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On Some Hermite-Hadamard Type Inequalities for Functions whose Power of the Absolute Value of Derivatives are (α, m)-Convex

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Abstract. In this paper we give a lemma which help us to formulate some inequalities of Hermite-Hadamard type for functions whose q-powers of the absolute values of second derivative is (α, m)- convex or s-convex. What become this inequalities for other type of convexities as P-convex or quasi-convex functions will be studied also here. Moreover, two inequalities will be presented using Barnes-Gudunova-Levin inequality. Then a similar lemma for two variables will be presented for partial differentiable mapping and an related to the right side of Hermite-Hadamard type inequality for co-ordinated convex functions in two variables are obtained.

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1. Introduction

We recall the well-known Holder′s integral inequality which can be stated as follows, see [18], [9] and then Theorem 2.1, see [9].

Theorem 1. *If* $f(x) \geq 0$, $g(x) \geq 0$ *and* $f(x) \in L^p[a, b]$, $g(x) \in L^q[a, b]$ *and* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\int_a^b f(x)g(x)dx \leq \left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}}. \tag{1}$$

Theorem 2. *If the conditions of Theorem 1 are satisfied and $t > 0$ then*

$$\int_a^b f(x)g(x)dx \le C(p,t)\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}}\left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}}. \tag{2}$$

where $C(p,t) = \frac{1}{p}t^{\frac{1}{p}-1} + (1-\frac{1}{p})t^{\frac{1}{p}}$.

We also need to recall the definition of s-convex functions in the second sense and Theorem 2.3 from [1].

Definition 1. *A function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$, where $\mathbb{R}_+ = [0,\infty)$ is said to be s-convex on I if the inequality $f(\alpha x + \beta y) \le \alpha^s f(x) + \beta^s f(y)$ holds for all $x,\ y \in I$ and $\alpha,\ \beta \in [0,1]$ with $\alpha + \beta = 1$ and for some fixed $s \in (0,1]$.*

Now we recall the notion of quasi-convex functions which also generalizes the notion of convex function and then we present Theorem 2.3 from [2].

Definition 2. *A function $f : [a,b] \to \mathbb{R}$ is said quasi-convex on $[a,b]$ if*

$$f(\lambda x + (1-\lambda)y) \le \sup\{f(x), f(y)\},$$

for all $x,\ y \in [a,b]$ and $\lambda \in [0,1]$.

Definition 3. *The function $f : [0,b] \to \mathbb{R}$ is said to be m-convex where $m \in [0,1]$ if for every $x, y \in [0,b]$ and $t \in [0,1]$ we have:*

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y).$$

As a generalization of previous notion we have the following:

Definition 4. *The function $f : [0,b] \to \mathbb{R},\ b > 0$ is said to be (α, m) -convex, where $(\alpha, m) \in [0,1]^2$ if we have*

$$f(tx + m(1-t)y) \le t^\alpha f(x) + m(1-t^\alpha)f(y)$$

for all $x, y \in [0,b]$ and $t \in [0,1]$.

Definition 5. *Let $I \subseteq \mathbb{R}$ be an interval. The function $f : I \to \mathbb{R}$ is said to belong to the class $P(I)$ (or P- convex) if it is nonnegative and for all $x,\ y \in I$ and $\lambda \in [0,1]$ satisfies the inequality*

$$f(\lambda x + (1-\lambda)y) \le f(x) + f(y).$$

We will use below in the proof of some theorem the following inequality, called Barnes-Gudunova-Levin inequality, see [29] and [24], [25], [26].

Let f, g be nonnegative concave functions on $[a,b]$. Then, for $p, q > 1$ we have

$$\left(\int_a^b f(x)^p dx\right)^{\frac{1}{p}}\left(\int_a^b g(x)^q dx\right)^{\frac{1}{q}} \le B(p,q)\int_a^b f(x)g(x)dx,$$

where

$$B(p,q) = \frac{6(b-a)^{(\frac{1}{p})+(\frac{1}{q})-1}}{(p+1)^{\frac{1}{p}}(q+1)^{\frac{1}{q}}}.$$

In the special case $q = p$ we have

$$\left(\int_a^b f(x)^p dx\right)^{\frac{1}{p}} \left(\int_a^b g(x)^p dx\right)^{\frac{1}{p}} \le B(p,p) \int_a^b f(x)g(x)dx,$$

with

$$B(p,p) = \frac{6(b-a)^{(\frac{2}{p})-1}}{(p+1)^{\frac{2}{p}}}.$$

We also need the following two results from [29] and [19]:

Remark 1. *(Remark 1.1 [29]) Observe that whenever, f^p is concave on $[a,b]$ the nonnegative function f is also concave on $[a,b]$, where $p > 1$.*

For $q > 1$ similarly if g^q is concave on $[a,b]$, the nonnegative function g is concave on $[a,b]$.

Lemma 1. *(Lemma 1 [19]) Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I°, $a,\ b \in I$ with $a < b$ and $f'' \in L[a,b]$. Then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{(b-a)^2}{2}\int_0^1 t(1-t)f''(ta+(1-t)b)dt.$$

Lemma 2. *(Lemma 2 [21]) Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I° such that $f'' \in L_1[a,b]$, where $a,\ b \in I$ with $a < b$ and $r \in \mathbb{R}^+$. Then the following equality holds:*

$$\frac{1}{r(r+1)}[f(a)+f(b)] + \frac{2}{r+1}f(\frac{a+b}{2}) - \frac{2}{r(b-a)}\int_a^b f(x)dx =$$

$$= (b-a)^2 \int_0^1 k(t) f''(tb+(1-t)a)dt$$

where

$$k(t) = \begin{cases} \frac{t}{r}(\frac{1}{r+1} - t), & t \in [0, \frac{1}{2}) \\ (1-t)(\frac{t}{r} - \frac{1}{r+1}), & t \in [\frac{1}{2}, 1]. \end{cases} \tag{1.1}$$

Lemma 3. *(Lemma 2.1 [5]) Suppose that $f : I \subset \mathbb{R} \to \mathbb{R}$ is a twice differentiable function on I°, the interior of I. Assume that $a,\ b \in I^\circ$ with $a < b$ and f'' is integrable on $[a,b]$. Then the following equality holds,*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx =$$

$$= \frac{(b-a)^2}{16}\int_0^1 (1-t^2)(f''(\frac{1+t}{2}a+\frac{1-t}{2}b))+(f''(\frac{1-t}{2}a+\frac{1+t}{2}b))dt$$

Let $\Delta = [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$ be a bidimensional interval. We recall, see [13] that a mapping $f:\Delta\to\mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x+(1-\lambda)z, \lambda y+(1-\lambda)w) \le \lambda f(x,y)+(1-\lambda)f(z,w),$$

holds for all $(x,y),\ (z,w)\in\Delta$ and $\lambda\in[0,1]$. Now we recal below also the definition of co-ordinated convex functions, see [13].

Definition 6. *A function $f:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ if the inequality*

$$f(tx+(1-t)y, su+(1-s)w) \le tsf(x,u)+t(1-s)f(x,w)+s(1-t)f(y,u)+(1-t)(1-s)f(y,w),$$

holds for all $t,s\in[0,1]$ and $(x,u),(y,w)\in\Delta$.

Also we need to recall the following the definition of quasi-convex functions on the co-ordinates, see [14] and we will use in our proof the second, which is the formal definition.

Definition 7. *A function $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is said to be quasi-convex on Δ if the inequality*

$$f(\lambda x+(1-\lambda)z, \lambda y+(1-\lambda)w) \le \max\{f(x,y), f(z,w)\},$$

holds for all $\lambda\in[0,1]$ and $(x,y),(z,w)\in\Delta$.

Definition 8. *A function $f:\Delta\to\mathbb{R}$ is said to be quasi-convex on the co-ordinates on Δ if*

$$f(tx+(1-t)z, sy+(1-s)w) \le \max\{f(x,y), f(x,w), f(z,y), f(z,w)\}$$

for all $(x,y),(z,w)\in\Delta$ $t,s\in[0,1]$.

A formal definition for co-ordinates s- convex functions in the second sense, which can be found in [15], is given below:

Definition 9. *A mapping $f:\Delta\to\mathbb{R}$ is said to be s-convex in the second sense on the co-ordinates on Δ if the inequality*

$$f(tx+(1-t)y, ru+(1-r)w) \le$$

$$\le t^s r^s f(x,u)+t^s(1-r)^s f(x,w)+r^s(1-t)^s f(y,u)+(1-t)^s(1-r)^s f(y,w),$$

holds for all $t,r\in[0,1]$, $(x,u),(y,w)\in\Delta$ and for some fixed $s\in(0,1]$.

2. Hermite-Hadamard's type inequalities for (α, m)- convex functions and s-convex functions in the second sense

In the following we will give a lemma similar to Lemma 1 from [17] but for $f^{''}$ instead of $f^{'}$.

