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New Exact Solutions of Broer-Kaup (BK) and Whitham Broer-Kaup (WBK)Systems via the First Integral Method

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Abstract. In this paper, the first integral method is used for constructing traveling wave solutions of some important nonlinear systems namely Broer-Kaup (BK) system and Whitham Broer-Kaup (WBK) system. As results, various types of traveling wave solutions are formally obtained for the these systems. The power of this manageable method is confirmed by applying it to these selected nonlinear systems. This method can also be applied to non integrable equations as well as integrable ones .

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1. Introduction

Over the four decades or so , nonlinear partial differential equations (NPDEs) have been the subject of extensive studies in various branches of nonlinear sciences . A special class of analytical solutions , the so called traveling waves , for NPDEs is of fundamental importance because lots of mathematical-physical models are often described by such wave phenomena.
Therefore , the investigation of traveling wave solutions is becoming more and more attractive in nonlinear sciences nowadays. However , not all equations posed of these

models are solvable. As a result ,many new techniques have been successfully developed by diverse groups of mathematicians and physicists , such as the inverse scattering method[1] , group theoretic methods[2,3] ,geometric methods [4,5] , the homogeneous balance method [6-12],the variable separation method [13,14] , the generalized hyperbolic function [15,16] , the exp function method [17] and etc.. .
Feng in his pioneering work [18] introduced the first integral method for a reliable treatment of NPDEs. This method was further developed by the same author in [18-24]and some other mathematicians[25-29].The interest in the present work is to implement the first integral method to stress its power in handling nonlinear PDEs. ,so that we can apply it for solving various types of these equations.
In Section 2,we describe this method for finding exact travelling wave solutions of nonlinear evolution equations. In Section 3 , we illustrate this method in detail with the Broer-Kaup (BK) system and Whitham Broer-Kaup (WBK) system .In Section 4, we give some conclusions.

2. The first integral method

K. Hosseini et . al. in [30] have summarized the first integral method in the following steps:

Step 1:
Consider the following nonlinear system of partial differential equations with independent variables x and t and dependent variables u and v

$$F_1(u,v, u_t , v_t , u_x , v_x , u_{tt} , v_{tt} , u_{xx} , v_{xx} , ...) = 0 ,$$
$$F_2(u , v , u_t , v_t , u_x , v_x , u_{tt} , v_{tt} , u_{xx} , v_{xx} , ...) = 0 . \qquad (2.1)$$

Applying the transformations $u(x,t) = u(\xi)$ and $v(x,t) = v(\xi)$, where $\xi = x - ct + \varsigma$, where ς is an arbitrary constant , converts (2.1) into a system of ordinary differential equations (ODEs) as follows

$$G_1(u , v , u' , v' , ...) = 0 ,$$
$$G_2(u , v , u' , v' , ...) = 0 , \qquad (2.2)$$

where the prime denotes the derivatives with respect to the same variable ξ.

Step 2:
Using some mathematical operations , the system (2.2) is converted into a second – order ODE as

$$D(u , u' , u'') = 0 . \qquad (2.3)$$

Step 3:
By introducing new variables $X = u(\xi)$ and $Y = u'(\xi)$, (2.3) changes into a system of ODEs as the following system

$$\begin{cases} X' = Y , \\ Y' = H(X,Y) . \end{cases} \tag{2.4}$$

Step 4 :
Now, the Division Theorem which is based on ring theory of commutative algebra , is adopted to obtain one first integrals to (2.4) , which reduces (2.3) to a first – order integrable ordinary differential equation. Finally , an exact solution to (2.1) is established , through solving the resulting first – order integrable differential equation.
Division Theorem . Suppose that $P(w, z)$, $Q(w, z)$ are polynomials in $C(w, z)$ and $P(w,z)$ is irreducible in $C(w,z)$. If $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exists a polynomial $G(w,z)$ in $C(w,z)$ such that

$$Q(w,z)=P(w,z)G(w,z)$$

3. Applications

3.1. Broer –Kaup (BK) system.

Consider the Broer-Kaup (BK) system [31]

$$\begin{aligned} & u_t + u u_x + v_x = 0 , \\ & v_t + u_x + (u v)_x + u_{xxx} = 0. \end{aligned} \tag{3.1.1}$$

