_k - E\xi_k)}{D_k(t)}. \tag{2.38}$$

and

$$p^{'}(t;x_{n+1},\cdots,x_{n+f(n)}) = \prod_{k=n+1}^{n+f(n)} p_k^{'}(t;x_k). \tag{2.39}$$

Define random variables:

$$\Lambda_n^{'}(t;\omega) = \frac{p^{'}(t;\xi_{n+1},\cdots,\xi_{n+f(n)})}{p(\xi_{n+1},\cdots,\xi_{n+f(n)})}. \tag{2.40}$$

By lemma 1, we have

$$\limsup_n \frac{1}{f(n)}\log\Lambda_n^{'}(t;\omega) \le 0, \quad a.s. \tag{2.41}$$

(2.38)-(2.41) and (1.3) yield

$$\log\Lambda_n^{'}(t;\omega) = t\sum_{k=n+1}^{n+f(n)}[\xi_k - E\xi_k] - \sum_{k=n+1}^{n+f(n)}\log D_k(t) + \log\Lambda_n(\omega). \tag{2.42}$$

Thus

$$\limsup_n \frac{1}{f(n)}\{\sum_{k=n+1}^{n+f(n)} t[\xi_k - E\xi_k] - \sum_{k=n+1}^{n+f(n)}\log D_k(t) + \log\Lambda_n(\omega)\} \le 0, \quad a.s. \tag{2.43}$$

This and (1.4) yields

$$\begin{aligned}
&\limsup_n \frac{t}{f(n)}\sum_{k=n+1}^{n+f(n)}[\xi_k - E\xi_k] \\
&\le \limsup_n \frac{1}{f(n)}\sum_{k=n+1}^{n+f(n)}\log D_k(t) - \gamma(\omega), \quad a.s.\ \omega \in \mathcal{J}
\end{aligned} \tag{2.44}$$

Noticing that $|\xi_k - E\xi_k| \le 2$ and $0 \le e^x - 1 - x \le cx^2$ for $|x| < 2$, we have

$$\begin{aligned}
0 &\le D_k(t) - 1 = E[\exp t(\xi_k - E\xi_k) - 1 - t(\xi_k - E\xi_k)] \\
&\le cE(\xi_k - E\xi_k)^2 = cVar\xi_k.
\end{aligned} \tag{2.45}$$

Hence

$$0 \le \limsup_n (f(n))^{-1} \sum_{k=n+1}^{n+f(n)} [D_k(t) - 1]$$

$$\le c \limsup_n (f(n))^{-1} \sum_{k=n+1}^{n+f(n)} Var\xi_k = c\sigma, \quad a.s.\ \omega \in \mathcal{J} \tag{2.46}$$

From the same inequality $0 \le \log x \le x - 1 (x \ge 1)$ we analogously find that

$$\limsup_n (f(n))^{-1} \sum_{k=n+1}^{n+f(n)} \log D_k(t) \le c\sigma, \quad a.s.\ \omega \in \mathcal{J} \tag{2.47}$$

(2.44) and (2.47) imply that

$$\limsup_n \frac{t}{f(n)} \sum_{k=n+1}^{n+f(n)} [\xi_k - E\xi_k] \le -\gamma(\omega) + c\sigma, \quad a.s.\ \omega \in \mathcal{J}. \tag{2.48}$$

Putting $t = \pm 1$, we have by (2.48)

$$\limsup_n \frac{1}{f(n)} \sum_{k=n+1}^{n+f(n)} [\xi_k - E\xi_k] \le -\gamma(\omega) + c\sigma, \quad a.s.\ \omega \in \mathcal{J}. \tag{2.49}$$

and

$$\liminf_n \frac{1}{f(n)} \sum_{k=n+1}^{n+f(n)} [\xi_k - E\xi_k] \ge \gamma(\omega) - c\sigma, \quad a.s.\ \omega \in \mathcal{J}. \tag{2.50}$$

These complete the proof.

□

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Received: March, 2012