Lemma 4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°, the interior if I where $a,\ b \in I$ with $a < b$. If $f^{''} \in L[a,b]$ then the following equality holds:*

$$-\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}+\frac{1}{b-a}\int_a^b f(x)dx=$$

$$=\frac{(b-a)^2}{128}[\int_0^1 t^2 f^{''}(t\frac{3a+b}{4}+(1-t)a)dt+\int_0^1 (t-1)^2 f^{''}(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4})dt+$$

$$+\int_0^1 t^2 f^{''}(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2})dt+\int_0^1 (t-1)^2 f^{''}(tb+(1-t)\frac{a+3b}{4})dt].$$

Proof. We shall denote

$$I_1=\int_0^1 t^2 f^{''}(t\frac{3a+b}{4}+(1-t)a)dt,$$

$$I_2=\int_0^1 (t-1)^2 f^{''}(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4})dt,$$

$$I_3=\int_0^1 t^2 f^{''}(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2})dt$$

and

$$I_4=\int_0^1 (t-1)^2 f^{''}(tb+(1-t)\frac{a+3b}{4})dt.$$

Then by calculuses we note that

$$I_1=\frac{4}{b-a}f^{'}(\frac{3a+b}{4})-\frac{32}{(b-a)^2}f(\frac{3a+b}{4})+\frac{32}{(b-a)^2}\int_0^1 f(t\frac{3a+b}{4}+(1-t)a)dt,$$

$$I_2=-\frac{4}{b-a}f^{'}(\frac{3a+b}{4})-\frac{32}{(b-a)^2}f(\frac{3a+b}{4})+\frac{32}{(b-a)^2}\int_0^1 f(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4})dt,$$

$$I_3=\frac{4}{b-a}f^{'}(\frac{a+3b}{4})-\frac{32}{(b-a)^2}f(\frac{a+3b}{4})+\frac{32}{(b-a)^2}\int_0^1 f(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2})dt,$$

$$I_4=-\frac{4}{b-a}f^{'}(\frac{a+3b}{4})-\frac{32}{(b-a)^2}f(\frac{a+3b}{4})+\frac{32}{(b-a)^2}\int_0^1 f(tb+(1-t)\frac{a+3b}{4})dt,$$

and using the substitutions $x=t\frac{3a+b}{4}+(1-t)a$, $x=t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}$, $x=t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}$ and $x=tb+(1-t)\frac{a+3b}{4}$ respectively and summing we will obtain:

$$-\frac{64}{(b-a)^2}f(\frac{3a+b}{4})-\frac{64}{(b-a)^2}f(\frac{a+3b}{4})+\frac{128}{(b-a)^3}\int_a^b f(x)dx=I_1+I_2+I_2+I_4.$$

□

We will use this lemma for obtaining a new Hermite-Hadamard type inequality similar to Theorem 2 from [17].

Theorem 3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°, the interior if I where $a,\ b \in I$ with $a < b$ and $f'' \in L_1[a,b]$. If $|f''|^q$ is (α, m)-convex on $[a,b]$ for some fixed $q > 1$ then the following equality holds:*

$$|\frac{f(\frac{3a+b}{4}) + f(\frac{a+3b}{4})}{2} - \frac{1}{b-a}\int_a^b f(x)dx| \leq \frac{(b-a)^2}{128}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}} \cdot$$

$$\cdot[C(p,l_1)(|f''(\frac{3a+b}{4})|^q + \alpha m|f''(\frac{a}{m})|^q)^{\frac{1}{q}} + C(p,l_2)(m|f''(\frac{a+b}{2m})|^q + \alpha|f''(\frac{3a+b}{4})|^q)^{\frac{1}{q}}$$

$$+C(p,l_3)(|f''(\frac{a+3b}{4})|^q + m\alpha|f''(\frac{a+b}{2m})|^q)^{\frac{1}{q}} + C(p,l_4)(m|f''(\frac{b}{m})|^q + \alpha|f''(\frac{a+3b}{4})|^q)^{\frac{1}{q}}],$$

where $C(p,l)$ is as in Theorem 2.

Proof. We will apply Lemma 4, Theorem 2 and the definition of (α, m)- convex functions for $|f''|^q$ obtaining the following:

$$|\frac{f(\frac{3a+b}{4}) + f(\frac{a+3b}{4})}{2} - \frac{1}{b-a}\int_a^b f(x)dx| \leq$$

$$\leq \frac{(b-a)^2}{128}[C(p,l_1)\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\int_0^1 (t^\alpha|f''(\frac{3a+b}{4})|^q + m(1-t^\alpha)|f''(\frac{a}{m})|^q)dt\right)^{\frac{1}{q}} +$$

$$+C(p,l_2)\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\int_0^1 (mt^\alpha|f''(\frac{a+b}{2m})|^q + (1-t^\alpha)|f''(\frac{3a+b}{4})|^q)dt\right)^{\frac{1}{q}} +$$

$$+C(p,l_3)\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\int_0^1 (t^\alpha|f''(\frac{a+3b}{4})|^q + m(1-t^\alpha)|f''(\frac{a+b}{2m})|^q)dt\right)^{\frac{1}{q}} +$$

$$+C(p,l_4)\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\int_0^1 (mt^\alpha|f''(\frac{b}{m})|^q + (1-t^\alpha)|f''(\frac{a+3b}{4})|^q)dt\right)^{\frac{1}{q}}].$$

Then by calculuses we obtain the inequality from theorem.

□

Another variant of previous theorem is the following:

Theorem 4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°, the interior if I where $a,\ b \in I$ with $a < b$ and $f'' \in L_1[a,b]$. If $|f''|^q$ is (α, m)-convex on $[a,b]$ for some fixed $q > 1$ then the following equality holds:*

$$|\frac{f(\frac{3a+b}{4}) + f(\frac{a+3b}{4})}{2} - \frac{1}{b-a}\int_a^b f(x)dx| \leq \frac{(b-a)^2}{128}\left(\frac{1}{\alpha+q+1}\right)^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \cdot$$

$$\cdot[C(p,l_1)(|f''(\frac{3a+b}{4})|^q + \frac{\alpha m}{q+1}|f''(\frac{a}{m})|^q)^{\frac{1}{q}} +$$

$$+C(p,l_2)(m\frac{\alpha q}{\alpha+q}\beta(\alpha,q)|f''(\frac{a+b}{2m})|^q+(\frac{\alpha+q+1}{q+1}-\frac{\alpha q}{\alpha+q}\beta(\alpha,q))|f''(\frac{3a+b}{4})|^q)^{\frac{1}{q}}$$

$$+C(p,l_3)(|f''(\frac{a+3b}{4})|^q+\frac{m\alpha}{q+1}|f''(\frac{a+b}{2m})|^q)^{\frac{1}{q}}+$$

$$+C(p,l_4)(m\frac{\alpha q}{\alpha+q}\beta(\alpha,q)|f''(\frac{b}{m})|^q+(\frac{\alpha+q+1}{q+1}-\frac{\alpha q}{\alpha+q}\beta(\alpha,q))|f''(\frac{a+3b}{4})|^q)^{\frac{1}{q}}],$$

where $C(p,l)$ is as in Theorem 2 and $\beta(\alpha,q)=\int_0^1 x^{\alpha-1}(1-x)^{q-1}$ is the Euler's function.

Proof. By using Lemma 4 we will have:

$$|\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}-\frac{1}{b-a}\int_a^b f(x)dx|\leq\frac{(b-a)^2}{128}[\int_0^1 t^2|f''(t\frac{3a+b}{4}+(1-t)a)|dt+$$

$$+\int_0^1(t-1)^2|f''(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4})|dt+\int_0^1 t^2|f''(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2})|dt+$$

$$+\int_0^1(t-1)^2|f''(tb+(1-t)\frac{a+3b}{4})|dt].$$

Now using Theorem 2 we obtain:

$$|\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}-\frac{1}{b-a}\int_a^b f(x)dx|\leq\frac{(b-a)^2}{128}\cdot$$

$$\cdot[C(p,l_1)(\int_0^1 t^p dt)^{\frac{1}{p}}(\int_0^1 t^q|f''(t\frac{3a+b}{4}+(1-t)a)|^q dt)^{\frac{1}{q}}+$$

$$+C(p,l_2)(\int_0^1(1-t)^p dt)^{\frac{1}{p}}(\int_0^1(1-t)^q|f''(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4})|^q dt)^{\frac{1}{q}}+$$

$$+C(p,l_3)(\int_0^1 t^p dt)^{\frac{1}{p}}(\int_0^1 t^q|f''(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2})|^q dt)^{\frac{1}{q}}+$$

$$+C(p,l_4)(\int_0^1(1-t)^p dt)^{\frac{1}{p}}(\int_0^1(1-t)^q|f''(tb+(1-t)\frac{a+3b}{4})|^q dt)^{\frac{1}{q}}]$$

and using the definition of (α,m)- convexity for $|f''|^q$ we will have

$$|\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}-\frac{1}{b-a}\int_a^b f(x)dx|\leq\frac{(b-a)^2}{128}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\cdot$$

$$\cdot[C(p,l_1)(\int_0^1 t^q(t^\alpha|f''(\frac{3a+b}{4})|^q+m(1-t^\alpha)|f''^{(\frac{a}{m})}|^q)dt)^{\frac{1}{q}}+$$

$$+C(p,l_2)(\int_0^1(1-t)^q(mt^\alpha|f''(\frac{a+b}{2m})|^q+(1-t^\alpha)|f''(\frac{3a+b}{4})|^q)dt)^{\frac{1}{q}}+$$

$$+C(p,l_3)(\int_0^1 t^q(t^\alpha|f''(\frac{a+3b}{4})|^q+m(1-t^\alpha)|f''^{(\frac{a+b}{2m})}|^q)dt)^{\frac{1}{q}}+$$

$$+C(p,l_4)(\int_0^1(1-t)^q(mt^\alpha|f''(\frac{b}{m})|^q+(1-t^\alpha)|f''(\frac{a+3b}{4})|^q)dt)^{\frac{1}{q}}].$$

By calculuses we find the inequality from the theorem.