Applying the transformations $u(x,t) = u(\xi)$, $v(x,t) = v(\xi)$, and the wave variable $\xi = x - ct + \varsigma$,where ς is an arbitrary constant, converts (3.1.1) into a system of ordinary differential equations as follows

$$-c u' + u u' + v' = 0 , \tag{3.1.2}$$

$$-c v' + u' + (u v)' + u''' = 0 , \tag{3.1.3}$$

where prime denotes the derivative with respect to the same variable ξ .
Integration of (3.1.2) yields

$$v(\xi) = c\, u(\xi) - \frac{1}{2} (u(\xi))^2 + \alpha \tag{3.1.4}$$

, where α is an arbitrary integration constant. Integrating (3.1.3) and substituting $v(\xi)$, we obtain :

$$u'' = \frac{1}{2} u^3 - \frac{3}{2} c u^2 + (c^2 - \alpha - 1) u + c \alpha + \beta , \tag{3.1.5}$$

, where β is an arbitary integration constant. By introducing new variables $X = u(\xi)$ and $Y = u'(\xi)$,(3.1.5) changes into a system of ODEs

$$\begin{cases} X' = Y , \\ Y' = \frac{1}{2} X^3 - \frac{3}{2} c X^2 + (c^2 - \alpha - 1) X + c\alpha + \beta \end{cases} \tag{3.1.6}$$

According to the first integral method , we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (3.1.6) and $P(X,Y)=\sum_{i=0}^{m} a_i(X)Y^i$ is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$P[X(\xi),Y(\xi)]=\sum_{i=0}^{m} a_i(X)\ Y^i \tag{3.1.7}$$

where $a_i(X)$, $(i=0,1,2,...,m)$ are polynomials of X and $a_m(X)\neq 0$.

Eq.(3.1.7) is called the first integral to (3.1.6) , due to the Division Theorem , there exists a polynomial $h(X)+g(X)Y$ in the complex domain $C[X,Y]$ such that

$$\frac{dP}{d\xi}=\frac{\partial P}{\partial X}\frac{\partial X}{\partial \xi}+\frac{\partial P}{\partial Y}\frac{\partial P}{\partial \xi},$$

$$=[h(X)+g(X)Y]\ \sum_{i=0}^{m} a_i(X)\ Y^i \tag{3.1.8}$$

Case I:

Suppose that $m=1$ in (3.1.7) , by equating the coefficients of Y^i $(i=0,1,2)$ on both sides of Eq.(3.1.8) , we have

$$a_1'(X)=g(X)\ \ a_1(X)\ , \tag{3.1.9}$$

$$a_0'(X)=h(X)\ a_1(X)+\ g(X)\quad a_0(X)\ , \tag{3.1.10}$$

$$a_1(X)\left(\ \frac{3}{2}X^3-\frac{9}{2}\ c\ X^2\ +\ 3\,(c^2-\alpha)\ X\ +\ 3\,(\,c\,\alpha+\beta\,)\ \right)=h(X)\ \ a_0(X)\ , \tag{3.1.11}$$

Since $a_i(X)(i=0,1)$ are polynomials , then from (3.1.9) we deduce that $a_1(X)$ is constant and $g(X)=0$ .For simplicity , take $a_1(X)=1$.

Balancing the degrees of $h(X)$ and $a_0(X)$, we conclude that $\deg(h(X))=1$ only.

Suppose that $h(X)=AX+B$, and $A\neq 0$, then we find $a_0(X)$ as :

$$a_0(X)\ =\frac{A}{2}X^2\ +\ B\ X\ +\ D$$

Substituting $a_0(X),a_1(X)$ and $h(X)$ in Eq.(3.1.11) and setting all the coefficients of powers X to be zero , then we obtain a system of nonlinear algebraic equations and by solving it , we obtain :

$$\beta=-\ B\ ,\ \ D=-1-\alpha\ ,\ \ A=1\ ,\ \ c\ =\ -\ B\ , \tag{3.1.12}$$

$$\beta=\ B\ ,\ \ D=1+\alpha\ ,\ \ A=-1\ ,\ \ c=\ \ B\ , \tag{3.1.13}$$

Setting (3.1.12) in (3.1.13) leads to

$$Y+(\frac{1}{2}X^2 + B\ X - 1-\alpha)=0,$$

$$Y+(-\frac{1}{2}X^2 + B\ X + 1+\alpha)=0.$$

Now, by combining these equations with (3.1.6) , a first order ordinary differential equation is derived, which by solving this equation and considering $X=u(\xi)$ and $u(x,t)=u(\xi)$, we obtain :

