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# On Strictly Transitive Operator Algebras

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**Abstract.** In this paper we search for the strictly transitive operator algebras and the strictly cyclic vectors on real or complex Banach spaces. Also we get some conclusions by using Open Mapping Theorem for the dual algebras about being $WOT$-dense in $L(X')$. Moreover we investigate the properties for the elements of dual algebra about being strictly cyclic vectors (functionals).

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## 1. Introduction

Throughout this work $X$ is a real or complex Banach space and $X'$ is the dual space of $X$. Let $L(X)$ denote the space of all continuous linear operators on $X$ and $\Lambda$ be a subalgebra of $L(X)$, the algebra of all bounded linear operators on $X$. Similarly, its dual algebra $\Lambda^*$ of $L(X')$ which is existing from the adjoints of the operators of $\Lambda$, is defined by

$\Lambda^* = \{T' \in L(X')|T \in \Lambda\}$ . In addition, $\Lambda^*$ is an operator algebra.

For $x \in X$ the orbit of $x$ under $\Lambda$ is defined by $\Lambda x = \{Ax|A \in \Lambda\}$. Also, we say that $x_0 \in X$ is cyclic for $\Lambda$ if $\Lambda x_0$ is dense in $X$, that is, $\overline{\Lambda x_0} = X$. $x_0$ is called strictly cyclic if $\Lambda x_0 = X$. We can write a similar definition for the elements of dual space $X'$ of $X$. We know that the elements of $X'$ are the linear functionals but we can consider them like vectors at the same time. Hence, we can say that a linear functional $x' \in X'$ is cyclic for $\Lambda^*$ if $\Lambda^* x'$ is dense in $X'$. Also, $x' \in X'$ is called strictly cyclic if $\Lambda^* x' = X'$. We can make the definition of $\Lambda^* x'$ as following ;

$\Lambda^* x' = \{Ax' | A \in \Lambda^*\}$.

Due to the this definition we can say that $\Lambda^* x' \subseteq X'$. Let take an element $A_1 x' \in \Lambda^* x'$, since $A_1 \in \Lambda^*$ it is the adjoint of an operator of $\Lambda$, let say $T_1$. Then, $A_1 x' = x' \circ T_1 \in X'$.

## 2. Strictly Transitive Operator Algebras

Now we will give some basic definitions about the operator algebras and their duals. $B_X$ will denote the closed unit ball of $X$.

**Definition 1.** Let $X$ be a Banach space, $X'$ be the topological dual of $X$ and $T \in L(X)$. Then its adjoint operator $T' \in L(X')$ is defined via the duality formula:

$< x, T'x' >=< Tx, x' >$ for all $x \in X$ and $x' \in X'$.

**Definition 2.** Let $X$ be a Banach space. It is known that $L(X)$ is a vector space with the usual algebraic operations. In addition to this, $L(X)$ is an operator algebra and multiplication is defined by the composition like for $T, S \in L(X)$, $T.S = T \circ S \in L(X)$. A vector subspace $\Lambda$ of $L(X)$ is called an algebra of operators ( or a subalgebra of $L(X)$ ) whenever $T, S \in \Lambda$ imply $T \circ S \in \Lambda$.

We have said that $\Lambda^*$ is an operator algebra above. To get this, all $S', T' \in \Lambda^*$ must imply $S'T' \in \Lambda^*$ , [1]. Since $\Lambda$ is an algebra, for all $T, S \in \Lambda$ we know that the composition(multiplication) of them $TS \in \Lambda$ . On the other hand, we have $S'T' = (TS)'$ and $\Lambda^*$ is existing from the adjoint operators of $\Lambda$, so that $S'T' \in \Lambda^*$.

**Definition 3 .** Let $\Lambda \subseteq L(X)$ be an algebra of operators. $\Lambda$ is transitive if it has no common invariant subspaces. In other words, we can say that the algebra $\Lambda$ is called transitive (strictly transitive) if every non-zero element of $X$ is cyclic (respectively, strictly cyclic).

**Definition 4 .** Let $\Lambda \subseteq L(X)$ be an algebra of operators. $\Lambda$ is said to be $n$-transitive for some $n \geq 1$ if every linearly independent $n$-tuple in $(X')^k$ is cyclic for the algebra $\Lambda^{(n)} = \{A \oplus ... \oplus A | A \in \Lambda\}$.