□

For functions with the power of the second derivative in absolute value s-convex we will obtain the following result:

Theorem 5. *Let $f : I \subseteq [0, \infty) \to \mathbb{R}$ be a twice differentiable function on I°, the interior if I where $a,\ b \in I$ with $a < b$ and $f'' \in L_1[a, b]$. If $|f''|^{p/(p-1)}$, $(p > 1)$ is s- convex on $[a, b]$ for some fixed $s \in (0, 1]$ then the following equality holds:*

$$|\frac{f(\frac{3a+b}{4}) + f(\frac{a+3b}{4})}{2} - \frac{1}{b-a}\int_a^b f(x)dx| \leq \frac{(b-a)^2}{128}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+q+1}\right)^{\frac{1}{q}} \cdot$$
$$\cdot[C(p, l_1)(|f''(\frac{3a+b}{4})|^q + \frac{sq}{s+q}\beta(s, q)|f''(\frac{a+b}{2})|^q)^{\frac{1}{q}} +$$
$$+C(p, l_2)(\frac{sq}{s+q}\beta(s, q)|f''(\frac{a+b}{2})|^q + |f''(\frac{3a+b}{4})|^q)^{\frac{1}{q}} +$$
$$+C(p, l_3)(|f''(\frac{a+3b}{4})|^q + \frac{sq}{s+q}\beta(s, q)|f''(\frac{a+b}{2})|^q)^{\frac{1}{q}} +$$
$$+C(p, l_4)(\frac{sq}{s+q}\beta(s, q)|f''(b)|^q + |f''(\frac{a+3b}{4})|^q)^{\frac{1}{q}}],$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $C(p, l)$ is as in Theorem 2 and $\beta(\alpha, q) = \int_0^1 x^{\alpha-1}(1-x)^{q-1}$ is the Euler's function.

Proof. We use Lemma 4, Theorem 2 and the definition of s-convexity in the second sense and the proof will be as in the previous theorem. □

3. Hermite-Hadamard's type inequalities for P- convex functions and quasi-convex functions

If we consider now P- convexity instead of (α, m)- convexity then we will obtain the following result:

Theorem 6. *Let $f : I \to \mathbb{R}$ be a twice differentiable function on I°. Assume that $p \in \mathbb{R}$, $p > 1$ such that $|f''|^{\frac{p}{p-1}|}$ is a P - convex function on I. Suppose that $a,\ b \in I^\circ$ with $a < b$ and $f'' \in L_1[a, b]$. Then we have:*

$$|\frac{f(\frac{3a+b}{4}) + f(\frac{a+3b}{4})}{2} - \frac{1}{b-a}\int_a^b f(x)dx| \leq \frac{(b-a)^2}{128}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}} \cdot$$
$$\cdot[C(p, l_1)(|f''(\frac{3a+b}{4})|^q + |f''(a)|^q)^{\frac{1}{q}} + C(p, l_2)(|f''(\frac{a+b}{2})|^q + |f''(\frac{a+3b}{4})|^q)^{\frac{1}{q}} +$$
$$+C(p, l_3)(|f''(\frac{a+3b}{4})|^q + |f''(\frac{a+b}{2})|^q)^{\frac{1}{q}} + C(p, l_4)(|f''(b)|^q + |f''(\frac{a+3b}{4})|^q)^{\frac{1}{q}}],$$

where $\frac{1}{p} + \frac{1}{q} = 1$, and $C(p, l)$, $l > 0$ is as in Theorem 2.

Proof. We apply Lemma 4 and Theorem 2. □

Theorem 7. *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I°. Assume that $p \in \mathbb{R},\ p > 1$ such that $|f''|^{\frac{p}{p-1}|}$ is a quasi-convex function on I. Suppose that $a,\ b \in I^\circ$ with $a < b$ and $f'' \in L^1[a,b]$. Then we have:*

$$|\frac{f(\frac{3a+b}{4}) + f(\frac{a+3b}{4})}{2} - \frac{1}{b-a}\int_a^b f(x)dx| \le \frac{(b-a)^2}{128}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\cdot$$

$$\cdot[C(p,l_1)\sup\{|f''(\frac{3a+b}{4})|,|f''(a)|\} + C(p,l_2)\sup\{|f''(\frac{a+b}{2})|,|f''(\frac{a+3b}{4})|\}+$$

$$+C(p,l_3)\sup\{|f''(\frac{a+3b}{4})|,|f''(\frac{a+b}{2})|\} + C(p,l_4)\sup\{|f''(b)|,|f''(\frac{a+3b}{4})|\}],$$

where $\frac{1}{p}+\frac{1}{q} = 1$, and $C(p,l),\ l > 0$ is as in Theorem 2.

4. Some Hermite-Hadamard's type inequalities for (α, m)-convex functions and for concave functions

We give an analog of Lemma 1 from [30] for the second derivative of f. Then using this lemma and Theorem 1 we give some variant of right hand left Hermite-Hadamard inequality for functions whose powers of second derivative in absolute value are (α,m)- convex.

Lemma 5. *Let $f : I^\circ \to \mathbb{R},\ I^\circ \subset [0,\infty)$ be a twice differentiable function on I° where $a,\ b \in I^\circ,\ a < b$. If $f'' \in L[a,b]$. Then we have:*

$$\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2}) = \frac{(a-b)^2}{2}[\int_0^{\frac{1}{2}} t^2 f''(ta+(1-t)b)dt+$$

$$+\int_{\frac{1}{2}}^1 (t-1)^2 f''(ta+(1-t)b)dt].$$

Proof. As in the proof of Lemma 1, [30] we note that

$$\int_0^{\frac{1}{2}} t^2 f''(ta+(1-t)b)dt = \frac{f'(\frac{a+b}{2})}{4(a-b)} - \frac{1}{(a-b)^2}f(\frac{a+b}{2}) + \frac{2}{(a-b)^2}\int_0^{\frac{1}{2}} f(ta+(1-t)b)dt$$

and

$$\int_{\frac{1}{2}}^1 (t-1)^2 f''(ta+(1-t)b)dt = -\frac{f'(\frac{a+b}{2})}{4(a-b)} - \frac{1}{(a-b)^2}f(\frac{a+b}{2}) + \frac{2}{(a-b)^2}\int_{\frac{1}{2}}^1 f(ta+(1-t)b)dt$$

using the integration by parts.

Using the last two equalities and the substitution $x = ta+(1-t)b$ we obtain:

$$\frac{2}{(a-b)^3}\int_a^b f(x)dx - \frac{2}{(a-b)^2}f(\frac{a+b}{2}) = \int_0^{\frac{1}{2}} t^2 f''(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (t-1)^2 f''(ta+(1-t)b)dt$$

i.e. the desired inequality.

□

Theorem 8. *Let $f : I \subset [0, b^*] \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a, b]$ where $a, b \in I^\circ$ with $a < b$, $b^* > 0$. If $|f''|$ is (α, m) - convex on $[a, b]$ for $(\alpha, m) \in (0, 1) \times (0, 1)$ then the following inequality holds:*

$$|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})| \leq \frac{(a-b)^2}{2}[m|f''(\frac{a}{m})|(\frac{1}{\alpha+3} - \frac{2}{\alpha+2} + \frac{1}{\alpha+1} - $$

$$-\frac{1}{(\alpha+1)(\alpha+2)2^{\alpha+1}}) + |f''(b)|(\frac{1}{12} - \frac{1}{\alpha+3} + \frac{2}{\alpha+2} - \frac{1}{\alpha+1} + \frac{1}{(\alpha+1)(\alpha+2)2^{\alpha+1}})].$$

Proof. Using now Lemma 5 and then Definition 4 we will obtain:

$$|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})| \leq \frac{(a-b)^2}{2}[\int_0^{\frac{1}{2}} t^2|f''(ta+(1-t)b)|dt +$$

$$+\int_{\frac{1}{2}}^1 (t-1)^2|f''(ta+(1-t)b)|dt] \leq \frac{(a-b)^2}{2}[\int_0^{\frac{1}{2}} t^2(mt^\alpha|f''(\frac{a}{m})| + (1-t^\alpha)|f''(b)|)dt +$$

$$+\int_{\frac{1}{2}}^1 (t-1)^2(mt^\alpha|f''(\frac{a}{m})| + (1-t^\alpha)|f''(b)|)dt].$$

By calculus we obtain

$$|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})| \leq \frac{(a-b)^2}{2}[m|f''(\frac{a}{m})|(\frac{1}{\alpha+3} - \frac{2}{\alpha+2} + \frac{1}{\alpha+1} - $$

$$-\frac{1}{(\alpha+1)(\alpha+2)2^{\alpha+1}}) + |f''(b)|(\frac{1}{12} - \frac{1}{\alpha+3} + \frac{2}{\alpha+2} - \frac{1}{\alpha+1} + \frac{1}{(\alpha+1)(\alpha+2)2^{\alpha+1}})].$$

□

Next we will find a lower bound for the expression from the Hermite-Hadamard inequality for functions which satisfy the conditions from the Barnes-Gudunova-Levin inequality.