$$u_1(x,t)=-B-i\sqrt{2+B^2+2\alpha}\ \tan[\frac{1}{2}i\sqrt{2+B^2+2\alpha}\ (x+Bt+\varsigma-2\xi_0)]\ ,$$

$$u_2(x,t)=B+i\sqrt{2+B^2+2\alpha}\ \tan[\frac{1}{2}i\sqrt{2+B^2+2\alpha}\ (x-Bt+\varsigma-2\xi_0)]\ ,$$

where ξ_0 is an arbitrary integration constant.

Also, by considering the solutions of two-first order differential equations and $X=u(\xi)$ and $v(x,t)=v(\xi)$, we obtain :

$$v_1(x,t)=B\left(B+i\sqrt{2+B^2+2\alpha}\ \tan[\frac{1}{2}i\sqrt{2+B^2+2\alpha}\ (x+Bt+\varsigma-2\xi_0)]\right)$$
$$-\frac{1}{2}\left(B+i\sqrt{2+B^2+2\alpha}\ \tan[\frac{1}{2}i\sqrt{2+B^2+2\alpha}\ (x+t+\varsigma-2\xi_0)]\right)^2+\alpha,$$

$$v_2(x,t)=B\left(B+i\sqrt{2+B^2+2\alpha}\ \tan[\frac{1}{2}i\sqrt{2+B^2+2\alpha}\ (x-Bt+\varsigma-2\xi_0)]\right)$$
$$-\frac{1}{2}\left(B+i\sqrt{2+B^2+2\alpha}\ \tan[\frac{1}{2}i\sqrt{2+B^2+2\alpha}\ (x-Bt+\varsigma-2\xi_0)]\right)^2+\alpha,$$

where ξ_0 is an arbitrary integration constant. Thus two solutions (u_1,v_1) *and* (u_2,v_2) are obtained for system (3.1.1).

Case II:

Also , it can be shown that when $m=2$ and $\deg h(x)=1$, the following traveling wave solutions of the Broer-Kaup (BK) system are obtained and can be written as

$$u_3(x,t)=c-i\sqrt{c^2-D}\ \tan[\frac{1}{2}i\sqrt{c^2-D}\ (x-ct+\varsigma-2\xi_0)],$$

$$v_3(x,t)=c\ [c-i\sqrt{c^2-D}\ \tan[\frac{1}{2}i\sqrt{c^2-D}\ (x-ct+\varsigma-2\xi_0)]]$$
$$-\frac{1}{2}[c-i\sqrt{c^2-D}\ \tan[\frac{1}{2}i\sqrt{c^2-D}\ (x-ct+\varsigma-2\xi_0)]]^2+\alpha$$

$$u_4(x,t) = c + i\sqrt{c^2+D}\ \tan\ [\frac{1}{2}\, i\sqrt{c^2+D}\ (x-ct+\varsigma-2\ \xi_0)],$$

$$v_4(x,t) = c\ [c+i\sqrt{c^2+D}\ \tan\ [\frac{1}{2}\, i\sqrt{c^2+D}\ (x-c\,t+\varsigma-2\ \xi_0)]]$$
$$-\frac{1}{2}\ [c-\ i\sqrt{c^2-D}\ \tan\ [\frac{1}{2}\, i\sqrt{c^2-D}\ (x-ct+\varsigma-2\ \xi_0)]]^2+\ \alpha$$

where ξ_0 is an arbitrary integration constant.

