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New Soliton Solutions of Some Important Nonlinear Systems via He's Variational Principle

S. I. El-Ganaini [1,2,*]

[1] Mathematics Department , Faculty of Science and Humanity Studies at Dawadmi , Shaqra University, Saudi Arabia
[2] Mathematics Department , Faculty of Science, Damanhour University, Egypt.
* e-mail address :ganaini5533@hotmail.com

Abstract. In this paper, we establish new soliton solutions for nonlinear equations. The **He's** semi-inverse variational principle is used to construct new soliton solutions of these equations.We apply **He's** semi-inverse variational principle to establish a variational theory for the dispersive long wave system and Maccari system.The condition for continuation of the new solitary solution is obtained.

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1. Introduction

There have been various approaches to search for soliton solutions for nonlinear partial differential equations(NPDEs) . These methods include the inverse scattering method [1] ,the truncated Painleve expansion [2] , Hirota 's bilinear method [3] , Backlund transformation method [4] , Darboux transformation [5] , Lie symmetries method [6] , algebra method [7] , sine-cosine method [8] , tanh-coth method [9] , Jacobi elliptic function method [10] , Homotopy perturbation method [11,12] , Luapanov's artificial small parameter method , δ- expansion method , Adomian decomposition method , variational iteration method , Homotopy analysis method , Homotopy Pade method [13-16] first integral method [17-25] and so on.

The main aim of this paper is to present new types of solitary wave (soliton)solutions for the dispersive long wave system and Maccari system.To the best of author's knowledge no attempt has been made for the dispersive long wave system and Maccari system in this form of solitary solutions.

In Sections 2 ,we describe He's semi – inverse principle for finding solitary wave solutions of nonlinear evolution equations .
In Sections 3 and 4 , we illustrate this method in detail with the dispersive long wave system and Maccari system , respectively .
In Section 5, we give some conclusions.

2. He's semi – inverse principle

Consider a general nonlinear PDE in the form

$$P(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, u_{xxx}, \ldots) = 0. \tag{2.1}$$

where P is a polynomial in its arguments.

A.Jabbari et.al. in [26] have been written the He's semi – inverse method in the following steps:

Step1. Seek solitary wave solutions of Eq.(2.1) by taking $u(x,t) = u(\xi)$, $\xi = x - ct + \varsigma$, where ς is an arbitrary constant , and transform Eq.(2.1) to the ordinary differential equation (ODE)

$$Q(U, U', U'', U''', \ldots) = 0. \tag{2.2}$$

, where the prime denotes the derivation with respect to ξ.

Step2. If possible , integrate Eq.(2.2) term by term one or more times. This yields constant(s) of integration. For simplicity , the integration constant(s) can be set to zero.

Step 3. According to He's semi – inverse method, we construct the following trial – functional

$$J(u) = \int L \, d\xi \,, \tag{2.3}$$

where L is an unknown function of u and its derivatives.

Step 4. By the Ritz method , we can obtain different forms of solitary wave solutions , such as $u(\xi) = A \, \operatorname{sech}(B\xi)$, $u(\xi) = A \, \operatorname{csch}(B\xi)$, $u(\xi) = A \tanh(B\xi)$, $u(\xi) = A \coth(B\xi)$ and so on.For example in this paper ,we search a solitary wave solution in the form

$$u(\xi) = A \, \operatorname{sech}(B\xi) \,, \tag{2.4}$$

where A and B are constants to be further determined. Substituting Eq.(2.4) into Eq.(2.3) and making J stationary with respect to A and B results in

$$\frac{\partial J}{\partial A}=0 \tag{2.5}$$

$$\frac{\partial J}{\partial B}=0 \tag{2.6}$$

Solving Eqs.(2.5) and (2.6) we obtain A and B . Hence the solitary wave solution (2.4) is well determined.