Theorem 9. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I°, a, $b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$. If f'' is nonnegative on $[a, b]$ and $(f'')^q$ is concave on $[a, b]$ for some fixed $m \in (0, 1]$, $\alpha \in (0, 1)$ and $q > 1$ then the following inequality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^2}{12}(q+1)^{\frac{1}{q}}\frac{1}{2^{\frac{1}{q}}}\frac{(\Gamma(p+1))^{\frac{1}{p}}}{((2p+1)...(p+2))^{\frac{1}{p}}}((f''(a))^q + (f''(b))^q)^{\frac{1}{q}}.$$

where $\Gamma(t) = \int_0^\infty e^{-x}x^{t-1}dx$, $t > 0$ is the function Γ of Euler, $\frac{1}{p} + \frac{1}{q} = 1$, and $\beta(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$ is the function β of Euler.

Proof. Using Lemma 1, see [19] and the Barnes-Gudunova-Levin inequality for f'' and the function $t(1-t),\ t \in [0,1]$ we obtain

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{(b-a)^2}{2}\int_0^1 t(1-t)f''(ta+(1-t)b)dt \geq$$

$$\geq \frac{1}{B(p,q)}\left(\int_0^1 (t(1-t))^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (f''(ta+(1-t)b))^q dt\right)^{\frac{1}{q}} \geq$$

$$\geq \frac{1}{B(p,q)}\left(\beta(p+1,p+1)\right)^{\frac{1}{p}}\left(\int_0^1 (t(f''(a))^q+(1-t)(f''(b))^q)dt\right)^{\frac{1}{q}} =$$

$$= \frac{1}{B(p,q)}\left(\beta(p+1,p+1)\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left((f''(a))^q+(f''(b))^q\right)^{\frac{1}{q}} =$$

$$= \left(\frac{1}{2}\right)^{\frac{1}{q}}\frac{1}{B(p,q)}\left(\frac{(\Gamma(p+1))^2}{\Gamma(2p+2)}\right)^{\frac{1}{p}}\left((f''(a))^q+(f''(b))^q\right)^{\frac{1}{q}}.$$

□

Remark 2. *(i) Under the above conditions, using the power-mean inequality, the inequality can be also written as:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^2}{12}(q+1)^{\frac{1}{q}}\frac{1}{2^{\frac{1}{q}}}\frac{(\Gamma(p+1))^{\frac{1}{p}}}{((2p+1)...(p+2))^{\frac{1}{p}}}(f''(a)+f''(b)).$$

(ii) If we consider in previous theorem $p=q>1$ not that $\frac{1}{p}+\frac{1}{q}=1$ then by Barnes-Gudunova-Levin inequality we have:

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^{3-(\frac{2}{p})}}{12}\frac{1}{2^{\frac{1}{q}}}\frac{(\Gamma(p+1))^{\frac{1}{p}}(p+1)^{\frac{1}{p}}}{(2p+1)^{\frac{1}{p}}...(p+2)^{\frac{1}{p}}}\left((f''(a))^p+(f''(b))^p\right)^{\frac{1}{p}}.$$

In the following theorem we will find a lower bound for the left member of the equality from Lemma 2, see [21] when $r \in (0,1)$ and f has some properties.

Theorem 10. *Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I° such that $f'' \in L_1[a,b]$, where $a,\ b \in I$ with $a<b$ and $r \in (0,1)$. If f'' is nonnegative on $[a,b]$ and $(f'')^q$ -is concave on $[a,b]$ then the following inequality holds:*

$$\frac{1}{r(r+1)}[f(a)+f(b)] + \frac{2}{r+1}f(\frac{a+b}{2}) - \frac{2}{r(b-a)}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^2}{2r(r+1)}[\frac{1}{B(p_1,q)}(\frac{\Gamma(p_1+1)}{2(2p_1+1)...(p_1+1)})^{\frac{1}{p_1}}(\frac{1}{8}(f''(b))^q+\frac{3}{8}(f''(a))^q)^{\frac{1}{q}}+$$

$$+\frac{1}{B(p_2,q_2)}(\frac{\Gamma(p_2+1)}{2(2p_2+1)...(p_2+1)})^{\frac{1}{p_2}}(\frac{3}{8}(f''(b))^q+\frac{1}{8}(f''(a))^q)^{\frac{1}{q}}].$$

where $p_1,\ q,\ p_2>1$ *and* $B(p,q)$ *is given in Barnes-Gudunova-Levin inequality.*

Proof. From Lemma 2 we can notice that

$$\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f(\frac{a+b}{2})-\frac{2}{r(b-a)}\int_a^b f(x)dx=$$

$$=(b-a)^2[\int_0^{\frac{1}{2}}\frac{t}{r}(\frac{1}{r+1}-t)dt+\int_{\frac{1}{2}}^1(1-t)(\frac{t}{r}-\frac{1}{r+1})dt].$$

Taking into account that for $r\in(0,1)$, $k(t)>0$ and that $\frac{t}{r}(\frac{1}{r+1}-t)$ and $(1-t)(\frac{t}{r}-\frac{1}{r+1})$ are concave because its second derivative is negative we apply the Barnes-Gudunova-Levin inequality obtaining:

$$\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f(\frac{a+b}{2})-\frac{2}{r(b-a)}\int_a^b f(x)dx\geq$$

$$\geq(b-a)^2[\frac{1}{B(p_1,q)}\left(\int_0^{\frac{1}{2}}(\frac{t}{r}(\frac{1}{r+1}-t))^{p_1}dt\right)^{\frac{1}{p_1}}\left(\int_0^{\frac{1}{2}}(f''(tb+(1-t)a))^q dt\right)^{\frac{1}{q}}+$$

$$+\frac{1}{B(p_2,q)}\left(\int_{\frac{1}{2}}^1((1-t)(\frac{t}{r}-\frac{1}{r+1}))^{p_2}dt\right)^{\frac{1}{p_2}}\left(\int_{\frac{1}{2}}^1(f''(tb+(1-t)a))^q dt\right)^{\frac{1}{q}}]\geq$$

$$\geq\frac{(b-a)^2}{r(r+1)}[\frac{1}{B(p_1,q)}(\int_0^{\frac{1}{2}}t^{p_1}(1-t(r+1))^{p_1}dt)^{\frac{1}{p_1}}(\int_0^{\frac{1}{2}}(t(f''(b))^q+(1-t)(f''(a))^q)dt)^{\frac{1}{q}}+$$

$$+\frac{1}{B(p_2,q)}(\int_{\frac{1}{2}}^1(1-t)^{p_2}(t(r+1)-r)^{p_2}dt)^{\frac{1}{p_2}}(\int_{\frac{1}{2}}^1(t(f''(b))^q+(1-t)(f''(a))^q)dt)^{\frac{1}{q}}]=$$

$$=\frac{(b-a)^2}{r(r+1)}[\frac{1}{B(p_1,q)}(\frac{1}{8}(f''(b))^q+\frac{3}{8}(f''(a))^q)^{\frac{1}{q}}(\int_0^{\frac{1}{2}}t^{p_1}(1-t(r+1))^{p_1}dt)^{\frac{1}{p_1}}+$$

$$+\frac{1}{B(p_2,q)}(\frac{3}{8}(f''(b))^q+\frac{1}{8}(f''(a))^q)^{\frac{1}{q}}(\int_{\frac{1}{2}}^1(1-t)^{p_2}(t(r+1)-r)^{p_2}dt)^{\frac{1}{p_2}}].$$

If we denote $\int_0^{\frac{1}{2}}t^p(1-t(r+1))^p dt$ by $I_1(p)$ and $\int_{\frac{1}{2}}^1(1-t)^p(t(r+1)-r)^p dt$ by $I_2(p)$ then using the substitution $u=1-t$ we have $I_2(p)=\int_0^{\frac{1}{2}}u^p((1-u)(r+1)-r)^p du=I_1(p)$.