3.2.Whitham – Broer – Kaup (WBK) system

Consider the Whitham – Broer- Kaup (WBK) system[32]

$$u_t + u\ u_x + v_x + \mu\ u_{xx} = 0\ ,$$
$$v_t + (u\,v)_x + \alpha\ u_{xxx} - \mu\ v_{xx} = 0. \tag{3.2.1}$$

Applying the transformations $u(x,t)=u(\xi)$, $v(x,t)=v(\xi)$, and the wave variable $\xi = x-ct+\varsigma$, converts (3.2.1) into a system of ordinary differential equations as follows

$$c\,u' + u\ u' + v' + \mu\ u'' = 0\ , \tag{3.2.2}$$

$$c\ v' + (u\,v)' + \alpha\,u''' - \mu\ v'' = 0\ . \tag{3.2.3}$$

Integrating (3.2.2) and (3.2.3) once, we get

$$c\ u + \frac{1}{2}\,u^2 + v + \mu\ u' = C_1\ , \tag{3.2.4}$$

$$c\ v + u\ v + \alpha\ u'' - \mu\ v' = C_2\ , \tag{3.2.5}$$

where C_1 and C_2 are integration constants.

From (3.2.4), we deduce that

$$v(\xi) = C_1 - \frac{1}{2}\,u^2 - \mu\ u' - c\ u\ . \tag{3.2.6}$$

Substituting Eq.(3.2.6) into (3.2.5) yields

$$u'' = [\frac{1}{2\,(\alpha+\mu^2)}]\,u^3 + [\frac{3\,c}{2\,(\alpha+\mu^2)}]\ u^2 - \ [\frac{C_1-c^2}{\alpha+\mu^2}]\ u + \frac{C_2-\ C_1\ c}{\alpha+\mu^2}. \tag{3.2.7}$$

where C_2 is an arbitrary integration constant.

By introducing new variables $X = u(\xi)$ and $Y = u'(\xi)$,(3.2.7) changes into a system of ODEs

$$\begin{cases} X' = Y \\ Y' = [\frac{1}{2(\alpha+\mu^2)}]X^3 + [\frac{3c}{2(\alpha+\mu^2)}]X^2 - [\frac{C_1 - c^2}{\alpha+\mu^2}]X + \frac{C_2 - C_1 c}{\alpha+\mu^2}. \end{cases} \quad (3.2.8)$$

According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (3.2.8) and $P(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i$ is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$P[X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X)Y^i = 0 \quad (3.2.9)$$

,where $a_i(X)$, $(i = 0,1,2,...,m)$ are polynomials of X and $a_m(X) \neq 0$.

Eq.(3.2.9) is called the first integral to (3.2.8) ,due to the Division Theorem , there exists a polynomial $h(X) + g(X)Y$ in the complex domain $C[X,Y]$ such that

$$\frac{dP}{d\xi} = \frac{\partial P}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial P}{\partial Y}\frac{\partial P}{\partial \xi},$$

$$= [h(X) + g(X)Y] \sum_{i=0}^{m} a_i(X) Y^i \quad (3.2.10)$$

Now, suppose that $m = 1$ in (3.2.9) , by equating the coefficients of Y^i $(i = 0,1,2)$ on both sides of Eq.(3.2.10) , we have

$$a_1'(X) = g(X) \ a_1(X), \quad (3.2.11)$$

$$a_0'(X) = h(X) \ a_1(X) + g(X) \ a_0(X), \quad (3.2.12)$$

$$a_1(X) \ [\ [\frac{1}{2(\alpha+\mu^2)}]X^3 + [\frac{3c}{2(\alpha+\mu^2)}]X^2 - [\frac{C_1 - c^2}{\alpha+\mu^2}]X + \frac{C_2 - C_1 c}{\alpha+\mu^2}] = h(X) \ a_0(X) \ .(3.3.13)$$

Since $a_i(X)(i = 0,1)$ are polynomials , then from (3.2.11) we deduce that $a_1(X)$ is constant and $g(X) = 0$ .For simpilcitly , take $a_1(X) = 1$.

Balancing the degrees of $h(X)$ and $a_0(X)$, we conclude that $\deg(h(X)) = 1$ only.