3. Dispersive long wave system

Consider the following dispersive long wave system [27,28]

$$\begin{aligned} &u_t + u\,u_x + v_x = 0, \\ &v_t + (u\,v)_x + \frac{1}{3}u_{xxx} = 0. \end{aligned} \tag{3.1}$$

By considering the transformations $u(x,t)=u(\xi)$, $v(x,t)=v(\xi)$, and the wave variable $\xi = x-ct+\varsigma$, where c, ς are arbitrary constants , (3.1.1) changes into a system of ordinary differential equations as follows

$$-c\,u' + u\,u' + v' = 0\ , \tag{3.2}$$

$$-c\,v' + (u\,v)' + \frac{1}{3}u''' = 0\ , \tag{3.3}$$

where prime denotes the derivative with respect to the same variable ξ.
Integration of (3.2) yields

$$v(\xi) = c\,u(\xi) - \frac{1}{2}(u(\xi))^2 + \alpha, \tag{3.4}$$

, where α is an arbitrary integration constant. Integrating (3.3) and substituting $v(\xi)$, we obtain

$$u'' = \frac{3}{2}u^3 - \frac{9}{2}c\,u^2 + 3(c^2-\alpha)u + 3(c\,\alpha+\beta) \tag{3.5}$$

, where β is an arbitrary integration constant.
We set $\alpha=0$ and $\beta=0$ for simplicity , Eq.(3.5) reduces

$$u'' - \frac{3}{2}u^3 + \frac{9}{2}c\,u^2 - 3\,c^2\,u = 0 \tag{3.6}$$

By He's semi – inverse principle [32] , we can obtain the following variational formulation

$$J = \int_0^{\infty}[-\frac{1}{2}(u')^2 - \frac{3}{8}u^4 + \frac{3}{2}c\,u^3 - \frac{3}{2}c^2\,u^2]d\xi \tag{3.7}$$

By a Ritz-like method , we search a solitary wave solution in the form

$$u(\xi)=A\operatorname{sech}(B\xi)\ , \tag{3.8}$$

where A and B are unknown constants to be further determined .

Substituting Eq.(3.8) into Eq.(3.7) , we have

$$J=\int_0^\infty[-\frac{A^2B^2}{2}\operatorname{sech}^2(B\xi)\tanh^2(B\xi)-\frac{3A^4}{8}\operatorname{sech}^4(B\xi)+\frac{3c\ A^3}{2}\operatorname{sech}^3(B\xi)-\frac{3c^2A^2}{2}\operatorname{sech}^2(B\xi)]d\xi$$

$$=-\ \frac{A^2[6A^2+4(B^2+9c^2)-9Ac\pi]}{24B} \tag{3.9}$$

Making J stationary with A and B results in

$$\frac{\partial J}{\partial A}=\frac{A[-8(3A^2+B^2+9c^2)+27Ac\pi]}{24B}=0 \tag{3.10}$$

$$\frac{\partial J}{\partial B}=\frac{A^2\ [6A^2-4B^2+36c^2-9Ac\pi]}{24B^2}=0 \tag{3.11}$$

From Eqs.(3.10) and (3.11) , we get

$$A=\frac{1}{8}c[5\pi\ \pm\ i\sqrt{256-\ 25\ \pi^2}\,]\ ,\ B=\frac{1}{8}\sqrt{3}\ c\ \sqrt{64-5\pi^2\ \mp\ i\,\pi\sqrt{256-25\ \pi^2}}\,.$$

The soliton solutions of the dispersive long wave system Eq.(3.1) are, therefore , obtained as follows

$$u(x,t)=\frac{1}{8}c\ [\ 5\,\pi\ \pm\ i\sqrt{256-25\ \pi^2}\,]\ \operatorname{sech}[\ \frac{1}{8}\sqrt{3}c\sqrt{64-5\pi^2\mp\ i\pi\ \sqrt{256-25\ \pi^2}}\ (x-ct+\varsigma)] \tag{3.12}$$

$$v(x,t)\ =\frac{1}{8}c^2\ [\ 5\,\pi\ \pm\ i\sqrt{256-25\ \pi^2}\,]\ \operatorname{sech}[\ \frac{1}{8}\sqrt{3}c\sqrt{64-5\pi^2\mp\ i\pi\ \sqrt{256-25\ \pi^2}}\ (x-ct+\varsigma)]$$
$$-\ \frac{1}{2}\left(\frac{1}{8}c\ [\ 5\,\pi\ \pm\ i\sqrt{256-25\ \pi^2}\,]\ \operatorname{sech}[\ \frac{1}{8}\sqrt{3}c\sqrt{64-5\pi^2\mp\ i\pi\ \sqrt{256-25\ \pi^2}}\ (x-ct+\varsigma)]\right)^2 \tag{3.13}$$