Therefore we obtain,

$$\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f(\frac{a+b}{2})-\frac{2}{r(b-a)}\int_a^b f(x)dx\geq$$

$$\geq\frac{(b-a)^2}{r(r+1)}[\frac{1}{B(p_1,q)}I_1(p_1)^{\frac{1}{p_1}}(\frac{1}{8}(f''(b))^q+\frac{3}{8}(f''(a))^q)^{\frac{1}{q}}+$$

$$+\frac{1}{B(p_2,q)}I_1(p_2)^{\frac{1}{p_2}}(\frac{3}{8}(f^{''}(b))^q+\frac{1}{8}(f^{''}(a))^q)^{\frac{1}{q}}].$$

Now we consider

$$I_1(p)=\int_0^{\frac{1}{2}}t^p(1-t(r+1))^p dt,\ r\in(0,1),\ p>1.$$

Then it is easy to see that $t^p(1-t(r+1))^p > t^p(1-2t)^p \geq 0$ and

$$\int_0^{\frac{1}{2}}t^p(1-t(r+1))^p dt > \int_0^{\frac{1}{2}}t^p(1-2t)^p dt.$$

If we use substitution $2t=u$ in last integral, we obtain

$$I=\int_0^{\frac{1}{2}}t^p(1-2t)^p dt=\frac{1}{2^{p+1}}\int_0^1 u^p(1-u)^p du=\frac{1}{2^{p+1}}\beta(p+1,p+1).$$

Thus

$$I_1(p)>\frac{1}{2^{p+1}}\beta(p+1,p+1)=\frac{\Gamma(p+1)}{2^{p+1}(2p+1)...(p+1)}$$

and then the inequality becomes:

$$\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f(\frac{a+b}{2})-\frac{2}{r(b-a)}\int_a^b f(x)dx\geq$$

$$\geq\frac{(b-a)^2}{2r(r+1)}[\frac{1}{B(p_1,q)}(\frac{\Gamma(p_1+1)}{2(2p_1+1)...(p_1+1)})^{\frac{1}{p_1}}(\frac{1}{8}(f^{''}(b))^q+\frac{3}{8}(f^{''}(a))^q)^{\frac{1}{q}}+$$

$$+\frac{1}{B(p_2,q)}(\frac{\Gamma(p_2+1)}{2(2p_2+1)...(p_2+1)})^{\frac{1}{p_2}}(\frac{3}{8}(f^{''}(b))^q+\frac{1}{8}(f^{''}(a))^q)^{\frac{1}{q}}].$$

□

Remark 3. *(i) Under the above conditions if $p_1=p_2=p$ then the inequality becomes:*

$$\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f(\frac{a+b}{2})-\frac{2}{r(b-a)}\int_a^b f(x)dx\geq$$

$$\geq\frac{(b-a)^2}{r(r+1)B(p,q)}(\frac{\beta(p+1,p+1)}{2^{p+1}})^{\frac{1}{p}}[(\frac{1}{8}(f^{''}(b))^q+\frac{3}{8}(f^{''}(a))^q)^{\frac{1}{q}}+(\frac{3}{8}(f^{''}(b))^q+\frac{1}{8}(f^{''}(a))^q)^{\frac{1}{q}}]$$

(ii) Under the above conditions if $p_1=p_2=q_1=q_2=p$ then the inequality becomes:

$$\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f(\frac{a+b}{2})-\frac{2}{r(b-a)}\int_a^b f(x)dx\geq$$

$$\geq\frac{(b-a)^{3-(2/p)}}{12r(r+1)2^{4/p}}(\frac{\Gamma(p+1)(p+1)}{(2p+1)...(p+2)})^{\frac{1}{p}}[(f^{''}(b))^p+3(f^{''}(a))^p)^{\frac{1}{p}}+(3(f^{''}(b))^p+8(f^{''}(a))^p)^{\frac{1}{p}}].$$

Next theorem will give a lower bound for the left member of the equality from Lemma 3, see[5].

Theorem 11. *Suppose that $f : I \subset \mathbb{R} \to \mathbb{R}$ is a twice differentiable function on I°, the interior of I. Assume that $a,\ b \in I^\circ$ with $a < b$ and f'' is integrable on $[a, b]$. If f'' is nonnegative on $[a, b]$ and $(f'')^q$ is concave, $q > 1,\ p_1 > 1,\ p_2 > 1$ then the following inequality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^2}{16}[\frac{(\frac{\sqrt{\pi}\Gamma(p_1+1)}{\Gamma(p_1+\frac{3}{2})})^{\frac{1}{p_1}}}{B(p_1,q)}(\frac{3}{4}(f''(a))^q + \frac{1}{4}(f''(b))^q)^{\frac{1}{q}} +$$

$$+\frac{(\frac{\sqrt{\pi}\Gamma(p_2+1)}{\Gamma(p_2+\frac{3}{2})})^{\frac{1}{p_2}}}{B(p_2,q)}(\frac{1}{4}(f''(a))^q + \frac{3}{4}(f''(b))^q)^{\frac{1}{q}}].$$

Proof. Using Lemma 3 and the Barnes-Gudunova-Levin inequality we have

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx =$$

$$= \frac{(b-a)^2}{16}\int_0^1 (1-t^2)(f''(\frac{1+t}{2}a + \frac{1-t}{2}b)) + (f''(\frac{1-t}{2}a + \frac{1+t}{2}b))dt \geq$$

$$\geq \frac{(b-a)^2}{16}[\frac{1}{B(p_1,q)}(\int_0^1 (1-t^2)^{p_1}dt)^{\frac{1}{p_1}}(\int_0^1 (f''(\frac{1+t}{2}a + \frac{1-t}{2}b))^q dt)^{\frac{1}{q}} +$$

$$+\frac{1}{B(p_2,q)}(\int_0^1 (1-t^2)^{p_2}dt)^{\frac{1}{p_2}}(\int_0^1 (f''(\frac{1-t}{2}a + \frac{1+t}{2}b))^q dt)^{\frac{1}{q}}] \geq$$

$$\geq \frac{(b-a)^2}{16}[\frac{1}{B(p_1,q)}I(p_1)^{\frac{1}{p_1}}(\int_0^1 (\frac{1+t}{2}(f''(a))^q + \frac{1-t}{2}(f''(b))^q)dt)^{\frac{1}{q}} +$$

$$+\frac{1}{B(p_2,q)}I(p_2)^{\frac{1}{p_2}}(\int_0^1 (\frac{1-t}{2}(f''(a))^q + \frac{1+t}{2}(f''(b))^q)dt)^{\frac{1}{q}}],$$

where we denoted $\int_0^1 (1-t^2)^p dt$ by $I(p)$. By calculus we obtain,

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^2}{16}[\frac{I(p_1)^{\frac{1}{p_1}}}{B(p_1,q)}(\frac{3}{4}(f''(a))^q+\frac{1}{4}(f''(b))^q)^{\frac{1}{q}}+\frac{I(p_2)^{\frac{1}{p_2}}}{B(p_2,q)}(\frac{1}{4}(f''(a))^q+\frac{3}{4}(f''(b))^q)^{\frac{1}{q}}].$$

But by substitution $t = \sqrt{1-s}$ we obtain $I(p) = \frac{1}{2}\int_0^1 s^p \frac{1}{\sqrt{1-s}}ds = \frac{1}{2}\beta(p+1, \frac{1}{2}) = \frac{\sqrt{\pi}\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}$, and replacing in previous inequality,

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^2}{16}[\frac{(\frac{\sqrt{\pi}\Gamma(p_1+1)}{\Gamma(p_1+\frac{3}{2})})^{\frac{1}{p_1}}}{B(p_1,q)}(\frac{3}{4}(f''(a))^q + \frac{1}{4}(f''(b))^q)^{\frac{1}{q}} +$$

$$+\frac{\left(\frac{\sqrt{\pi}\Gamma(p_2+1)}{\Gamma(p_2+\frac{3}{2})}\right)^{\frac{1}{p_2}}}{B(p_2,q)}\left(\frac{1}{4}(f''(a))^q+\frac{3}{4}(f''(b))^q\right)^{\frac{1}{q}}].$$

□

Remark 4. *(i) Under the above conditions, if* $p_1 = p_2$ *we have:*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^2}{16}\frac{\left(\frac{\sqrt{\pi}\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}\right)^{\frac{1}{p}}}{B(p,q)}\left[\left(\frac{3}{4}(f''(a))^q+\frac{1}{4}(f''(b))^q\right)^{\frac{1}{q}}+\left(\frac{1}{4}(f''(a))^q+\frac{3}{4}(f''(b))^q\right)^{\frac{1}{q}}\right].$$

(ii) Under the above conditions, if $p_1 = p_2 = q = p$ *we have:*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^2}{16}\frac{\left(\frac{\sqrt{\pi}\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}\right)^{\frac{1}{p}}}{B(p,p)}\left[\left(\frac{3}{4}(f''(a))^p+\frac{1}{4}(f''(b))^p\right)^{\frac{1}{p}}+\left(\frac{1}{4}(f''(a))^p+\frac{3}{4}(f''(b))^p\right)^{\frac{1}{p}}\right].$$

5. Some Hermite-Hadamard's type inequalities for differentiable co-ordinated convex, quasi-convex and s-convex functions

We will give an analog of Lemma 1 see [11] for functions on rectangle from the plane $\mathbb{R}^2$. This result will be used then in the proof of next theorems.