**Theorem 5 .**(Open Mapping Theorem) If $X$, $Y$ are Banach spaces and $T : X \to Y$ is continuous linear surjection, then $T$ is open, that is, $T(G)$ is open in $Y$ whenever $G$ is open in $Y$, [2].

We characterize the Open Mapping Lemma[4], which is the standard lemma for the Open Mapping Theorem, for dual spaces.

**Theorem 6 .**(Open Mapping Lemma[4]) Let $X$, $Y$ are normed spaces and $T : X \to Y$ is a bounded operator and $T'$ is the adjoint of $T$. ($T' : Y' \to X'$)Assume that there exist $K > 0$ and $0 < \varepsilon < 1$ such that

$$B_{X'} \subseteq KT'(B_{Y'}) + \varepsilon B_{X'}. \tag{1}$$

Then $T'$ is a surjective open map. In particular, (1) is satisfied if $\overline{T'(B_{Y'})}$ has non-empty interior.

**Proof.** Initially, fix $f \in B_{X'}$ and select $y_1 \in KB_{Y'}$ such that $\|f - T'y_1\| \leq \varepsilon$ and say $f_1 = f - T'y_1$ so

$\|f_1\| = \|f - T'y_1\| \leq \varepsilon$ then $f_1 \in \varepsilon B_{X'}$.

Also we have $y_2 \in \varepsilon KB_{Y'}$ such that $\|f_1 - T'y_2\| \leq \varepsilon^2$, say $f_2 = f_1 - T'y_2$ so

$\|f_2\| = \|f_1 - T'y_2\| \leq \varepsilon^2$ then $f_2 \in \varepsilon^2 B_{X'}$.

If we continue analogously, we get $f_n \in \varepsilon^n B_{X'}$ . Therefore $\|f_n\| \leq \varepsilon^n$ and also we get

$y_1 \in KB_{Y'}$, $y_2 \in \varepsilon KB_{Y'}$,....,$y_n \in \varepsilon^{n-1} KB_{Y'}$ so that for all $n$

$$\|y_n\| \leq \varepsilon^{n-1} K. \tag{2}$$

Moreover, from the equalities:

$f_1 = f - T'y_1$ and $f_2 = f_1 - T'y_2$ we get $f_2 = f - T'y_1 - T'y_2$. By continuing, $f_n = f - T'(y_1 + y_2 + ... + y_n)$ so we get

$$f_n = f - T' \textstyle\sum_{j=1}^{n} y_j \text{ and } \|f_n\| \leq \varepsilon^n. \tag{3}$$

We know that $X'$ and $Y'$ are complete since they are the topological dual spaces of $X$ and $Y$, respectively. In this manner, since $Y'$ are complete and from (2), we can say that $\sum_{j=1}^{\infty} y_j$ converges to an element $y \in Y'$. Furthermore, we get $T'y = f$ as $T'$ is continuous. Hence $T'$ is surjective. Also from (2) we had $\|y_n\| \leq \varepsilon^{n-1} K$, if we define

$M = K \sum_{j=0}^{\infty} \varepsilon^j = \frac{K}{1-\varepsilon}$ for $\|y\| \leq M$

also if we consider the equality $f = T'y$ we get $y \in MB_{Y'}$.Then $f \in MT'B_{Y'}$ which leads to $B_{X'} \subseteq MT'(B_{Y'})$. Finally, $T'$ is open.

Now we will give a proposition about some properties of strictly cyclic elements of dual algebras.

**Proposition 7 .**([4]) Suppose that $\Lambda^*$ of $L(X')$ is the norm closed dual algebra for a subalgebra $\Lambda$ of $L(X)$ and $f_0 \in X'$ with $f_0 \neq 0$. Then the following are equivalent.

(i) $f_0$ is strictly cyclic.

(ii) There is a constant $K > 0$ so that for all $y \in B_{X'}$ there exists an operator $A \in \Lambda^*$ with $\|A\| \leq K$ such that $Af_0 = y$.

(iii) There exist $K > 0$ and $0 < \varepsilon < 1$ such that for all $y \in B_{X'}$ there is an operator $A \in \Lambda^*$ with $\|A\| \leq K$ and $\|y - Af_0\| < \varepsilon$.