We search another soliton solution in the form[33]

$$u(\xi)=D\ \operatorname{sech}^2(F\ \xi) \tag{3.14}$$

where D and F are unknown constants to be further determined .
Substituting Eq.(3.14) into Eq.(3.7) , we get the following form

$$J\ =-\ \frac{D^2[105c^2-84cD+18D^2+28\ F^2]}{105F} \tag{3.15}$$

Making J stationary with D and F results in

$$\frac{\partial J}{\partial D}=-\frac{2D[105c^2-126cD+36D^2+28F^2]}{105F}=0 \tag{3.16}$$

$$\frac{\partial J}{\partial F}=\frac{D^2[105c^2-84cD+18D^2-28F^2]}{105F^2}=0 \tag{3.17}$$

From Eqs.(3.16) and (3.17) , we get

$D=\frac{1}{18}c[35 \pm i\sqrt{35}]$, $F=\frac{1}{6}\ c\ \sqrt{10 \mp i\sqrt{35}}.$

Therefore ,the soliton solutions are, therefore , obtained for the dispersive long wave system Eq.(3.1) as follows

$$u(x,t)=\frac{1}{18}c[35 \pm i\sqrt{35}]\ \mathrm{sec}h^2[\frac{1}{6}\ c\ \sqrt{10 \mp i\sqrt{35}}\ (x-ct+\varsigma)] \tag{3.18}$$

$$\begin{aligned} v(x,t) &= \frac{1}{18}c^2[35 \pm i\sqrt{35}]\ \mathrm{sec}h^2[\frac{1}{6}\ c\ \sqrt{10 \mp i\sqrt{35}}\ (x-ct+\varsigma)] \\ &\quad - \frac{1}{2}\left(\frac{1}{18}c[35 \pm i\sqrt{35}]\ \mathrm{sec}h^2[\frac{1}{6}\ c\ \sqrt{10 \mp i\sqrt{35}}\ (x-ct+\varsigma)]\right)^2 \end{aligned} \tag{3.19}$$

4. Maccari system

We consider (2+1)-dimensional soliton equation in the following form[29,30]

$$\begin{aligned} &i\,u_t + u_{xx} + u\,v = 0\ , \\ &v_t + v_y + (u\,u^*)_x = 0 \end{aligned} \tag{4.1}$$

where $i=\sqrt{-1}$, $u(x,y,t)$ is complex function and $v(x,y,t)$ is real function.
Eq (3.2.1) is also called Maccari system [31].
We assume that (4.1) has the solution in the form[29,30]

$$u(x,y,t)=e^{i\eta}\ \phi(\xi)\quad ,v(x,y,t)=v(\xi)\ , \tag{4.2}$$

$$\eta=kx+\ell y+\lambda t+\eta_0 \qquad ,\ \xi=K(x+Ly-2kt+\varsigma)\ , \tag{4.3}$$

where $\phi(\xi)$ and $v(\xi)$ are real functions , $k,\ell,\lambda,\eta_0,\varsigma,K$ and L are real constants.
Substituting (4.2),(4.3) into (4.1) , we have

$$\begin{cases} K^2\ \phi''(\xi)\ -(\lambda+k^2)\ \phi(\xi)\ +\ \phi(\xi)\ \ v(\xi)\ =0\ , & (4.4) \\ (L-2k)\ v'(\xi)\ +\ (\phi^2(\xi))'\ =0. & (4.5) \end{cases}$$

In order to simplify ordinary differential Eqs.(4.4), (4.5) , integrating (4.5) , yields

$$v(\xi)=\frac{1}{2k-L}\phi^2(\xi)+c,\quad L\neq 2k \tag{4.6}$$

where c is an integration constant.
Inserting (3.2.6) into (3.2.4) we have,

$$\phi''(\xi) = -\left(\frac{c-\lambda-k^2}{K^2}\right)\phi(\xi) + \left(\frac{1}{K^2(L-2k)}\right)\phi^3(\xi). \tag{4.7}$$

where prime denotes the derivative with respect to the same variable ξ.