Lemma 6. *Let* $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be a partial differentiable mapping on* $\Delta := [a,b]\times[c,d]$ *with* $a < b,\ c < d$. *If* $\frac{\partial^2 f}{\partial r \partial t} \in L(\Delta)$, *then the following identity takes place:*

$$\frac{64}{(b-a)(d-c)}[f(\frac{3a+b}{4},\frac{3c+d}{4})+f(\frac{3a+b}{4},\frac{c+3d}{4})+f(\frac{a+3b}{4},\frac{3c+d}{4})+$$

$$+f(\frac{a+3b}{4},\frac{c+3d}{4})-\frac{2}{d-c}\int_c^d(f(\frac{3a+b}{4},y)+f(\frac{a+3b}{4},y))dy-\frac{2}{b-a}\int_a^b(f(x,\frac{3c+d}{4})+$$

$$+f(x,\frac{c+3d}{4}))dx+\frac{4}{(b-a)(c-d)}\int_a^b\int_c^d f(x,y)dxdy]=$$

$$=\sum_{i=0,j=0}^{3}(-1)^{i+j}\int_0^1\int_0^1 s_i(t)s_j(r)\frac{\partial^2 f}{\partial r\partial t}(tx_{i+1}+(1-t)x_i, ry_{j+1}+(1-r)y_j)dtdr$$

where

$$s_i : [0,1] \to \mathbb{R},\ s_i(t) = t,\ i \in \{0,2\},$$
$$s_i : [0,1] \to \mathbb{R},\ s_i(t) = 1-t,\ i \in \{1,3\}$$

$x_i = \frac{(4-i)a+ib}{4},\ i = \overline{0,4}$ *and* $y_j = \frac{(4-j)c+jd}{4},\ j = \overline{0,4}$.

Proof. We denote

$$m_i(t) = tx_{i+1} + (1-t)x_i,\ i = \overline{0,3},$$
$$n_j(r) = ry_{j+1} + (1-r)y_j,\ j = \overline{0,3},$$

and

$$I_{ij} = \int_0^1 \int_0^1 s_i(t)s_j(r)\frac{\partial^2 f}{\partial r \partial t}(tx_{i+1} + (1-t)x_i, ry_{j+1} + (1-r)y_j)dtdr.$$

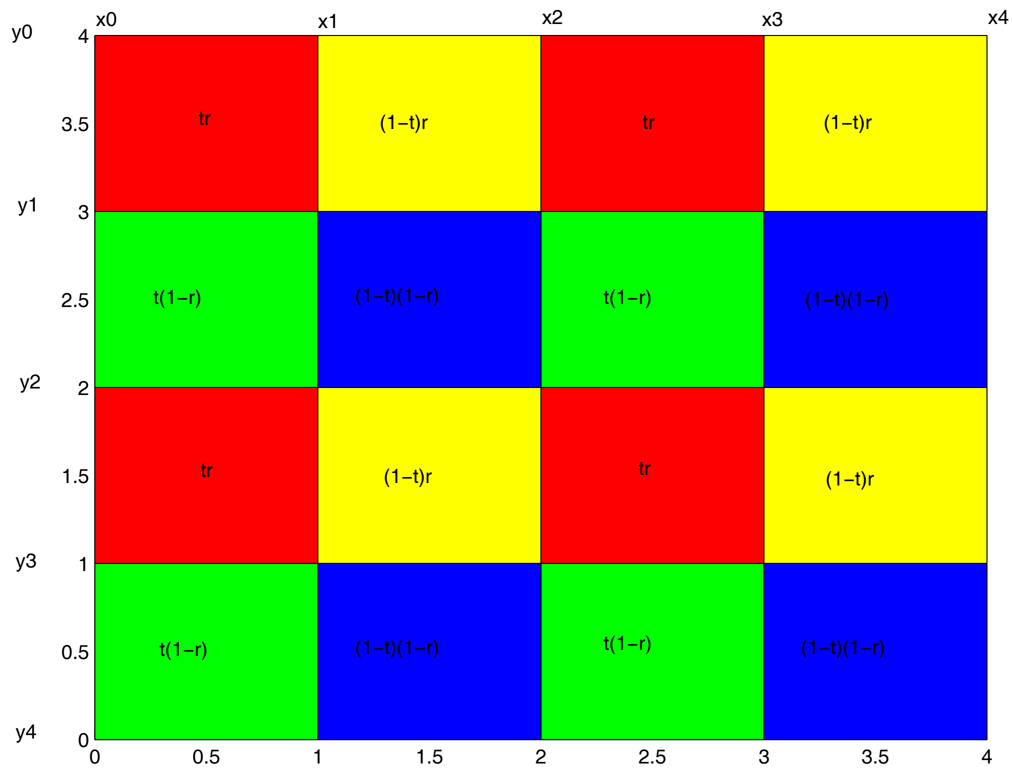

FIGURE 1

Now by integration by parts, we have

$$I_{ij} = \int_0^1 \int_0^1 s_i(t)s_j(r)\frac{\partial^2 f}{\partial r \partial t}(m_i(t), n_j(r))dtdr =$$

$$= \int_0^1 s_j(r)\frac{4}{b-a}[s_i(t)\frac{\partial f}{\partial r}(m_i(t), n_j(r))|_0^1 - \int_0^1 (-1)^i \frac{\partial f}{\partial r}(m_i(t), n_j(r))dt]dr =$$

$$= \frac{4}{b-a}\int_0^1 s_j(r)[\frac{1-(-1)^{i+1}}{2}\frac{\partial f}{\partial r}(m_i(1), n_j(r)) - \frac{1-(-1)^i}{2}\frac{\partial f}{\partial r}(m_i(0), n_j(r)) -$$

$$-(-1)^i \int_0^1 \frac{\partial f}{\partial r}(m_i(t), n_j(r))dt]dr = \frac{4}{(b-a)}[\frac{1-(-1)^{i+1}}{2}\int_0^1 s_j(r)\frac{\partial f}{\partial r}(m_i(1), n_j(r))dr -$$

$$-\frac{1-(-1)^i}{2}\int_0^1 s_j(r)\frac{\partial f}{\partial r}(m_i(0),n_j(r))dr-(-1)^i\int_0^1\int_0^1 s_j(r)\frac{\partial f}{\partial r}(m_i(t),n_j(r))dtdr]=$$

$$=\frac{16}{(b-a)(d-c)}[\frac{1-(-1)^{i+1}}{2}(\frac{1-(-1)^{j+1}}{2}f(m_i(1),n_j(1))-\frac{1-(-1)^j}{2}f(m_i(1),n_j(0))-$$

$$-(-1)^j\int_0^1 f(m_i(1),n_j(r))dr)-\frac{1-(-1)^i}{2}(\frac{1-(-1)^{j+1}}{2}f(m_i(0),n_j(1))-$$

$$-\frac{1-(-1)^j}{2}f(m_i(0),n_j(0))-(-1)^j\int_0^1 f(m_i(0),n_j(r)dr)+$$

$$+(-1)^{i+1}\int_0^1(\frac{1-(-1)^{j+1}}{2}f(m_i(t),n_j(1))-\frac{1-(-1)^j}{2}f(m_i(t),n_j(0))-$$

$$-\int_0^1(-1)^j f(m_i(t),n_j(r))dr)dt]=\frac{16}{(b-a)(d-c)}[\frac{(1-(-1)^{i+1})(1-(-1)^{j+1})}{4}\cdot$$

$$\cdot f(m_i(1),n_j(1))-\frac{(1-(-1)^{i+1})(1-(-1)^j)}{4}f(m_i(1),n_j(0))-\frac{(1-(-1)^i)(1-(-1)^{j+1})}{4}\cdot$$

$$\cdot f(m_i(0),n_j(1))+\frac{(1-(-1)^i)(1-(-1)^j)}{4}f(m_i(0),n_j(0))+(-1)^{j+1}\frac{1-(-1)^{i+1}}{2}\cdot$$

$$\cdot\int_0^1 f(m_i(1),n_j(r))dr+(-1)^j\frac{1-(-1)^i}{2}\int_0^1 f(m_i(0),n_j(r))dr+(-1)^{i+1}\frac{1-(-1)^{j+1}}{2}\cdot$$

$$\cdot\int_0^1 f(m_i(t),n_j(1))dt+(-1)^i\frac{1-(-1)^j}{2}f(m_i(t),n_j(0))+(-1)^{i+j}\int_0^1\int_0^1 f(m_i(t),n_j(r))dtdr].$$