**Proof.** Firstly, let define $T : \Lambda^* \to X'$ with $T(A) = Af_0 \in X'$. It can be easily verified that $T$ is a bounded linear operator between $\Lambda^*$ and $X'$. We first assumed that $\Lambda^*$ is norm closed so it is a Banach space.

$(i) \Rightarrow (ii)$ Now consider the Open Mapping Theorem which says that a bounded linear surjection between Banach spaces is open. From (i) we know that $f_0$ is strictly cyclic which means that $\Lambda^* f_0 = X'$. Then we have $T : \Lambda^* \to X'$ is a surjection and open. So we can easily say that for all $y \in B_{X'}$ one can find a constant $K > 0$ and an operator $A \in \Lambda^*$ with $\|A\| \leq K$ such that $Af_0 = y$.

$(ii) \Rightarrow (iii)$ Assume that there exists a constant $K > 0$ so that for all $y \in B_{X'}$ there is an operator $A \in \Lambda^*$ with $\|A\| \leq K$ such that $Af_0 = y$. Then we can easily get the presence of $0 < \varepsilon < 1$ such that $\|y - Af_0\| < \varepsilon$.

$(iii) \Rightarrow (i)$ Again consider the map $T : \Lambda^* \to X'$ which is linear and defined by $T(A) = Af_0$ for $A \in \Lambda^*$. We know that $\Lambda^*$ and $X'$ are the Banach spaces and $T$ is bounded. Then $T$ is a surjective open map from the Open Mapping Lemma. Thus $T(\Lambda^*) = X'$ which means

$$T(\Lambda^*) = \Lambda^* f_0 = \{T(A) | A \in \Lambda^*\} = X'.$$

Finally, $f_0$ is strictly cyclic.

**Corollary 8 .**([4]) If $\Lambda^*$ is norm closed, then the set of strictly cyclic vectors for $\Lambda^*$ is open.

**Proof.** Suppose that $f_0$ is a strictly cyclic vector for $\Lambda^*$ and $K$ is defined in Proposition.7(ii). Take $0 < \delta < \frac{1}{K}$ and $f \in X'$ such that $\|f - f_0\| < \delta$. Since $X'$ is the dual space and the elements of it are the linear functionals,

$\|f - f_0\| = sup_{x \in B_X}|(f - f_0)(x)| < \delta.$

Take $y \in B_{X'}$ and $A \in \Lambda^*$ with $\|A\| \leq K$ and $Af_0 = y$ . Then

$\|Af - y\| = sup_{x \in B_X}|(Af - y)(x)| = sup_{x \in B_X}|Af(x) - y(x)| =$

$sup_{x \in B_X}|Af(x) - Af_0(x)| = sup_{x \in B_X}|A(f - f_0)(x)| = \|A(f - f_0)\| \leq \|A\|.\|f - f_0\| \leq K\delta.$

We know that $\delta < \frac{1}{K}$ so $K\delta = 1$. Then

$\|Af - y\| = \|y - Af\| < 1.$

Therefore we can say that $f$ is strictly cyclic due to the Proposition.7. Note that we take $\|f - f_0\| < \delta$ so $f \in B(f_0, f)$ that means, $f$ belongs to the open ball centered at $f_0$. Finally, the set of strictly cyclic vectors for $\Lambda^*$ is open.

Following this result we can give a version of an alternate proof of a result due to [3].

**Corollary 9 .**([3], [4]) Suppose that $X$ is a complex Banach space and $\Lambda^*$ is a transitive operator algebra on $X'$ ( for a subalgebra $\Lambda$ on $X$ ) with a strictly cyclic vector, then $\Lambda^*$ is $WOT$-dense in $L(X')$. ( WOT means weak operator topology )

**Proof.** Firstly, we can assume that $\Lambda^*$ is normed closed by replacing $\Lambda^*$ by $\overline{\Lambda^*}$. Let $f_0$ be a strictly cyclic vector for $\Lambda^*$. One can find $\delta > 0$ such that $\|f - f_0\| < \delta$ which means $f$ is strictly cyclic due to the Corollary.8 .