Suppose that $h(X) = AX + B$, and $A \neq 0$, then we find $a_0(X)$:

$$a_0(X) = \frac{A}{2}X^2 + BX + D$$

Substituting $a_0(X), a_1(X)$ and $h(X)$ in Eq.(3.2.13) and setting all the coefficients of powers X to be zero , then we obtain a system of nonlinear algebraic equations and by solving it , we obtain:

$$C_2 = 0 \ , \quad C_1 = -D\sqrt{\alpha+\mu^2} \quad , \quad A = \frac{1}{\sqrt{\alpha+\mu^2}} \quad , \quad B = \frac{c}{\sqrt{\alpha+\mu^2}} \quad , \tag{3.2.14}$$

$$C_2 = 0 \ , \quad C_1 = D\sqrt{\alpha+\mu^2} \quad , \quad A = \frac{-1}{\sqrt{\alpha+\mu^2}} \quad , \quad B = \frac{-c}{\sqrt{\alpha+\mu^2}} \quad , \tag{3.2.15}$$

Similarly , as procedures explained in previous sections , we obtain the traveling wave solutions of the Whitham – Broer-Kaup (WBK) system and can be written as

$$u_1(x,t) = -c - \sqrt{-c^2+2D\sqrt{\alpha+\mu^2}} \ \tan\left[\frac{\sqrt{-c^2+2D\sqrt{\alpha+\mu^2}}\ (x-ct+\varsigma-2\sqrt{\alpha+\mu^2}\ \xi_0)}{2\sqrt{\alpha+\mu^2}}\right],$$

$$v_1(x,t) = -D\sqrt{\alpha+\mu^2} - \frac{1}{2}[-c-\sqrt{-c^2+2D\sqrt{\alpha+\mu^2}} \ \times$$

$$\times \tan[\frac{\sqrt{-c^2+2D\sqrt{\alpha+\mu^2}}\ (x-ct+\varsigma-2\sqrt{\alpha+\mu^2}\ \xi_0)}{2\sqrt{\alpha+\mu^2}}]]^2$$

$$-\mu\left[\frac{-2D+\dfrac{c^2}{\sqrt{\alpha+\mu^2}}}{1+\cos[\dfrac{\sqrt{-c^2+2D\sqrt{\alpha+\mu^2}}\ (x+ct+\varsigma-2\sqrt{\alpha+\mu^2}\xi_0)}{\sqrt{\alpha+\mu^2}}]}\right]$$

$$-c\,[-c-\sqrt{-c^2+2D\sqrt{\alpha+\mu^2}} \ \tan[\frac{\sqrt{-c^2+2D\sqrt{\alpha+\mu^2}}\ (x-ct+\varsigma-2\sqrt{\alpha+\mu^2}\ \xi_0)}{2\sqrt{\alpha+\mu^2}}]],$$

and

$$u_2(x,t) = -c + \sqrt{-c^2-2D\sqrt{\alpha+\mu^2}} \ \tan[\frac{\sqrt{-c^2-2D\sqrt{\alpha+\mu^2}}\ (x-ct+\varsigma-2\sqrt{\alpha+\mu^2}\ \xi_0)}{2\sqrt{\alpha+\mu^2}}],$$

$$v_2(x,t)=D\sqrt{\alpha+\mu^2}-\frac{1}{2}[u_2(x,t)=-c+\sqrt{-c^2-2D\sqrt{\alpha+\mu^2}}\times$$
$$\times\tan[\frac{\sqrt{-c^2-2D\sqrt{\alpha+\mu^2}}\ (x-ct+\varsigma-2\sqrt{\alpha+\mu^2}\,\xi_0)}{2\sqrt{\alpha+\mu^2}}]]^2$$
$$+\mu\ [\frac{(c^2+2D\sqrt{\alpha+\mu^2})\sec[\frac{\sqrt{-c^2-2D\sqrt{\alpha+\mu^2}}\ (x+ct+\varsigma-2\sqrt{\alpha+\mu^2}\,\xi_0)}{2\sqrt{\alpha+\mu^2}}]^2}{-c\ [u_2(x,t)=-c+\sqrt{-c^2-2D\sqrt{\alpha+\mu^2}}\ \times\tan[\frac{\sqrt{-c^2-2D\sqrt{\alpha+\mu^2}}\ (x-ct+\varsigma-2\sqrt{\alpha+\mu^2}\,\xi_0)}{2\sqrt{\alpha+\mu^2}}]]}]$$

where ξ_0 is an arbitrary integration constant.

4. Conclusion

In this paper , the first integral method was applied successfully to obtain solutions of some important nonlinear systems namely Broer-Kaup (BK) system and Whitham Broer-Kaup (WBK) system . Also, we conclude that the proposed method is powerfull, effective and can be extended to solve more other nonlinear equations which may arise in nonlinear sciences and this will be done elsewhere .

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