If we make use of the substitution $x=tx_{i+1}+(1-t)x_i,\ i=\overline{0,3}$, $t\in[0,1]$ (that means $t\in[0,1]\Rightarrow x\in[x_i,x_{i+1}]$ and $dx=\frac{b-a}{4}dt$) $y=ry_{j+1}+(1-r)y_j,\ j=\overline{0,3}$, $r\in[0,1]$ (that means $r\in[0,1]\Rightarrow y\in[y_j,y_{j+1}]$ and $dy=\frac{d-c}{4}$) and take into account that $m_1(0)=m_0(1)=x_1,\ m_3(0)=m_2(1)=x_3$ and $n_0(1)=n_1(0)=y_1,\ n_3(0)=n_2(1)=y_3$ we obtain

$$\sum_{i=0,j=0}^{3}(-1)^{i+j}\int_0^1\int_0^1 s_i(t)s_j(r)\frac{\partial^2 f}{\partial r\partial t}(tx_{i+1}+(1-t)x_i,ry_{j+1}+(1-r)y_j)dtdr=$$

$$=\frac{16}{(b-a)(d-c)}[4f(\frac{3a+b}{4},\frac{3c+d}{4})+4f(\frac{3a+b}{4},\frac{c+3d}{4})+4f(\frac{a+3b}{4},\frac{3c+d}{4})+$$

$$+4f(\frac{a+3b}{4},\frac{c+3d}{4})+\frac{16}{(b-a)(d-c)}\sum_{i,j=0}^{3}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}f(x,y)dxdy+$$

$$+\frac{4}{d-c}(\sum_{i=0}^{3}\frac{-1+(-1)^{i+1}}{2}\sum_{j=0}^{3}\int_{y_j}^{y_{j+1}}f(x_{i+1},y)dy+\sum_{i=3}^{3}\frac{-1+(-1)^i}{2}\sum_{j=1}^{3}\int_{y_j}^{y_{j+1}}f(x_i,y)dy)+$$

$$+\frac{4}{b-a}(\sum_{j=0}^{3}\frac{-1+(-1)^{j+1}}{2}\sum_{i=0}^{3}\int_{x_i}^{x_{i+1}}f(x,y_{j+1})dx+\sum_{j=3}^{3}\frac{-1+(-1)^j}{2}\sum_{i=1}^{3}\int_{x_i}^{x_{i+1}}f(x,y_j)dx))=$$

$$= \frac{16}{(b-a)(d-c)}[4f(\frac{3a+b}{4}, \frac{3c+d}{4}) + 4f(\frac{3a+b}{4}, \frac{c+3d}{4}) + 4f(\frac{a+3b}{4}, \frac{3c+d}{4}) +$$

$$+4f(\frac{a+3b}{4}, \frac{c+3d}{4}) + \frac{16}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dxdy +$$

$$+\frac{4}{d-c}(\sum_{i=0}^{3} \frac{-1+(-1)^{i+1}}{2} \int_c^d f(x_{i+1},y)dy + \sum_{i=3}^{3} \frac{-1+(-1)^{i}}{2} \int_c^3 f(x_i,y)dy) +$$

$$+\frac{4}{b-a}(\sum_{j=0}^{3} \frac{-1+(-1)^{j+1}}{2} \int_a^b f(x,y_{j+1})dx + \sum_{j=3}^{3} \frac{-1+(-1)^{j}}{2} \int_a^b f(x,y_j)dx)].$$

□

Theorem 12. *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a,b] \times [c,d]$ with $a < b$, $c < d$. If $|\frac{\partial^2 f}{\partial r \partial t}|$ is convex on the co-ordinates on Δ then the following inequality holds:*

$$\frac{64}{(b-a)(d-c)}[f(\frac{3a+b}{4}, \frac{3c+d}{4}) + f(\frac{3a+b}{4}, \frac{c+3d}{4}) + f(\frac{a+3b}{4}, \frac{3c+d}{4}) +$$

$$+f(\frac{a+3b}{4}, \frac{c+3d}{4}) - \frac{2}{d-c}\int_c^d (f(\frac{3a+b}{4}, y) + f(\frac{a+3b}{4}, y))dy - \frac{2}{b-a}\int_a^b (f(x, \frac{3c+d}{4}) +$$

$$+f(x, \frac{c+3d}{4}))dx + \frac{4}{(b-a)(c-d)} \int_a^b \int_c^d f(x,y)dxdy] \leq$$

$$\leq \frac{4}{9}[|\frac{\partial^2 f}{\partial r \partial t}(\frac{3a+b}{4}, \frac{3c+d}{4})| + |\frac{\partial^2 f}{\partial r \partial t}(\frac{3a+b}{4}, \frac{c+3d}{4})| + |\frac{\partial^2 f}{\partial r \partial t}(\frac{a+3b}{4}, \frac{3c+d}{4})| +$$

$$+|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+3b}{4}, \frac{c+3d}{4})|] + \frac{1}{36}[|\frac{\partial^2 f}{\partial r \partial t}(a,c)| + |\frac{\partial^2 f}{\partial r \partial t}(a,d)| + |\frac{\partial^2 f}{\partial r \partial t}(b,c)| + |\frac{\partial^2 f}{\partial r \partial t}(b,d)| +$$

$$+2|\frac{\partial^2 f}{\partial r \partial t}(a, \frac{c+d}{2})| + 2|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+b}{2}, c)| + 2|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+b}{2}, d)| + 2|\frac{\partial^2 f}{\partial r \partial t}(b, \frac{c+d}{2})| +$$

$$+4|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+b}{2}, \frac{c+d}{2})|] + \frac{1}{9}[(|\frac{\partial^2 f}{\partial r \partial t}(\frac{3a+b}{4}, c)| + |\frac{\partial^2 f}{\partial r \partial t}(\frac{a+3b}{4}, c)| + |\frac{\partial^2 f}{\partial r \partial t}(\frac{3a+b}{4}, d)| +$$

$$+|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+3b}{4}, d)| + 2|\frac{\partial^2 f}{\partial r \partial t}(\frac{3a+b}{4}, \frac{c+d}{2})| + 2|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+3b}{4}, \frac{c+d}{2})|) + (|\frac{\partial^2 f}{\partial r \partial t}(a, \frac{3c+d}{4})| +$$

$$+|\frac{\partial^2 f}{\partial r \partial t}(a, \frac{c+3d}{4})| + |\frac{\partial^2 f}{\partial r \partial t}(b, \frac{3c+d}{4})| + |\frac{\partial^2 f}{\partial r \partial t}(b, \frac{c+3d}{4})| + 2|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+b}{2}, \frac{3c+d}{4})| +$$

$$+2|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+b}{2}, \frac{c+3d}{4})|)].$$

Proof. Using previous lemma and Definition 6 we have

$$I=\frac{64}{(b-a)(d-c)}[f(\frac{3a+b}{4},\frac{3c+d}{4})+f(\frac{3a+b}{4},\frac{c+3d}{4})+f(\frac{a+3b}{4},\frac{3c+d}{4})+$$

$$+f(\frac{a+3b}{4},\frac{c+3d}{4})-\frac{2}{d-c}\int_c^d(f(\frac{3a+b}{4},y)+f(\frac{a+3b}{4},y))dy-\frac{2}{b-a}\int_a^b(f(x,\frac{3c+d}{4})+$$

$$+f(x,\frac{c+3d}{4}))dx+\frac{4}{(b-a)(c-d)}\int_a^b\int_c^d f(x,y)dxdy]\leq$$

$$\leq\sum_{i,j=0}^{3}\int_0^1\int_0^1 s_i(t)s_j(r)|\frac{\partial^2 f}{\partial r\partial t}(tx_{i+1}+(1-t)x_i,ry_{j+1}+(1-r)y_j)|dtdr\leq$$

$$\leq\sum_{i,j=0}^{3}\int_0^1\int_0^1 s_i(t)s_j(r)[tr|\frac{\partial^2 f}{\partial r\partial t}(x_{i+1},y_{j+1})|+t(1-r)|\frac{\partial^2 f}{\partial r\partial t}(x_{i+1},y_j)|+$$

$$+r(1-t)|\frac{\partial^2 f}{\partial r\partial t}(x_i,y_{j+1})|+(1-t)(1-r)|\frac{\partial^2 f}{\partial r\partial t}(x_i,y_j)|]=$$

$$=\sum_{i,j=0}^{3}[D_{i+1,j+1}\int_0^1\int_0^1 trs_i(t)s_j(r)dtdr+D_{i+1,j}\int_0^1\int_0^1 t(1-r)s_i(t)s_j(r)dtdr+$$

$$+D_{i,j+1}\int_0^1\int_0^1(1-t)rs_i(t)s_j(r)dtdr+D_{i,j}\int_0^1\int_0^1(1-t)(1-r)s_i(t)s_j(r)dtdr]=$$

$$=\sum_{i,j\in\{0,2\}}[D_{i+1,j+1}\int_0^1\int_0^1 t^2r^2dtdr+D_{i+1,j}\int_0^1\int_0^1 t^2(1-r)rdtdr+$$