We proceed as in previous section , so by By He's semi – inverse principle [32] , we have the variational formulation

$$J = \int_0^\infty [-\frac{1}{2}(\varphi')^2 - \left(\frac{1}{4K^2\ (L-2k)}\right)\varphi^4 + \left(\frac{c-\lambda-k^2}{2K^2}\right)\ \varphi^2]d\xi \tag{4.8}$$

By a Ritz-like method , we search a solitary wave solution in the form

$$\varphi(\xi) = A\sec h(B\xi)\ , \tag{4.9}$$

where A and B are unknown constants to be further determined .
Substituting Eq.(4.9) into Eq.(4.8) , we have

$$J = \frac{A^2\,(A^2-(2k-L)(-3c+3k^2+B^2K^2+3\lambda))}{6BK^2\,(2k-L)} \tag{4.10}$$

Making J stationary with A and B results in

$$\frac{\partial J}{\partial A} = \frac{A\,[2A^2-(2k-L)\,(-3c+3k^2+B^2K^2+3\lambda)]}{3BK^2\,(2k-L)} = 0 \tag{4.11}$$

$$\frac{\partial J}{\partial B} = -\frac{A^2\ [A^2+(2k-L)\,(3c-3k^2+B^2K^2-3\lambda)]}{6B^2K^2\,(2k-L)} = 0 \tag{4.12}$$

From Eqs.(4.11) and (4.12) , we get

$$A = \sqrt{2}\ \sqrt{(2k-L)(-c+k^2+\lambda)}\ \ ,\ B = \frac{\sqrt{-c+k^2+\lambda}}{K}\ .$$

The soliton solutions of the Maccari system Eq.(4.1) are, therefore , obtained as follows

$$u(x,y,t) = \sqrt{2}\ \sqrt{(2k-L)(-c+k^2+\lambda)}\sec h[\sqrt{-c+k^2+\lambda}\ (x+Ly-2kt+\varsigma)]\exp[i(kx+\ell y+\lambda t+\eta_0)] \tag{4.13}$$

$$v(x,y,t) = \frac{1}{2k-L}\left(2(2k-L)(-c+k^2+\lambda)\sec h^2[\sqrt{-c+k^2+\lambda}\ (x+Ly-2kt+\varsigma)]\right)+c \tag{4.14}$$

Similarly, we can search another soliton solution in the form

$$\varphi(\xi) = D\ \sec h^2(F\ \xi) \tag{4.15}$$

where D and F are unknown constants to be further determined . Substituting Eq.(4.15) into Eq.(4.8) , we get the following form

$$J = \frac{D^2\left[-28F^2 - \dfrac{12D^2}{K^2\,(-2k+L)} + \dfrac{35(c-k^2-\lambda)}{K^2}\right]}{105F} \tag{4.16}$$

For making J stationary with D and F

$$\frac{\partial J}{\partial D} = -\frac{2D[24D^2 - 7(2k-L)(-5c+5k^2+4F^2K^2+5\lambda)]}{105F(2k-L)} = 0 \tag{4.17}$$

$$\frac{\partial J}{\partial F} = \frac{D^2\,[-12D^2 + 7(2k-L)(-5c+5k^2-4F^2K^2+5\lambda)]}{105F^2K^2(2k-L)} = 0 \tag{4.18}$$

From Eqs.(4.17) and (4.18) , we get

$$D = \frac{1}{3}\sqrt{\frac{35}{2}}\,\sqrt{(2k-L)\;(-c+k^2+\lambda)}\,,\quad F = \frac{\sqrt{\frac{5}{3}}\;\sqrt{-c+k^2+\lambda}}{2K}$$

Therefore ,the soliton solutions are, therefore , obtained for the Maccari system Eq. (4.1) as follows

$$u(x,y,t) = \frac{1}{3}\sqrt{\frac{35}{2}}\,\sqrt{(2k-L)\;(-c+k^2+\lambda)}\,\operatorname{sec}h^2\left[\frac{\sqrt{\frac{5}{3}}\;\sqrt{-c+k^2+\lambda}}{2}(x+Ly-2kt+\varsigma)\right] \times \exp[i(kx+\ell y+\lambda t+\eta_0)] \tag{4.20}$$

$$v(x,y,t) = \frac{1}{2k-L}\left(\frac{1}{3}\sqrt{\frac{35}{2}}\,\sqrt{(2k-L)\;(-c+k^2+\lambda)}\,\operatorname{sec}h^2\left[\frac{\sqrt{\frac{5}{3}}\;\sqrt{-c+k^2+\lambda}}{2}(x+Ly-2kt+\varsigma)\right]\right)^2 + c \tag{4.21}$$

5. Conclusion

In this paper, we have used a variational principle algorithm for the dispersive long wave system and Maccari system. We have obtained new types of solitary solutions for the considered systems. He's variational principle is a very dominant instrument to find the soliton solutions for various nonlinear equations and this will be done in a forthcoming paper.

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