Take $0 \neq y \in X'$. As $\Lambda^*$ is a transitive algebra, we can find $A \in \Lambda^*$ such that $\|Ay - f_0\| < \delta$. Following, $Ay$ is strictly cyclic. Therefore, $y$ is also strictly cyclic. That means all non-zero $y \in X'$ is strictly cyclic ; hence $\Lambda^*$ is strictly transitive and finally, $WOT$-dense in $L(X')$.

Next result shows that transitive algebras always have operators which are almost zero on prescribed vectors, [4].

**Proposition 10 .**([4]) Let $X$ be a complex Banach space, $X'$ be the dual of $X$ and $\Lambda^*$ be a transitive operator algebra on $X'$ for a subalgebra $\Lambda$ on $X$. Then for all $f \in X'$ and $\varepsilon > 0$ there is an $A \in \Lambda^*$ with $\|A\| = 1$ and $\|Af\| < \varepsilon$.

**Proof.** Without loss of generality, we can assume that $\Lambda^*$ is $WOT$-closed. Note that the conclusions which holds for $\overline{\Lambda^*}^{WOT}$, also holds for $\Lambda^*$.

Now, fix $f \in X'$ and $0 < \varepsilon < 1$ and choose $\delta = \frac{\varepsilon}{4}$. We can take a $A \in \overline{\Lambda^*}^{WOT}$ such that $\|A\| = 1$ and $\|Af\| < \varepsilon$.

Then we can write $\|Ay\| \geq 1 - \varepsilon$ for some $y \in X'$ with $\|y\| = 1$.

As $\overline{\Lambda^*}^{WOT} = \overline{\Lambda^*}^{SOT}$, ( SOT means strong operator topology )

there exists $B \in \Lambda^*$ such that

$\|(A - B)f\| < \delta$ and $\|(A - B)y\| < \delta$.

Following these inequalities we get

$\|Bf\| < 2\delta$ and $\|By\| \geq 1 - 2\delta$, so that $\|B\| \geq 1 - 2\delta$.

Take $C = \frac{B}{\|B\|}$, so $\|Cf\| < \frac{2\delta}{1-2\delta}$.

Therefore we get $C \in \Lambda^*$ with $\|C\| = \frac{\|B\|}{\|B\|} = 1$ and $\|Cf\| < \varepsilon$.

Here we must note that, the conclusion finishes the proof as $\Lambda^* = L(X')$. Otherwise $\Lambda^*$ would be a $WOT$-closed subalgebra of $L(X')$ and we could choose $\|f\| = 1$ and $\varepsilon > 0$ so that $\|Af\| > \varepsilon$ for $A \in \Lambda^*$ with $\|A\| = 1$. But the operator $T : \Lambda^* \to X'$ defined by $T(A) = Af$ for $A \in \Lambda^*$ is an isomorphism.

In particular, $\Lambda^* f$ is normed closed being isomorphic to $\Lambda^*$. Then $\Lambda^* f = X'$ as $\Lambda^*$ is transitive. This contradicts Corollary.9.

Now we will give an important proof since the generalization of this result due to it.

**Theorem 11 .**(Jiaosheng Jiang, [4] ) Let $\Lambda$ be a commutative transitive operator algebra on a complex Banach space. Then for all $x_0, x_1, ..., x_n \in X$ and $\varepsilon > 0$ there exists an operator $A \in \Lambda$ with $\|A\| = 1$ and $\|Ax_i\| < \varepsilon$ for all $0 \leq i \leq n$.

**Proof.** We may assume without loss of generality that $x_0 \neq 0$. By the transitivity of $\Lambda$, for each $1 \leq i \leq n$ we may choose $B_i \in \Lambda$ so that $\|B_i x_0 - x_i\| \leq \frac{\varepsilon}{2}$.

Let $\delta = min\{\varepsilon, \frac{\varepsilon}{2max_i\|B_i\|}\}$. By Proposition.10, we may choose $A \in \Lambda$ with $\|A\| = 1$ and $\|Ax_0\| < \delta$.

Then $\|Ax_0\| < \varepsilon$ and for all $0 \leq i \leq n$,

$\|Ax_i\| \leq \|A(x_i - B_i x_0)\| + \|AB_i x_0\|$

$\leq \|x_i - B_i x_0\| + \|B_i A x_0\| \leq \frac{\varepsilon}{2} + \|B_i\|\delta < \varepsilon$.