$$+D_{i,j+1}\int_0^1\int_0^1(1-t)tr^2dtdr+D_{i,j}\int_0^1\int_0^1(1-t)t(1-r)rdtdr]+$$

$$+\sum_{i,j\in\{1,3\}}[D_{i+1,j+1}\int_0^1\int_0^1 tr(1-t)(1-r)dtdr+D_{i+1,j}\int_0^1\int_0^1 t(1-t)(1-r)^2dtdr+$$

$$+D_{i,j+1}\int_0^1\int_0^1(1-t)^2r(1-r)dtdr+D_{i,j}\int_0^1\int_0^1(1-t)^2(1-r)^2dtdr]+$$

$$+\sum_{i\in\{0,2\},j\in\{1,3\}}[D_{i+1,j+1}\int_0^1\int_0^1 t^2r(1-t)dtdr+D_{i+1,j}\int_0^1\int_0^1 t^2(1-r)^2dtdr+$$

$$+D_{i,j+1}\int_0^1\int_0^1 t(1-t)r(1-r)dtdr+D_{i,j}\int_0^1\int_0^1 t(1-t)(1-r)^2dtdr]+$$

$$+\sum_{i\in\{1,3\},j\in\{0,2\}}[D_{i+1,j+1}\int_0^1\int_0^1 t(1-t)r^2dtdr+D_{i+1,j}\int_0^1\int_0^1 t(1-t)r(1-r)dtdr+$$

$$+D_{i,j+1}\int_0^1\int_0^1(1-t)^2r^2dtdr+D_{i,j}\int_0^1\int_0^1(1-t)^2r(1-r)dtdr],$$

where $D_{i,j} = |\frac{\partial^2 f}{\partial r \partial t}(x_i, y_j)|,\ i, j = \overline{0,4}$. By calculus we obtain:

$$I \leq \sum_{i,j\in\{0,2\}} [D_{i+1,j+1}\frac{1}{9} + D_{i+1,j}\frac{1}{18} + D_{i,j+1}\frac{1}{18} + D_{i,j}\frac{1}{36}] +$$

$$+ \sum_{i,j\in\{1,3\}} [D_{i+1,j+1}\frac{1}{36} + D_{i+1,j}\frac{1}{18} + D_{i,j+1}\frac{1}{18} + D_{i,j}\frac{1}{9}] +$$

$$+ \sum_{i\in\{0,2\},j\in\{1,3\}} [D_{i+1,j+1}\frac{1}{18} + D_{i+1,j}\frac{1}{9} + D_{i,j+1}\frac{1}{36} + D_{i,j}\frac{1}{18}] +$$

$$+ \sum_{i\in\{1,3\},j\in\{0,2\}} [D_{i+1,j+1}\frac{1}{18} + D_{i+1,j}\frac{1}{36} + D_{i,j+1}\frac{1}{9} + D_{i,j}\frac{1}{18}],$$

and then the inequality from theorem.

□

We formulate below now the a similar result for quasi-convex functions on co-ordinates.

Theorem 13. *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a, b] \times [c, d]$ with $a < b,\ c < d$. If $|\frac{\partial^2 f}{\partial r \partial t}|$ is quasi-convex on the co-ordinates on Δ then the following inequality holds:*

$$\frac{64}{(b-a)(d-c)}[f(\frac{3a+b}{4}, \frac{3c+d}{4}) + f(\frac{3a+b}{4}, \frac{c+3d}{4}) + f(\frac{a+3b}{4}, \frac{3c+d}{4}) +$$

$$+ f(\frac{a+3b}{4}, \frac{c+3d}{4}) - \frac{2}{d-c}\int_c^d (f(\frac{3a+b}{4}, y) + f(\frac{a+3b}{4}, y))dy - \frac{2}{b-a}\int_a^b (f(x, \frac{3c+d}{4}) +$$

$$+ f(x, \frac{c+3d}{4}))dx + \frac{4}{(b-a)(c-d)}\int_a^b\int_c^d f(x,y)dxdy] \leq$$

$$\leq \frac{1}{4}\sum_{i,j=0}^{3} \max\{|\frac{\partial^2 f}{\partial r \partial t}(x_{i+i}, y_{j+1})|, |\frac{\partial^2 f}{\partial r \partial t}(x_{i+1}, y_j)|, |\frac{\partial^2 f}{\partial r \partial t}(x_i, y_{j+1})|, |\frac{\partial^2 f}{\partial r \partial t}(x_i, y_j)|\}.$$

Proof. Using now Lemma 6 and Definition 8 we find that the left member is less than the following expressions:

$$\sum_{i,j=0}^{3}\int_0^1\int_0^1 s_i(t)s_j(r)|\frac{\partial^2 f}{\partial r \partial t}(tx_{i+1} + (1-t)x_i, ry_{j+1} + (1-r)y_j)|dtdr \leq$$

$$\leq \sum_{i,j=0}^{3}\int_0^1\int_0^1 s_i(t)s_j(r)\max\{|\frac{\partial^2 f}{\partial r \partial t}(x_{i+i}, y_{j+1})|, |\frac{\partial^2 f}{\partial r \partial t}(x_{i+1}, y_j)|, |\frac{\partial^2 f}{\partial r \partial t}(x_i, y_{j+1})|, |\frac{\partial^2 f}{\partial r \partial t}(x_i, y_j)|\}.$$

□

For s- convex functions on the co-ordinates we can formulate also the following result:

Theorem 14. *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partial differentiable mapping on Δ°, $\Delta = [a, b] \times [c, d]$ with $a < b$, $c < d$, $a, c \geq 0$ such that $\frac{\partial^2 f}{\partial r \partial t} \in L(\Delta)$. If $|\frac{\partial^2 f}{\partial r \partial t}|$ is s-convex on the co-ordinates on Δ then the following inequality holds:*

$$\frac{64}{(b-a)(d-c)}[f(\frac{3a+b}{4}, \frac{3c+d}{4}) + f(\frac{3a+b}{4}, \frac{c+3d}{4}) + f(\frac{a+3b}{4}, \frac{3c+d}{4})+$$

$$+f(\frac{a+3b}{4}, \frac{c+3d}{4}) - \frac{2}{d-c}\int_c^d (f(\frac{3a+b}{4}, y) + f(\frac{a+3b}{4}, y))dy - \frac{2}{b-a}\int_a^b (f(x, \frac{3c+d}{4})+$$

$$+f(x, \frac{c+3d}{4}))dx + \frac{4}{(b-a)(c-d)}\int_a^b\int_c^d f(x,y)dxdy] \leq$$

$$\leq \frac{1}{(s+2)^2}[|\frac{\partial^2 f}{\partial r \partial t}(\frac{3a+b}{4}, \frac{3c+d}{4})| + |\frac{\partial^2 f}{\partial r \partial t}(\frac{3a+b}{4}, \frac{c+3d}{4})| + |\frac{\partial^2 f}{\partial r \partial t}(\frac{a+3b}{4}, \frac{3c+d}{4})|+$$

$$+|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+3b}{4}, \frac{c+3d}{4})|] + \frac{1}{(s+1)^2(s+2)^2}[|\frac{\partial^2 f}{\partial r \partial t}(a,c)| + |\frac{\partial^2 f}{\partial r \partial t}(a,d)| + |\frac{\partial^2 f}{\partial r \partial t}(b,c)|+$$

$$+|\frac{\partial^2 f}{\partial r \partial t}(b,d)| + 2|\frac{\partial^2 f}{\partial r \partial t}(a, \frac{c+d}{2})| + 2|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+b}{2}, c)| + 2|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+b}{2}, d)| + 2|\frac{\partial^2 f}{\partial r \partial t}(b, \frac{c+d}{2})|+$$

$$+4|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+b}{2}, \frac{c+d}{2})|] + \frac{1}{(s+1)(s+2)^2}[(|\frac{\partial^2 f}{\partial r \partial t}(\frac{3a+b}{4}, c)| + |\frac{\partial^2 f}{\partial r \partial t}(\frac{a+3b}{4}, c)|+$$

$$+|\frac{\partial^2 f}{\partial r \partial t}(\frac{3a+b}{4}, d)| + |\frac{\partial^2 f}{\partial r \partial t}(\frac{a+3b}{4}, d)| + 2|\frac{\partial^2 f}{\partial r \partial t}(\frac{3a+b}{4}, \frac{c+d}{2})| + 2|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+3b}{4}, \frac{c+d}{2})|)+$$

$$+(|\frac{\partial^2 f}{\partial r \partial t}(a, \frac{3c+d}{4})| + |\frac{\partial^2 f}{\partial r \partial t}(a, \frac{c+3d}{4})| + |\frac{\partial^2 f}{\partial r \partial t}(b, \frac{3c+d}{4})| + |\frac{\partial^2 f}{\partial r \partial t}(b, \frac{c+3d}{4})|+$$

$$+2|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+b}{2}, \frac{3c+d}{4})| + 2|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+b}{2}, \frac{c+3d}{4})|)].$$

Proof. The proof will be as in Theorem 12. □

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