It is easy to see that this theorem can be also used for the commutative transitive dual algebra $\Lambda^*$ of a subalgebra $\Lambda$ on a complex Banach space $X$. Analogously we can say that for all $f_0, f_1, ..., f_n \in X'$ and $\varepsilon > 0$ there exists an operator $A \in \Lambda^*$ with $\|A\| = 1$ and $\|Af_i\| < \varepsilon$ for all $0 \leq i \leq n$.

To see this, first take $f_0 \neq 0$. Again, since $\Lambda^*$ is transitive, we may select $T_i \in \Lambda^*$ so that $\|T_i f_0 - f_i\| \leq \frac{\varepsilon}{2}$ for each $i = 1, 2, .., n$. Let $\delta = min\{\varepsilon, \frac{\varepsilon}{2max_i\|T_i\|}\}$.

Then according to Proposition.10, we have $K \in \Lambda^*$ with $\|K\| = 1$ and $\|Kf_0\| < \delta$. Then $\|Kf_0\| < \varepsilon$ and for all $i = 1, 2, .., n$.

$\|Kf_i\| = sup_{x \in B_X}|(Kf_i)(x)| = sup_{x \in B_X}|f_i S(x)|$. ( here $T$ is the adjoint of $S$ )

Due to this definition we get;

$\|Kf_i\| = \|Kf_i - KT_i f_0 + KT_i f_0\| \leq \|K(f_i - T_i f_0)\| + \|KT_i f_0\|$

$\leq \|K\|.\|f_i - T_i f_0\| + \|KT_i f_0\|$.

Now we must consider that $\|K\| = 1$ and $\|KT_i f_0\| = \|T_i K f_0\|$ since $\Lambda^*$ is a commutative algebra. Then,

$\|Kf_i\| \leq \|f_i - T_i f_0\| + \|T_i K f_0\| \leq \|f_i - T_i f_0\| + \|T_i\|.\|Kf_0\| \leq \frac{\varepsilon}{2} + \|T_i\|.\delta < \varepsilon$.

It is not known that if the conclusion of this proposition holds for general transitive algebras. We will give a partial answer to this problem.

**Proposition 12 .**([4]) Let $\Lambda^*$ be an operator algebra of a subalgebra $\Lambda$ of $L(X)$. If $\Lambda^*$ is $n$-transitive for some $n \geq 1$, then for all $f_1, f_2, ..., f_n \in X'$ and $\varepsilon > 0$ there exists an operator $A \in \Lambda^*$ with $\|A\| = 1$ and $\|Af_i\| < \varepsilon$ for all $i$.

**Proof.** Similar to the proof of Proposition.10, we may assume that $\Lambda^*$ is a $WOT$-closed proper subalgebra of $L(X')$. If the conclusion were incorrect, we could find $f_1, f_2, ..., f_n \in X'$ and $\varepsilon > 0$ such that

$max_{1 \leq i \leq n}\|Af_i\| \geq \varepsilon$ for all $A \in \Lambda^*$ with $\|A\| = 1$.

Without loss of generality, we may suppose that $f_1, f_2, ..., f_k$ are linearly independent for some $k \leq n$ and $f_j \in span\{f_1, f_2, ..., f_k\}$ whenever $k < j \leq n$. Then there is a $C > 1$ such that

$$max_{1 \leq i \leq n}\|Af_i\| \leq C.max_{1 \leq i \leq k}\|Af_i\|.$$

whenever $A \in \Lambda^*$ with $\|A\| = 1$, so that $max_{1 \leq i \leq k}\|Af_i\| \geq \frac{\varepsilon}{C}$.

Moreover, the map $T : \Lambda^* \rightarrow (X')^k$ defined by $T(A) = (Af_1, ..., Af_k)$ for $A \in \Lambda^*$ is an isomorphism. Therefore, $T(\Lambda^*)$ is closed. As $\Lambda^*$ is $n$-transitive, it is also $k$-transitive. Since $f_1, f_2, ..., f_k$ are linearly independent, $T(\Lambda^*) = (X')^k$. In particular, this implies that $f_1$ is strictly cyclic for $\Lambda^*$ and as $\Lambda^*$ is transitive, $\Lambda^*$ is $WOT$-dense in $L(X')$ due to the Corollary.9, which is a contradiction.

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